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POSITIVE SOLUTIONS OF DIFFUSIVE LOGISTIC EQUATIONS

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Dedicated to Professor Kunihiko Kajitani on the occasion of his 60th birthday

Abstract. This paper is devoted to static bifurcation theory for a class of *degenerate* boundary value problems for diffusive logistic equations with indefinite weights which model population dynamics in environments with spatial heterogeneity. The purpose of this paper is to discuss the changes that occur in the structure of the positive solutions as a parameter varies near the first eigenvalue of the linearized problem.

1. Introduction and Main Results

Let D be a bounded domain of Euclidean space $\mathbb{R}^N$, $N \geq 2$, with smooth boundary ∂D; its closure $\overline{D} = D \cup \partial D$ is an N-dimensional, compact smooth manifold with boundary. This paper is devoted to the study of the existence and uniqueness of positive solutions of the following semilinear elliptic boundary value problem:

$$\begin{cases} -\Delta u = \lambda(m(x) - h(x)\,u)u & \text{in } D, \\ Bu := a(x')\frac{\partial u}{\partial \mathbf{n}} + b(x')u = 0 & \text{on } \partial D. \end{cases} \tag{1.1}$$

Here:

(1) $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \cdots + \partial^2/\partial x_N^2$ is the usual Laplacian.

(2) λ is a real parameter.

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(3) $m(x)$ and $h(x)$ are real-valued Hölder continuous functions with exponent $0 < \theta < 1$ on $\overline{D}$.

(4) $a(x')$ and $b(x')$ are nonnegative smooth functions on ∂D.

(5) $\mathbf{n} = (n_1, n_2, \dots, n_N)$ is the unit exterior normal to the boundary ∂D.

A solution $u \in C^2(\overline{D})$ of problem (1.1) is said to be *nontrivial* if it does not identically equal zero on $\overline{D}$. We call a nontrivial solution u of problem (1.1) a *positive solution* if $u(x) \geq 0$ on $\overline{D}$.

We discuss our motivation and some of the modeling process leading to problem (1.1). The basic interpretation of the various terms in problem (1.1) is that $u(x)$ represents the population density of a species inhabiting the region D. The members of the population are assumed to move about D via the type of random walks occurring in Brownian motion which is modeled by the diffusive term $(1/\lambda)\Delta$; hence $1/\lambda$ represents the diffusion rate, so for small values of λ the population spreads more rapidly than for larger values of λ. The local rate of change in the population density is described by the density dependent term $m(x) - h(x)u$. In this term, $m(x)$ describes the rate at which the population would grow or decline at the location x in the absence of crowding or limitations on the availability of resources. The sign of $m(x)$ will be positive on favorable habitats for population growth and negative on unfavorable ones. Specifically, $m(x)$ may be considered as a food source or any resource which will be good in some areas and bad in some others. The term $-h(x)u$ describes the effects of crowding on the growth rate of the population at the location x; these effects are assumed to be independent of those determining the growth rate at low densities. The size of $h(x)$ describes the strength of the crowding effects.

On the other hand, in terms of biology, the functions $a(x')$ and $b(x')$ measure the hostility of the exterior of the domain. For example, if $a(x') \equiv 0$ and $b(x') \equiv 1$ on ∂D, then the (Dirichlet) boundary condition B represents that D is surrounded by a completely hostile exterior such that any member of the population which reaches the boundary dies immediately; in other words, the exterior of the domain is deadly to the population. If $a(x') \equiv 1$ and $b(x') \equiv 0$ on ∂D, then the (Neumann) boundary condition B represents that the boundary acts as a barrier, that is, individuals reaching the boundary simply return to the interior. If the exterior is hostile but not completely deadly, then the general boundary condition $Bu = a(\partial u)/(\partial \mathbf{n}) + bu = 0$ results.

In this paper we study problem (1.1) under the following two conditions on the functions $m(x)$, $a(x')$ and $b(x')$:

(H.1) The function $m(x)$ takes a *positive* value in D.

(H.2) $a(x') + b(x') > 0$ on ∂D, and $b(x') \not\equiv 0$ on ∂D.

Condition (H.1) implies that there exists a region endowed with a nice food source, while condition (H.2) implies that the exterior of the domain is not totally reflective, that is, the boundary condition B is not the pure Neumann condition. It is worth while pointing out here that problem (1.1) is a degenerate elliptic boundary value problem from an analytical point of view. This is due to the fact that the so-called Shapiro and Lopatinskii complementary condition is violated at the points $x' \in \partial D$ where $a(x') = 0$.

To study problem (1.1), we consider the following linearized eigenvalue problem:

$$\begin{cases} -\Delta\varphi = \lambda\, m(x)\, \varphi & \text{in } D, \\ B\varphi = 0 & \text{on } \partial D. \end{cases} \tag{1.2}$$

The next theorem asserts that the first eigenvalue of problem (1.2) is simple and its corresponding eigenfunction is positive, which is a generalization of a result due to Manes and Micheletti [14] (see [5, Theorem 1.13]) to the degenerate case:

Theorem 1.1. *If conditions* (H.1) *and* (H.2) *are satisfied, then the first eigenvalue $\lambda_1(m)$ of problem* (1.2) *is positive and simple, and its corresponding eigenfunction $\psi_1(x)$ may be chosen to be positive everywhere in D. Moreover, no other eigenvalues have positive eigenfunctions.*

Theorem 1.1 is proved by Taira [21, Theorem 1.1]. By the Rayleigh principle, we can prove that the first eigenvalue $\lambda_1(m)$ is characterized by the variational formula

$$\lambda_1(m) = \inf\left\{ \frac{(-\Delta\phi, \phi)}{\int_D m(x)\phi^2\, dx} : \phi \in H^2(D), B\phi = 0, \int_D m(x)\phi^2\, dx > 0 \right\}.$$

The main purpose of this paper is to discuss the changes that occur in the structure of the positive solutions as the parameter λ varies near the first eigenvalue $\lambda_1(m)$ under the condition:

(H.3) The function $m(x)$ attains both *positive* and *negative* values in D.

Assume that $h(x)$ is a function in the Hölder space $C^\theta(\overline{D})$, $0 < \theta < 1$, such that

$$h(x) \geq 0 \quad \text{on } \overline{D}.$$

We let

$$D_0(h) = \text{the interior of the set } \{x \in D : h(x) = 0\}.$$

We consider the case where $h(x) > 0$ on the boundary ∂D. More precisely, our structural condition on the function $h(x)$ is the following (see Figure 1.1 below):

(Z) The open set $D_0(h)$ consists of a *finite* number of connected components with smooth boundary, say $D_i(h)$, $1 \leq i \leq \ell$, which are bounded away from ∂D.

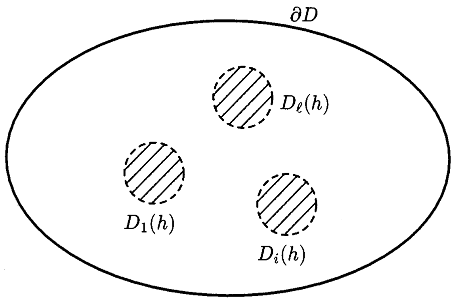

Figure 1.1

In the Dirichlet case, condition (Z) can be weakened so that the function $h(x)$ may vanish on the boundary ∂D (see [7, Theorem 3.5]).

We consider the Dirichlet eigenvalue problem in each connected component $D_i(h)$, $1 \leq i \leq \ell$,

$$\begin{cases} -\Delta\varphi = \lambda\, m(x)\, \varphi & \text{in } D_i(h), \\ \varphi = 0 & \text{on } \partial D_i(h), \end{cases} \tag{1.3}$$

and let

$$\lambda_1(D_i(h)) = \text{the } first\ eigenvalue \text{ of problem (1.3)}.$$

By the Rayleigh principle, we know that the first eigenvalue $\lambda_1(D_i(h))$ is given by the variational formula

$$\lambda_1(D_i(h)) = \inf\left\{ \int_{D_i(h)} |\nabla\psi|^2\, dx : \psi \in H_0^1(D_i(h)),\ \|\psi\|_{L^2(D_i(h))} = 1 \right\}.$$

Here $H_0^1(D_i(h))$ is the closure of smooth functions with compact support in $D_i(h)$ in the Sobolev space $H^1(D_i(h))$.

We let

$$\mu_1(D_0(h)) = \min\{\lambda_1(D_1(h)), \lambda_1(D_2(h)), \ldots, \lambda_1(D_\ell(h))\}.$$

Now we can state our main result which is a generalization of Fraile *et al.* [7, Theorem 3.5] to the degenerate case (cf. [17, Theorem 3.2], [20, Theorem 1.2]):

Theorem 1.2. *Assume that conditions* (H.2) *and* (H.3) *are satisfied, and that the function $h(x)$ satisfies condition* (Z) *and further that each set $\{x \in D_0^i(h) : m(x) > 0\}$, $1 \leq i \leq \ell$, has positive measure. Then problem* (1.1) *has a unique positive solution $u(\lambda)$ in the space $C^{2+\theta}(\overline{D})$ for every $\lambda \in (\lambda_1(m), \mu_1(D_0(h)))$. For any $\lambda \geq \mu_1(D_0(h))$, there exists no positive solution of problem* (1.1). *Moreover, we have*

$$\lim_{\lambda \to \mu_1(D_0(h))} \|u(\lambda)\|_{L^2(D)} = +\infty,$$

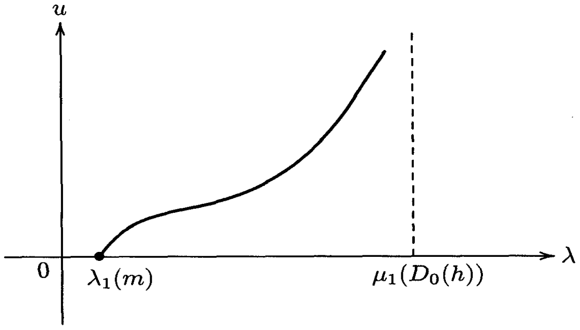

Figure 1.2

and also

$$\lim_{\lambda \to \lambda_1(m)} \|u(\lambda)\|_{C^{2+\theta}(\overline{D})} = 0.$$

Our situation may be represented schematically by the above bifurcation diagram 1.2.
Rephrased, Theorem 1.2 asserts that the models we consider predict persistence for a population if its diffusion rate $1/\lambda$ is below the critical value $1/\lambda_1(m)$ depending on the coefficient $m(x)$ describing the growth rate and if it is above the critical value $1/\mu_1(D_0(h))$ depending on the coefficient $h(x)$ describing the strength of the crowding effects. We remark that the first eigenvalue $\lambda_1(m)$ will tend to be smaller in situations where favorable and unfavorable habitats are closely intermingled (producing cancellation effects), and larger when the favorable region consists of a relatively small number of relatively large isolated components.

The rest of this paper is organized as follows. In Section 2 we summarize the basic definitions and results about ordered Banach spaces and the well-known Kreĭn-Rutman theorem for strongly positive, compact linear operators (Theorem 2.1), which enter naturally in connection with elliptic eigenvalue problems. In particular, we prove the comparison theorem for the first eigenvalues of Dirichlet and Neumann problems (Theorem 2.5). Sections 3 and 4 are devoted to the proof of Theorem 1.2. First, by using Green's formula we prove that if there exists a positive solution $u(\lambda) \in C^2(\overline{D})$ of problem (1.1), then we have $\lambda > \lambda_1(m)$ (Lemma 3.1). The existence of positive solutions of problem (1.1) near the point $(\lambda_1(m), 0)$ follows by applying local static bifurcation theory from a simple eigenvalue due to Crandall and Rabinowitz [4] (Lemma 3.2). Next, by making use of the implicit function theorem we prove that there exists a critical value $\overline{\lambda}(h) \in (\lambda_1(m), \mu_1(D_0(h))]$ such that problem (1.1) has a positive solution $u(\lambda)$ for all $\lambda \in (\lambda_1(m), \overline{\lambda}(h))$ (Lemma 3.3). The formula $\overline{\lambda}(h) = \mu_1(D_0(h))$ follows from the uniqueness of a bifurcation point and the comparison principle (Theorem 4.3).

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2. The Theory of Ordered Banach Spaces

A general class of semilinear second-order elliptic boundary value problems satisfies the maximum principle. Roughly speaking, this additional information means that the operators associated with the boundary value problems are compatible with the natural ordering of the underlying function spaces. Consequently, we are led to the study of nonlinear equations in the framework of ordered Banach spaces. This setting has the advantages that it takes into consideration in an optimal way the *a priori* information given by the maximum principle and that it is amenable to the methods of abstract functional analysis (cf. [1, 11]).

2.1. Ordered Banach spaces and the Kreĭn-Rutman theorem

Let X be a nonempty set. An ordering $\leq$ in X is a relation in X which is reflexive, transitive and antisymmetric. A nonempty set together with an ordering is called an ordered set.

Let V be a real vector space. An ordering $\leq$ in V is said to be *linear* if the following two conditions are satisfied:

(i) If x, $y \in V$ and $x \leq y$, then we have $x + z \leq y + z$ for all $z \in V$.

(ii) If x, $y \in V$ and $x \leq y$, then we have $\alpha x \leq \alpha y$ for all $\alpha \geq 0$.

A real vector space together with a linear ordering is called an *ordered vector space*.

If we let

$$Q = \{x \in V : x \geq 0\},$$

then it is easy to verify that the set Q has the following two conditions:

(iii) If x, $y \in Q$, then $\alpha x + \beta y \in Q$ for all α, $\beta \geq 0$.

(iv) If $x \neq 0$, then at least one of x and $-x$ does not belong to Q, that is, $Q \cap (-Q) = \{0\}$.

The set Q is called the *positive cone* of the ordering $\leq$.

Let E be a Banach space E with a linear ordering $\leq$. The Banach space E is called an *ordered Banach space* if the positive cone P is closed in E. For x, $y \in E$, we write

$$\begin{aligned} x \geq y \quad & \text{if } x - y \in P, \\ x > y \quad & \text{if } x - y \in P \setminus \{0\}. \end{aligned}$$

If the interior $\mathrm{int}(P)$ is nonempty, then we write

$$x \gg y \quad \text{if } x - y \in \mathrm{int}\,(P).$$

A linear operator $K : E \to E$ is said to be *strongly positive* if Kx belongs to $\mathrm{int}(P)$ for every $x \in P \setminus \{0\}$:

$$x > 0 \implies Kx \gg 0.$$

Then the well-known Kreĭn - Rutman theorem for strongly positive, compact linear operators reads as follows (see [12]):

Theorem 2.1. *Let* (E, P) *be an ordered Banach space with nonempty* $\mathrm{int}(P)$ *and* $K : E \to E$ *a linear operator. If* K *is strongly positive and compact, then we have the following* :

(1) $r := \lim_{n\to\infty} \sqrt[n]{\|K^n\|} > 0$, *and* r *is the unique eigenvalue of* K *having a positive eigenfunction* x.

(2) *The eigenvalue* r *is algebraically simple and* $x \gg 0$.

(3) *The eigenvalue* r *is greater than all the remaining eigenvalues* λ *of* K : $r > |\lambda|$.

The eigenvalue r is called the *principal eigenvalue* of K.

As an application of the Kreĭn - Rutman theorem, we consider the following nonhomogeneous equation: For given $h > 0$ in E, find an element $u \in E$ such that

$$\lambda u - Ku = h, \tag{2.1}$$

where λ is a real parameter.

The next theorem will play an important role in the proof of Theorem 1.2 in the sequel (see [1, 11]):

Theorem 2.2. *Let* $K : E \to E$ *be a strongly positive, compact linear operator and* r *its principal eigenvalue. Then we have the following* :

(i) *If* $\lambda > r$, *then equation* (2.1) *has a unique positive solution* u, *and* $u \gg 0$.

(ii) *If* $\lambda < r$, *then equation* (2.1) *has no positive solution.*

(iii) *If* $\lambda = r$, *then equation* (2.1) *has no solution.*

2.2. The comparison theorem for first eigenvalues

(I) First we consider the Dirichlet eigenvalue problem with indefinite weight function $m(x)$ and positive parameter λ:

$$\begin{cases} -\Delta\phi = \lambda\, m(x)\, \phi & \text{in } D, \\ \phi = 0 & \text{on } \partial D. \end{cases} \tag{2.2}$$

The next theorem asserts the existence of the first positive eigenvalue of problem (2.2) (see [14, 5]):

Theorem 2.3. *If condition* (H.1) *is satisfied*, *then the first eigenvalue* $\gamma_1(m)$ *of problem* (2.2) *is positive and simple*, *and its corresponding eigenfunction* $\phi_1(x)$ *may be chosen to be strictly positive everywhere in* D. *Moreover*, *no other eigenvalues have positive eigenfunctions.*

If $v(x)$ is a positive eigenfunction corresponding to the first eigenvalue $\mu_D(\lambda)$ of the Dirichlet problem

$$\begin{cases} (-\Delta - \lambda\, m(x))v = \mu_D(\lambda)v & \text{in } D, \\ v = 0 & \text{on } \partial D, \end{cases} \tag{2.3}$$

then it is easy to see that λ is the first eigenvalue $\gamma_1(m)$ of problem (2.2) with corresponding positive eigenfunction if and only if $\mu_D(\lambda) = 0$ is an eigenvalue of problem (2.3) with corresponding positive eigenfunction.

(II) Secondly we consider the Neumann eigenvalue problem with indefinite weight function $m(x)$ and positive parameter λ

$$\begin{cases} -\Delta\phi = \lambda\, m(x)\, \phi & \text{in } D, \\ \dfrac{\partial \phi}{\partial \mathbf{n}} = 0 & \text{on } \partial D. \end{cases} \tag{2.4}$$

The next theorem asserts the existence of the first eigenvalue of problem (2.4) (see [2, Theorem 3.13], [18, Theorems 2 and 3]):

Theorem 2.4. *If condition* (H.3) *is satisfied*, *then problem* (2.4) *admits a unique nonnegative eigenvalue* $\nu_1(m)$ *having a positive eigenfunction*, *and we have*

$$\begin{cases} \nu_1(m) > 0 & \text{if } \int_D m(x)\, dx < 0, \\ \nu_1(m) = 0 & \text{if } \int_D m(x)\, dx \geq 0. \end{cases}$$

If $w(x)$ is a positive eigenfunction corresponding to the first eigenvalue $\mu_N(\lambda)$ of the Neumann problem

$$\begin{cases} (-\Delta - \lambda\, m(x))w = \mu_N(\lambda)w & \text{in } D, \\ \dfrac{\partial w}{\partial \mathbf{n}} = 0 & \text{on } \partial D, \end{cases} \tag{2.5}$$

then it is easy to see that λ is the first eigenvalue $\nu_1(m)$ of problem (2.4) with corresponding positive eigenfunction if and only if $\mu_N(\lambda) = 0$ is an eigenvalue of problem (2.5) with corresponding positive eigenfunction.

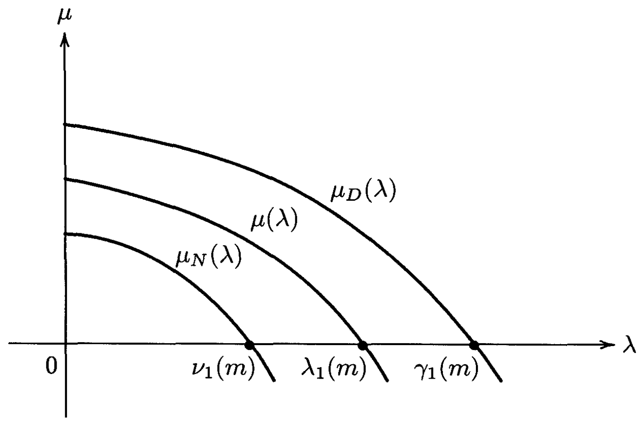


Figure 2.1

(III) Thirdly we consider the following eigenvalue problem:

$$\begin{cases} -\Delta u - \lambda\, m(x)\, u = \mu(\lambda) u & \text{in } D, \\ Bu = 0 & \text{on } \partial D. \end{cases} \tag{2.6}$$

Then it is easy to see that λ is the first eigenvalue $\lambda_1(m)$ of problem (1.2) with corresponding positive eigenfunction if and only if $\mu(\lambda) = 0$ is an eigenvalue of problem (2.6) with corresponding positive eigenfunction.

The main result of this subsection is the following (cf. [9, Proposition 17.7]):

Theorem 2.5. *For all $\lambda \geq 0$, we have* (*see Figure* 2.1)

$$\mu_N(\lambda) < \mu(\lambda) < \mu_D(\lambda).$$

In particular, *it follows that*

$$\nu_1(m) < \lambda_1(m) < \gamma_1(m).$$

Proof. (I) First we show that

$$\mu_D(\lambda) > \mu(\lambda). \tag{2.7}$$

If we take a constant $c > 0$ so large that

$$\begin{aligned} &c + \mu_D(\lambda) > 0, \\ &c - \lambda\, m(x) > 0 \quad \text{in } D, \end{aligned}$$

then, by Taira [19, Theorem 1.1], one can find a unique solution $u \in C^2(\overline{D})$ of the problem

$$\begin{cases} (-\Delta - \lambda\, m(x) + c)u = (\mu_D(\lambda) + c)v & \text{in } D, \\ Bu = 0 & \text{on } \partial D. \end{cases}$$

By the maximum principle (see [19, Proposition 1.6]), it follows that

$$u > 0 \quad \text{in } D.$$

Moreover we have the following:

Claim 2.6. $u \geq v$ *in* D.

Proof. Assume to the contrary that

$$\alpha = \min_{\overline{D}}(u - v) < 0.$$

However it follows that

$$\begin{cases} (-\Delta + c - \lambda\, m(x))v = (\mu_D(\lambda) + c)v & \text{in } D, \\ v = 0 & \text{on } \partial D. \end{cases}$$

Hence we have

$$\begin{cases} (\Delta + \lambda\, m(x) - c)(u - v) = 0 & \text{in } D, \\ u - v \geq 0 & \text{on } \partial D. \end{cases}$$

This implies that the function $u - v$ may take its negative minimum α at a point of D. Thus, applying the maximum principle we obtain that

$$u(x) - v(x) \equiv \alpha \quad \text{in } D.$$

Hence we have

$$0 = (\Delta + \lambda\, m(x) - c)(u - v) = (\lambda\, m(x) - c)\alpha > 0 \quad \text{in } D.$$

This contradiction proves Claim 2.6. ■

By Claim 2.6, it follows that

$$(-\Delta - \lambda\, m(x) + c)u = (\mu_D(\lambda) + c)v \leq (\mu_D(\lambda) + c)u \quad \text{in } D.$$

Hence we have

$$\begin{cases} (\mu_D(\lambda) - (-\Delta - \lambda\, m(x)))u \geq 0 & \text{in } D, \\ u > 0 & \text{in } D, \\ Bu = 0 & \text{on } \partial D. \end{cases}$$

Therefore the desired assertion (2.7) follows by applying Theorem 2.2 to our situation.

(II) Next we show that

$$\mu(\lambda) > \mu_N(\lambda). \tag{2.8}$$

Let $u(x)$ be a positive eigenfunction corresponding to the first eigenvalue $\mu(\lambda)$ of the problem

$$\begin{cases} (-\Delta - \lambda\, m(x))u = \mu(\lambda)u & \text{in } D, \\ Bu = 0 & \text{on } \partial D. \end{cases}$$

If we take a constant $d > 0$ so large that

$$\begin{aligned} &d + \mu(\lambda) > 0, \\ &d - \lambda\, m(x) > 0 \quad \text{in } D, \end{aligned}$$

then one can find a unique solution $w \in C^2(\overline{D})$ of the Neumann problem

$$\begin{cases} (-\Delta - \lambda\, m(x) + d)w = (\mu(\lambda) + d)u & \text{in } D, \\ \dfrac{\partial w}{\partial \mathbf{n}} = 0 & \text{on } \partial D. \end{cases}$$

By the maximum principle, it follows that

$$w > 0 \quad \text{in } D.$$

Moreover, we have the following:

Claim 2.7. *$w \geq u$ in D.*

Proof. Assume to the contrary that

$$\beta = \min_{\overline{D}}(w - u) < 0.$$

We remark that

$$(\Delta + \lambda\, m(x) - d)(w - u) = 0 \quad \text{in } D.$$

(a) If the function $w - u$ takes its negative minimum β at a point $x_0 \in D$, then, by applying the maximum principle, we obtain that

$$w(x) - u(x) \equiv \beta \quad \text{in } D.$$

Hence we have

$$0 = (\Delta + \lambda\, m(x) - d)(w - u) = (\lambda\, m(x) - d)\beta > 0 \quad \text{in } D.$$

This is a contradiction.

(b) If the function $w - u$ takes its negative minimum β at a point $x_0' \in \partial D$, then, by applying the boundary point lemma, we obtain that

$$\frac{\partial(w-u)}{\partial \mathbf{n}}(x_0') < 0.$$

This implies that

$$\frac{\partial u}{\partial \mathbf{n}}(x_0') > 0, \tag{2.9}$$

since we have

$$\frac{\partial w}{\partial \mathbf{n}} = 0 \quad \text{on } \partial D.$$

On the other hand, we have

$$0 = Bu(x_0') = a(x_0')\frac{\partial u}{\partial \mathbf{n}} + b(x_0')u(x_0'). \tag{2.10}$$

Hence it follows from condition (H.2) that

$$a(x_0') \implies u(x_0') = 0,$$

so that

$$0 \leq w(x_0') = w(x_0') - u(x_0') = \beta < 0.$$

This contradiction proves that

$$a(x_0') > 0.$$

Therefore, combining assertions (2.9) and (2.10) we find that

$$0 < \frac{\partial u}{\partial \mathbf{n}}(x_0') = -\frac{b(x_0')}{a(x_0')}u(x_0') \leq 0.$$

This contradiction proves Claim 2.7. ■

By Claim 2.7, it follows that

$$(-\Delta - \lambda\, m(x) + d)w = (\mu(\lambda) + d)u \leq (\mu(\lambda) + d)w \quad \text{in } D.$$

Hence we have

$$\begin{cases} (\mu(\lambda) - (-\Delta - \lambda\, m(x)))w \geq 0 & \text{in } D, \\ w > 0 & \text{in } D, \\ \dfrac{\partial w}{\partial \mathbf{n}} = 0 & \text{on } \partial D. \end{cases}$$

Therefore the desired assertion (2.8) follows by applying Theorem 2.2 to our situation.

Now the proof of Theorem 2.5 is complete. ■

3. Proof of Theorem 1.2 -(1)-

This section and the next section are devoted to the proof of Theorem 1.2.

(I) First we begin with the following lower bound on the parameter λ for the existence of positive solutions of problem (1.1):

Lemma 3.1. *If there exists a positive solution $u(\lambda) \in C^2(\overline{D})$ of problem* (1.1), *then we have*

$$\lambda > \lambda_1(m). \tag{3.1}$$

Proof. Let $\psi_1(x)$ be a positive eigenfunction corresponding to the first eigenvalue $\lambda_1(m)$:

$$\begin{cases} -\Delta\psi_1 = \lambda_1(m)\, m(x)\, \psi_1 & \text{in } D, \\ \psi_1 > 0 & \text{in } D, \\ B\psi_1 = 0 & \text{on } \partial D. \end{cases}$$

Then it follows from an application of Green's formula that

$$\begin{aligned} 0 &= \int_D \left(\Delta u(\lambda) + \lambda m(x)\, u(\lambda) - \lambda\, h(x)\, u(\lambda)^2\right) \psi_1\, dx \\ &= \int_D u(\lambda) \cdot \Delta\psi_1\, dx + \lambda \int_D m(x)\, u(\lambda)\, \psi_1\, dx - \lambda \int_D h(x)\, u(\lambda)^2\, \psi_1\, dx \\ &\quad + \int_{\partial D} \frac{\partial u(\lambda)}{\partial \mathbf{n}}\, \psi_1\, d\sigma - \int_{\partial D} u(\lambda)\, \frac{\partial \psi_1}{\partial \mathbf{n}}\, d\sigma \\ &= (\lambda - \lambda_1(m)) \int_D m(x)\, u(\lambda)\, \psi_1\, dx - \lambda \int_{\partial D} h(x)\, u(\lambda)^2\, \psi_1\, dx \\ &\quad + \int_{\partial D} \left(\frac{\partial u(\lambda)}{\partial \mathbf{n}}\, \psi_1 - u(\lambda)\, \frac{\partial \psi_1}{\partial \mathbf{n}} \right) d\sigma. \end{aligned} \tag{3.2}$$

However we recall that $u(\lambda)$ and ψ_1 satisfy the boundary conditions

$$\begin{pmatrix} \frac{\partial u(\lambda)}{\partial \mathbf{n}} & u(\lambda) \\ \frac{\partial \psi_1}{\partial \mathbf{n}} & \psi_1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{on } \partial D,$$

so that

$$\begin{vmatrix} \frac{\partial u(\lambda)}{\partial \mathbf{n}} & u(\lambda) \\ \frac{\partial \psi_1}{\partial \mathbf{n}} & \psi_1 \end{vmatrix} = 0 \quad \text{on } \partial D.$$

Therefore we obtain from formula (3.2) that

$$(\lambda_1(m) - \lambda) \int_D m(x)\, u(\lambda)\, \psi_1\, dx + \lambda \int_D h(x)\, u(\lambda)^2\, \psi_1\, dx = 0,$$

so that

$$\lambda - \lambda_1(m) = \frac{\lambda \int_D h(x)\, u(\lambda)^2 \psi_1 \, dx}{\int_D m(x)\, u(\lambda)\, \psi_1 \, dx} > 0.$$

This proves inequality (3.1). ∎

(II) Next we construct a positive solution $u(\lambda)$ of problem (1.1) for $\lambda > \lambda_1(m)$. To do so, we let

$$C_B^{2+\theta}(\overline{D}) = \{u \in C^{2+\theta}(\overline{D}) : Bu = 0 \text{ on } \partial D\},$$

and associate with problem (1.1) a nonlinear mapping $G(\lambda, u)$ of $\mathbb{R} \times C_B^{2+\theta}(\overline{D})$ into $C^\theta(\overline{D})$ as follows:

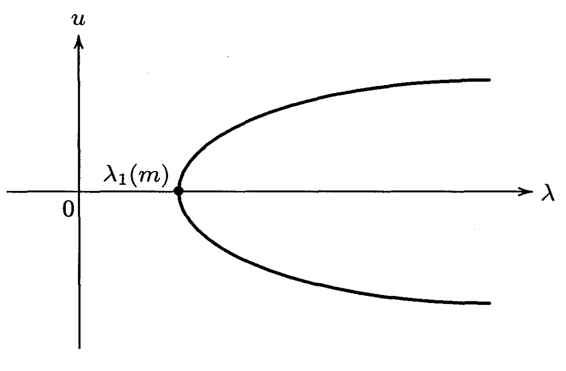
$$\begin{aligned} G : \mathbb{R} \times C_B^{2+\theta}(\overline{D}) &\longrightarrow C^\theta(\overline{D}) \\ (\lambda, u) &\longmapsto -\Delta u - \lambda m(x)\, u + \lambda h(x)\, u^2. \end{aligned}$$

It is clear that a function $u \in C^{2+\theta}(\overline{D})$ is a solution of problem (1.1) if and only if $G(\lambda, u) = 0$.

(II-a) First the next lemma proves the existence of positive solutions of problem (1.1) near the point $(\lambda_1(m), 0)$:

Lemma 3.2. *There exists a positive bifurcation solution curve* $(\lambda, u(\lambda))$ *of the equation* $G(\lambda, u) = 0$ *starting at* $(\lambda_1(m), 0)$.

Proof. The Crandall - Rabinowitz local bifurcation theorem [4, Theorem 1.7] may be employed to assert that the simplicity of $\lambda_1(m)$ guarantees the existence of a continuum of nontrivial solutions of problem (1.1) emanating from $(\lambda_1(m), 0)$, which can be expressed as the union of two subcontinua intersecting at $(\lambda_1(m), 0)$. By the maximum principle, it follows that these subcontinua are locally the strictly positive and the strictly negative solutions of problem (1.1) as in Figure 3.1.



Figrue 3.1

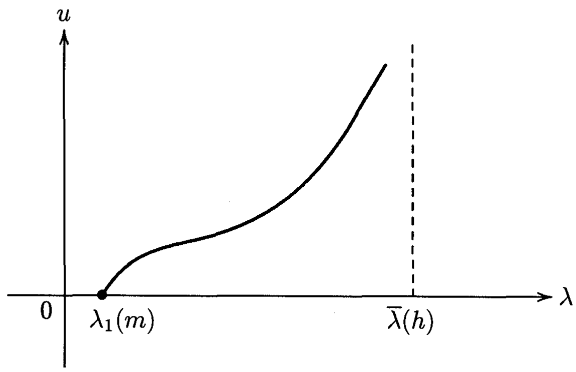


Figure 3.2

Moreover the Rabinowitz global bifurcation theorem [16, Theorem 1.10] tells us that the subcontinuum $\mathcal{C}$ of positive solutions emanating from $(\lambda_1(m), 0)$ is either unbounded or contains another bifurcation point $(\lambda_0, 0)$ with $\lambda_0 \neq \lambda_1(m)$.

However, we can prove that the subcontinuum $\mathcal{C}$ cannot contain a point $(\lambda_0, 0)$ with $\lambda_0 \neq \lambda_1(m)$; hence $\mathcal{C}$ must be unbounded (cf. [6, Theorem 29.2]). ■

(II-b) Secondly, by using the implicit function theorem we show (cf. [9, Theorem 27.1]) that there exists a critical value $\overline{\lambda}(h) \in (\lambda_1(m), +\infty]$ such that one can parametrize the bifurcation solution curve $(\lambda, u(\lambda))$ by λ, $\lambda_1(m) < \lambda < \overline{\lambda}(h)$, as a C^1 curve as in Figure 3.2.

Lemma 3.3. *There exists a constant* $\overline{\lambda}(h) \in (\lambda_1(m), +\infty]$ *such that we have a positive solution* $(\lambda, u(\lambda))$ *of the equation* $G(\lambda, u) = 0$ *for all* $\lambda \in (\lambda_1(m), \overline{\lambda}(h))$.

Proof. It suffices to prove that there exists a constant $\overline{\lambda}(h) \in (\lambda_1(m), +\infty]$ such that the Fréchet derivative

$$\begin{aligned} G_u(\lambda, u(\lambda)) : C_B^{2+\theta}(\overline{D}) &\longrightarrow C^\theta(\overline{D}) \\ v &\longmapsto -\Delta v - \lambda m(x)\, v + 2\lambda h(x)\, u(\lambda) v \end{aligned}$$

is an algebraic and topological isomorphism for all $\lambda \in (\lambda_1(m), \overline{\lambda}(h))$. Indeed, by using the implicit function theorem one can obtain a positive bifurcation solution curve $(\lambda, u(\lambda))$ of the equation $G(\lambda, u) = 0$ for all $\lambda \in (\lambda_1(m), \overline{\lambda}(h))$.

However, by applying Taira [19, Theorem 1.1] to our situation we find that the Fréchet derivative $G_u(\lambda, u(\lambda))$ is a Fredholm operator with index *zero*. Hence, to prove the lemma we have only to show that $G_u(\lambda, u(\lambda))$ is injective.

The next claim proves the injectivity and hence surjectivity of $G_u(\lambda, u(\lambda))$:

Claim 3.4. *We define a densely defined, closed linear operator* $\mathfrak{A}(\lambda) : L^2(D) \to L^2(D)$ *as follows.*

(a) *The domain of definition* $D(\mathfrak{A}(\lambda))$ *is the space*

$$D(\mathfrak{A}(\lambda)) = \{v \in H^2(D) : Bv = 0 \text{ on } \partial D\}.$$

(b) $\mathfrak{A}(\lambda)v = -\varDelta v + 2\lambda h(x)\, u(\lambda)\, v$, $v \in D(\mathfrak{A}(\lambda))$.

Then the first eigenvalue $\mu_1(\lambda)$ *of* $\mathfrak{A}(\lambda) - \lambda m(x)\, I$ *is* positive *for* $\lambda > \lambda_1(m)$; *in particular*, 0 *is not an eigenvalue of* $\mathfrak{A}(\lambda) - \lambda m(x)\, I$.

Proof. Let $\mu_1(\lambda)$ and $v_1(\lambda)$ be the first eigenvalue and corresponding eigenfunction of $\mathfrak{A}(\lambda) - \lambda I$, respectively:

$$(\mathfrak{A}(\lambda) - \lambda m(x) I) v_1(\lambda) = \mu_1(\lambda)\, v_1(\lambda),$$

or equivalently,

$$\begin{cases} (-\varDelta - \lambda m(x) + 2\lambda h(x)\, u(\lambda)) v_1(\lambda) = \mu_1(\lambda)\, v_1(\lambda) & \text{in } D, \\ B\, v_1(\lambda) = 0 & \text{on } \partial D. \end{cases}$$

By Theorem 1.1, one may assume that $v_1(\lambda) > 0$ in D. Then, just as in the proof of Lemma 3.1, we have, by Green's formula,

$$\begin{aligned}
&\mu_1(\lambda) \int_D u(\lambda)\, v_1(\lambda)\, dx \\
&= - \int_D (\varDelta v_1(\lambda) + \lambda m(x)\, v_1(\lambda) - 2\lambda h(x)\, u(\lambda)\, v_1(\lambda)) u(\lambda)\, dx \\
&= - \int_D v_1(\lambda)(\varDelta + \lambda m(x)) u(\lambda)\, dx + 2\lambda \int_D h(x)\, v_1(\lambda)\, u(\lambda)^2\, dx \\
&\quad - \int_{\partial D} \frac{\partial v_1(\lambda)}{\partial \mathbf{n}}\, u(\lambda)\, d\sigma + \int_{\partial D} v_1(\lambda)\, \frac{\partial u(\lambda)}{\partial \mathbf{n}}\, d\sigma \\
&= -\lambda \int_D h(x)\, u(\lambda)^2\, v_1(\lambda)\, dx + 2\lambda \int_D h(x)\, v_1(\lambda)\, u(\lambda)\, u(\lambda)\, dx \\
&\quad + \int_{\partial D} \left(v_1(\lambda) \frac{\partial u(\lambda)}{\partial \mathbf{n}} - \frac{\partial v_1(\lambda)}{\partial \mathbf{n}}\, u(\lambda) \right) d\sigma \\
&= \lambda \int_D h(x)\, u(\lambda)^2\, v_1(\lambda)\, dx.
\end{aligned}$$

Therefore we obtain that

$$\mu_1(\lambda) = \lambda \frac{\int_D h(x)\, u(\lambda)^2\, v_1(\lambda)\, dx}{\int_D u(\lambda)\, v_1(\lambda)\, dx} > 0.$$

This proves Claim 3.4. ■

The proof of Lemma 3.3 is complete. ■

4. Proof of Theorem 1.2 -(2)-

It remains to characterize the critical value $\overline{\lambda}(h)$ in Lemma 3.3 as follows:

$$\overline{\lambda}(h) = \mu_1(D_0(h)). \tag{4.1}$$

(I) First we consider the logistic *Dirichlet* problem

$$\begin{cases} -\Delta v = \lambda(m(x) - h(x)v)v & \text{in } D, \\ v = 0 & \text{on} \partial D. \end{cases} \tag{4.2}$$

Then we have the following generalization of Cantrell and Cosner [3, Theorems 2.1 and 2.3], Hess [9, Theorem 27.1] and Hess and Kato [10, Theorem 2] to the case where $h(x)$ may vanish in D:

Theorem 4.1. *Assume that* $h(x) \in C^\theta(\overline{D})$ *satisfies condition* (Z). *If* $m(x)$ *is a function in* $C^\theta(\overline{D})$ *such that each set* $\{x \in D_0^i(h) : m(x) > 0\}$, $1 \leq i \leq \ell$, *has positive measure*, *then problem* (4.2) *has a unique positive solution* $v(\lambda) \in C^{2+\theta}(\overline{D})$ *for every* $\lambda \in (\gamma_1(m), \mu_1(D_0(h)))$. *For any* $\lambda \geq \mu_1(D_0(h))$, *there exists no positive solution of problem* (4.2). *Moreover*, *we have*

$$\lim_{\lambda \to \mu_1(D_0(h))} \|v(\lambda)\|_{L^2(D)} = +\infty,$$

and also

$$\lim_{\lambda \to \gamma_1(m)} \|v(\lambda)\|_{C^{2+\theta}(\overline{D})} = 0.$$

Theorem 4.1 is proved by Taira [20, Theorem 1.2]. The situation may be represented schematically by the following bifurcation diagram 4.1:

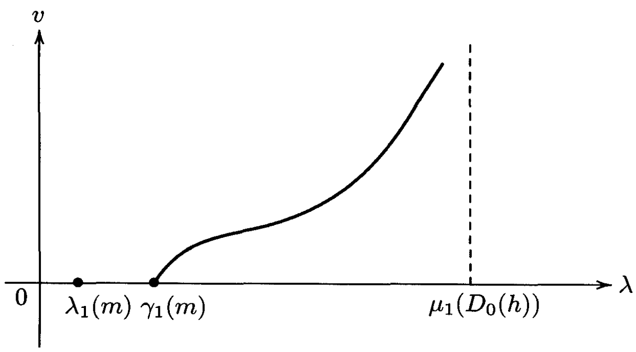


Figure 4.1

Remark 4.1. López-Gómez and Sabina de Lis [13] analyze the pointwise growth to infinity of positive solutions of the logistic Dirichlet problem under the condition that $m(x) \equiv 1$ in D (see [13, Theorems 4.2 and 4.3]). Moreover, García-Melián *et al*. [8] study the pointwise behavior and the uniqueness of positive solutions of nonlinear elliptic boundary value problems of general sublinear type, and give the exact limiting profile of the positive solutions (see [8, Theorem 3.1, Corollary 3.3 and Theorem 6.4]). Their numerical computations confirm and illuminate the above bifurcation diagram 4.1.

(II) Next we consider the logistic *Neumann* problem

$$\begin{cases} -\varDelta w = \lambda(m(x) - h(x)w)w & \text{in } D, \\ \dfrac{\partial w}{\partial \mathbf{n}} = 0 & \text{on } \partial D. \end{cases} \tag{4.3}$$

Then we have the following generalization of Hess [9, Theorem 27.1] and Senn [17, Theorem 2.4] to the case where $h(x)$ may vanish in D:

Theorem 4.2. *Assume that condition* (H.3) *is satisfied, and that* $h(x) \in C^\theta(\overline{D})$ *satisfies condition* (Z) *and further that each set* $\{x \in D_0^i(h) : m(x) > 0\}$, $1 \leq i \leq \ell$, *has positive measure. Then problem* (4.3) *has a unique positive solution* $w(\lambda) \in C^{2+\theta}(\overline{D})$ *for every* $\lambda \in (\nu_1(m),\ \mu_1(D_0(h)))$. *For any* $\lambda \geq \mu_1(D_0(h))$, *there exists no positive solution of problem* (4.3). *Moreover, we have*

$$\lim_{\lambda \to \mu_1(D_0(h))} \|w(\lambda)\|_{L^2(D)} = +\infty,$$

and also

$$\lim_{\lambda \to \nu_1(m)} \|w(\lambda) - c\|_{C^{2+\theta}(\overline{D})} = 0,$$

where

$$c = \max\left\{ \frac{\int_D m(x)\, dx}{\int_D h(x)\, dx}, 0 \right\}.$$

Theorem 4.2 is proved by Taira [20, Theorem 7.2]. The situation may be represented schematically by the following two bifurcation diagrams 4.2 and 4.3.

(III) The next comparison principle plays an essential role in the proof of formula (4.1):

Theorem 4.3. *Assume that conditions* (H.2) *and* (H.3) *are satisfied. If* $u(\lambda)$, $v(\lambda)$ *and* $w(\lambda)$ *are positive solutions problems* (1.1), (4.2) *and* (4.3), *respectively, then we have*

$$v(\lambda) \leq u(\lambda) \leq w(\lambda) \quad \text{on } \overline{D}.$$

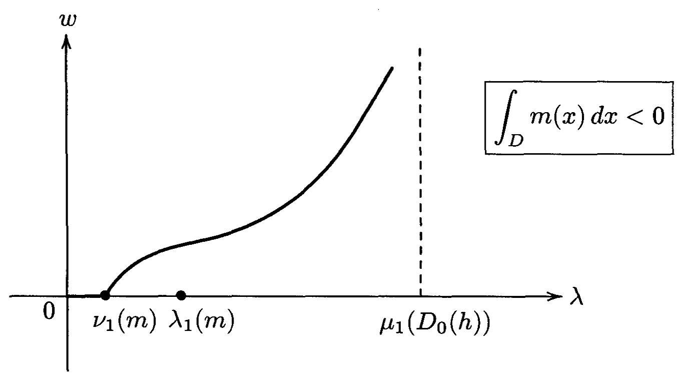


Figure 4.2

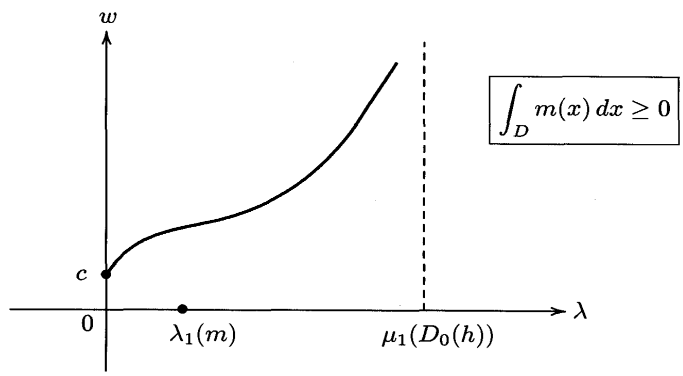


Figure 4.3

Proof. (i) First we show that

$$u(\lambda) \leq w(\lambda) \quad \text{on } \overline{D}. \tag{4.4}$$

Let

$$\varphi = u(\lambda) - w(\lambda),$$

and assume to the contrary that the set

$$D^+ = \{x \in D : \varphi(x) > 0\} = \{x \in D : u(\lambda)(x) > w(\lambda)(x)\}$$

is nonempty. Then it follows that

$$\begin{aligned}0 &= -\Delta\varphi - \lambda(m(x)u(\lambda) - h(x)u(\lambda)^2) + \lambda(m(x)w(\lambda) - h(x)w(\lambda)^2)\\ &= -\Delta\varphi - \lambda m(x)\varphi + \lambda h(x)(u(\lambda)^2 - w(\lambda)^2)\\ &= -\Delta\varphi - \lambda m(x)\varphi + \lambda h(x)(u(\lambda) + w(\lambda))\varphi \quad \text{in } D.\end{aligned}$$

Hence we have

$$\Delta\varphi + \lambda m(x)\varphi = \lambda h(x)(u(\lambda) + w(\lambda))\varphi \geq 0 \quad \text{in } D^+. \tag{4.5}$$

Let x_0 be a point of $\overline{D^+}$ such that

$$\varphi(x_0) = \max_{\overline{D^+}} \varphi(x) > 0. \tag{4.6}$$

Without loss of generality, one may assume that

$$\sup_{x\in D} m(x) \leq 1. \tag{4.7}$$

If we let

$$\Phi(x,t) = e^{-\lambda t}\,\varphi(x),$$

then it follows from inequality (4.5) that

$$\begin{aligned}\frac{\partial\Phi}{\partial t} - \Delta\Phi + \lambda(1 - m(x))\Phi &= e^{-\lambda t}(-\Delta\varphi - \lambda m(x)\varphi)\\ &\leq 0 \quad \text{in } D^+ \times (0,T).\end{aligned}$$

Here we remark by condition (4.7) that

$$\lambda(1 - m(x)) \geq 0 \quad \text{in } D,$$

and that

$$\max_{\overline{D^+}\times[0,T]} \Phi(x,t) = \max_{\overline{D^+}} \varphi(x) = \varphi(x_0) > 0.$$

(a) We consider the case where $x_0 \in D^+$: By applying the parabolic maximum principle (see [15, Theorem 3.3.7]) to our situation, we obtain from condition (4.6) that

$$\varphi(x) = \Phi(x,0) \equiv \Phi(x_0,0) = \varphi(x_0) > 0, \quad x \in D^+.$$

However this is a contradiction, since we have

$$\varphi(x) = 0 \quad \text{on } \partial D^+ \cap D.$$

(b) Next we consider the case where $x_0 \in \partial D \cap \partial D^+$: It follows from an application of the Hopf boundary point lemma (see [15, Theorem 3.3.7]) that

$$\frac{\partial \varphi}{\partial \mathbf{n}}(x_0) > 0. \tag{4.8}$$

However we have

$$0 = B\varphi(x_0) = a(x_0)\frac{\partial \varphi}{\partial \mathbf{n}}(x_0) + b(x_0)\varphi(x_0).$$

Thus, combining conditions (4.6) and (4.8) we obtain that

$$a(x_0) = b(x_0) = 0.$$

This contradicts condition (H.2).

Therefore we have proved assertion (4.4), since the set D^+ is empty.

(ii) Secondly we show that

$$v(\lambda) \leq u(\lambda) \quad \text{on } \overline{D}. \tag{4.9}$$

Let

$$\psi = v(\lambda) - u(\lambda),$$

and assume to the contrary that the set

$$E^+ = \{x \in D : \psi(x) > 0\} = \{x \in D : u(\lambda)(x) > w(\lambda)(x)\}$$

is nonempty. Then it follows that

$$\begin{aligned} 0 &= -\Delta\psi - \lambda(m(x)v(\lambda) - h(x)v(\lambda)^2) + \lambda(m(x)u(\lambda) - h(x)u(\lambda)^2) \\ &= -\Delta\psi - \lambda m(x)\psi + \lambda h(x)(v(\lambda)^2 - u(\lambda)^2) \\ &= -\Delta\psi - \lambda m(x)\psi + \lambda h(x)(v(\lambda) + u(\lambda))\psi \quad \text{in } D. \end{aligned}$$

Hence we have

$$\Delta\psi + \lambda m(x)\psi = \lambda h(x)(v(\lambda) + u(\lambda))\psi \geq 0 \quad \text{in } E^+. \tag{4.10}$$

Let x_0 be a point of $\overline{E^+}$ such that

$$\psi(x_0) = \max_{\overline{E^+}} \psi(x) > 0. \tag{4.11}$$

If we let

$$\Psi(x,t) = e^{-\lambda t}\, \psi(x),$$

then it follows from inequality (4.10) that

$$\begin{aligned}\frac{\partial \Psi}{\partial t} - \Delta\Psi + \lambda(1-m(x))\Psi &= e^{-\lambda t}(-\Delta\psi - \lambda m(x)\psi)\\ &\leq 0 \quad \text{in } D^+ \times (0,T).\end{aligned}$$

Here we recall by condition (4.7) that

$$\lambda(1-m(x)) \geq 0 \quad \text{in } D,$$

and that

$$\max_{\overline{E^+}\times[0,T]} \Psi(x,t) = \max_{\overline{E^+}} \psi(x) = \psi(x_0) > 0.$$

(a) We consider the case where $x_0 \in E^+$: By applying the parabolic maximum principle (see [15, Theorem 3.3.7]) to our situation we obtain that

$$\psi(x) = \Psi(x,0) \equiv \Psi(x_0,0) = \psi(x_0) > 0, \quad x \in E^+.$$

However this is a contradiction, since we have

$$\psi(x) = 0 \quad \text{on } \partial E^+ \cap D.$$

(b) Next we consider the case where $x_0 \in \partial D \cap \partial E^+$: We have, by condition (4.11),

$$0 < \psi(x_0) = v(\lambda)(x_0) - u(\lambda)(x_0) = -u(x_0) \leq 0.$$

This is a contradiction.

Therefore we have proved assertion (4.9), since the set E^+ is empty.

Now the proof of Theorem 4.3 is complete. ∎

(IV) The desired formula (4.1) follows by combining Theorem 2.5 and Theorems 4.1 through 4.3. Indeed, it suffices to note the following two bifurcation diagrams 4.4 and 4.5:

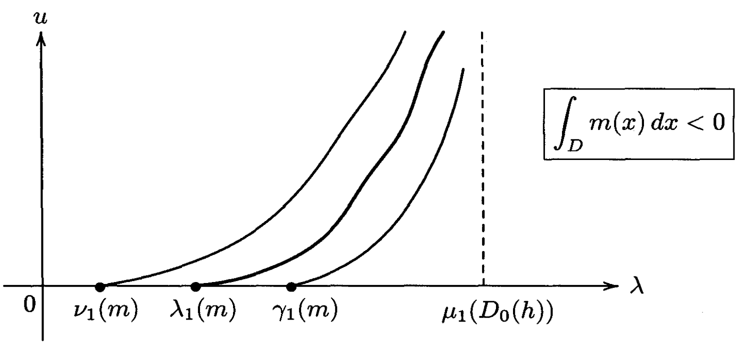


Figure 4.4

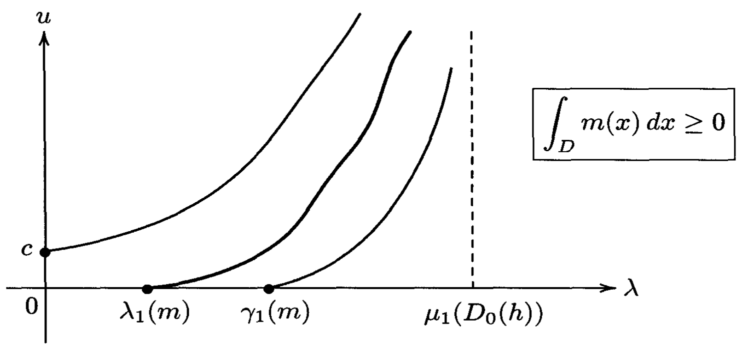


Figure 4.5

Now the proof of Theorem 1.2 is complete. ■

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