TAIWANESE JOURNAL OF MATHEMATICS
Vol. 5, No. 1, pp. 97-108, March 2001

FIXED POINT AND NON-RETRACT THEOREMS – CLASSICAL CIRCULAR TOURS

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Dedicated to the memory of Dr. Ming-Po Chen

Abstract. We show that the Brouwer fixed point theorem is equivalent to a number of results closely related to the Euclidean spaces or n-simplexes or n-balls. Among them are the Sperner lemma, the KKM theorem, some intersection theorems, various fixed point theorems, an intermediate value theorem, various non-retract theorems, the non-contractibility of spheres, and others.

1. Introduction

It is well-known that the Brouwer fixed point theorem has numerous equivalent formulations in various fields of mathematics such as topology, nonlinear analysis, equilibrium theory in economics, game theory, and others. In this article, we collect such formulations closely related to Euclidean spaces or n-simplexes or n-balls. For some other equivalent formulations of the Brouwer theorem, see [24, 25].

Originally, some of the results covered in this paper were treated as consequences of the Sperner lemma and the Knaster-Kuratowski-Mazurkiewicz (simply, KKM) theorem; see [1, 2]. Nowadays, however, under the strong influence of [9], many of them might be seen as essential applications of the homology theory. Our aim in this paper is to show that these results are easily accessible to any reader if he or she could understand the proofs of the Sperner lemma and the KKM theorem.

In Section 2, we introduce three classical results – the Brouwer theorem, the Sperner lemma, and the KKM theorem – as well as some variants of the KKM

Received January 5, 2000.
Communicated by P. Y. Wu.
2000 *Mathematics Subject Classification*: 46A55, 47H10, 54C15, 54H25, 55M20.
Key words and phrases: Fixed point, retract, contractible, sphere, simplex, essentiality, inward (outward) set.

theorem due to Sperner and Alexandroff-Pasynkoff. And we give a simple proof of the Brouwer theorem based on a variant of the KKM theorem. Actually these statements in Section 2 are all equivalent to each other, and hence we would have our first classical circular tour.

Section 3 deals with fixed point theorems, intermediate value theorems, various non-retract theorems, and the non-contractibility of a sphere. We will deduce one after another by giving transparent proofs. This would be our second classical circular tour which starts and ends with the Brouwer theorem.

2. The Mathematical Trinity and a Simple Proof of the Brouwer Theorem

In this section, first, we indicate that the three classical results – the Brouwer theorem, the Sperner lemma, and the KKM theorem – are mutually equivalent in the sense that each one can be deduced from another with or without aid of some minor results. Second, a particular form of the Knaster-Kuratowski-Mazurkiewicz theorem is used to give a simple proof of the Brouwer fixed point theorem.

Let $\Delta_n = v_0v_1\cdots v_n$ be an n-simplex and $\partial\Delta_n = \bigcup_{i=0}^{n} v_0v_1\cdots\widehat{v_i}\cdots v_n$ its boundary, that is, the union of $(n-1)$-faces of Δ_n.

The Brouwer fixed point theorem [6] as follows is one of the most well-known and useful theorems in topology:

Theorem (Brouwer). *A continuous map $f : \Delta_n \to \Delta_n$ has a fixed point $x_0 = f(x_0) \in \Delta_n$.*

There are a large number of different proofs of the Brouwer theorem; for the literature, see [24, 25].

One of the earlier proofs of the Brouwer theorem was given by Knaster, Kuratowski, and Mazurkiewicz (simply, KKM) [19] based on the following [28]:

Lemma (Sperner). *Let K be a simplicial subdivision of an n-simplex $v_0v_1\cdots v_n$. To each vertex of K, let an integer be assigned in such a way that whenever a vertex u of K lies on a face $v_{i_0}v_{i_1}\cdots v_{i_k}$ $(0 \le k \le n,\ 0 \le i_0 < i_1 < \cdots < i_k \le n)$, the number assigned to u is one of the integers $i_0, i_1, \cdots, i_k$. Then the total number of those n-simplexes of K whose vertices receive all $n+1$ integers $0, 1, \cdots, n$, is odd. In particular, there is at least one such n-simplex.*

For proofs of the Sperner lemma, see [10, 17, 28, 31]. The lemma was first applied to new proofs of the invariance theorems on dimensions and domains in [28] and, subsequently, to obtain the "closed" version of the following in [19]:

Theorem (KKM). *Let F_i $(0 \leq i \leq n)$ be $n+1$ closed [resp. open] subsets of an n-simplex $v_0v_1\cdots v_n$. If the inclusion relation*

$$v_{i_0}v_{i_1}\cdots v_{i_k} \subset F_{i_0} \cup F_{i_1} \cup \cdots \cup F_{i_k}$$

holds for all faces $v_{i_0}v_{i_1}\cdots v_{i_k}$ $(0 \leq k \leq n,\ 0 \leq i_0 < i_1 < \cdots < i_k \leq n)$, then $\bigcap_{i=0}^n F_i \neq \emptyset$.

For proofs of the closed version using the Sperner lemma, see [10, 17, 19, 31]. The KKM theorem was used in [19] to obtain one of the most direct proofs of the Brouwer theorem. Therefore, it was conjectured that those three theorems are mutually equivalent. This was clarified by Yoseloff [30]. In fact, those three theorems are regarded as a sort of mathematical trinity. All are extremely important and have many applications.

$$\begin{tikzcd}
& \text{Brouwer} \arrow[dl, "1974"'] & \\
\text{Sperner} \arrow[rr, dashed, "1929"'] & & \text{KKM} \arrow[ul, "1929"']
\end{tikzcd}$$

The "open" version of the KKM theorem was due to Kim [18] and Shih and Tan [27], and later, Lassonde [22] showed that the closed and open versions of the KKM theorem can be derived from each other.

We give here a simple proof of the equivalency of the closed and the open versions:

The open version follows from the closed version. In fact, by Shih [26, Theorem 1], if G_i $(0 \leq i \leq n)$ are open sets satisfyng the inclusion relation in the KKM theorem with $G_i = F_i$, then there exist $n+1$ closed sets $F_i \subset G_i$ $(0 \leq i \leq n)$ satisfying the hypothesis of the KKM theorem.

Conversely, for any $\varepsilon > 0$, let G_i^ε be the open ε-neighborhood of F_i $(0 \leq i \leq n)$ with respect to the Euclidean metric on the n-simplex. Then, by the open version, there exists an $x_\varepsilon \in \bigcap_{i=0}^n G_i^\varepsilon \neq \emptyset$. We may assume that the net $\{x_\varepsilon\}$ converges to a limit x_0. Note that $x_0 \in \bigcap_{i=0}^n F_i$. This completes our proof.

In [19], it was noted that the closed version of the following particular form of the KKM theorem was used by Sperner [28] in order to prove the invariance of dimension:

Theorem 1 (Sperner). *Let F_i $(0 \leq i \leq n)$ be $n+1$ closed [resp. open] sets covering an n-simplex $\Delta_n = v_0v_1\cdots v_n$. If, for each i, F_i is disjoint from the $(n-1)$-face $v_0v_1\cdots\widehat{v_i}\cdots v_n$, then $\bigcap_{i=0}^n F_i \neq \emptyset$.*

The open valued version is due to Stromquist [29]. For reader's convenience, we show that Theorem 1 follows from the KKM theorem as in [19]:

Proof. It suffices to show that F_i $(0 \le i \le n)$ satisfy the requirement of the KKM theorem. For any face $v_{i_0} v_{i_1} \cdots v_{i_k}$ $(0 \le k \le n,\ 0 \le i_0 < i_1 < \cdots < i_k \le n)$, we have

$$v_{i_0} v_{i_1} \cdots v_{i_k} \subset F_{i_0} \cup F_{i_1} \cup \cdots \cup F_{i_k}$$

since $v_j \notin \{v_{i_0}, v_{i_1}, \cdots, v_{i_k}\}$ implies

$$F_j \cap v_{i_0} v_{i_1} \cdots v_{i_k} \subset F_j \cap v_0 v_1 \cdots \widehat{v_j} \cdots v_n = \emptyset.$$

This completes our proof. ■

In $\Delta_n = v_0 v_1 \cdots v_n$, the $(n-1)$-faces are denoted as follows:

$$A_0 := v_0 v_1 \cdots v_{n-1} \quad \text{and} \quad A_i := v_i \cdots v_n v_0 \cdots v_{i-2} \quad \text{for} \quad 1 \le i \le n.$$

The closed version of the following is due to Alexandroff and Pasynkoff [3] and noted by Fan [11-13]:

Theorem 2 (Alexandroff and Pasynkoff). *Let X_i $(0 \le i \le n)$ be $n+1$ closed [resp. open] sets covering an n-simplex $\Delta_n = v_0 v_1 \cdots v_n$ such that $A_i \subset X_i$ for each i. Then $\bigcap_{i=0}^n X_i \neq \emptyset$.*

Proof. Suppose that $X_0 \cap X_1 \cap \cdots \cap X_n = \emptyset$. Set

$$F_{i-1} := \Delta_n \backslash X_i \text{ for } i = 1, \cdots, n; \quad \text{and } F_n := \Delta_n \backslash X_0.$$

Then $\{F_0, \cdots, F_n\}$ is an open [resp. closed] cover of Δ_n, and

$$v_0 \cdots \widehat{v_{i-1}} \cdots v_n = A_i \subset X_i = \Delta_n \backslash F_{i-1} \text{ for } i = 1, \cdots, n;$$
$$v_0 \cdots v_{n-1} \widehat{v_n} = A_0 \subset X_0 = \Delta_n \backslash F_n.$$

Since $F_i \cap v_0 \cdots \widehat{v_i} \cdots v_n = \emptyset$ for $i = 0, 1, \cdots, n$, we have by Theorem 1 that

$$\emptyset \neq F_0 \cap \cdots \cap F_n = \Delta_n \backslash (X_0 \cup \cdots \cup X_n),$$

contrary to $\Delta_n = X_0 \cup \cdots \cup X_n$. Therefore, $\bigcap_{i=0}^n X_i \neq \emptyset$. ■

The closed version of Theorem 2 was first applied to the essentiality of the identity map of the boundary of a simplex in [3]; see also [8]. The open version of Theorem 2 was noted by Lassonde [22].

Fan [11, 12] noted that each of Theorems 1 and 2 can be easily derived from the other, and obtained generalizations of Theorems 1 and 2 with some applications in [11-13].

For completeness we give the following:

Proof of Theorem 1 *using Theorem* 2. Suppose that $F_0 \cap \cdots \cap F_n = \emptyset$. Set

$$X_{i+1} := \Delta_n \backslash F_i \text{ for } i = 0, 1, \cdots, n-1; \quad \text{and } X_0 := \Delta_n \backslash F_n.$$

Then $\{X_0, \cdots, X_n\}$ is a cover of Δ_n. Since

$$A_{i+1} = v_0 \cdots \widehat{v_i} \cdots v_n \subset \Delta_n \backslash F_i = X_{i+1} \text{ for } i = 0, 1, \cdots, n-1;$$
$$A_0 = v_0 \cdots v_{n-1} \widehat{v_n} \subset \Delta_n \backslash F_n = X_0,$$

we have $\emptyset \neq X_0 \cap \cdots \cap X_n = \Delta_n \backslash (F_0 \cup \cdots \cup F_n)$, contrary to $\Delta_n = F_0 \cup \cdots \cup F_n$. Therefore, $\bigcap_{i=0}^n F_i \neq \emptyset$. ■

From Theorem 2, we can easily deduce the Brouwer fixed point theorem:

Proof of the Brouwer theorem. Suppose that a continuous map $f : \Delta_n \to \Delta_n$ has no fixed point. For each $x \in \Delta_n$, let $g(x)$ be defined by either $g(x) = x$ if $x \in \partial \Delta_n$, or the point of $\partial \Delta_n$ such that x lies on the line segment from $f(x)$ to $g(x)$, if $x \notin \partial \Delta_n$. Then $g : \Delta_n \to \partial \Delta_n$ is a continuous map such that $g|_{\partial \Delta_n} = \mathrm{id}_{\partial \Delta_n}$. Let $X_i := g^{-1}(A_i)$ for each $(n-1)$-face A_i in $\partial \Delta_n$. Then X_i $(0 \leq i \leq n)$ are $n+1$ closed sets covering Δ_n such that $A_i \subset X_i$. In fact, $A_i = g(A_i)$ implies

$$X_i = g^{-1}(A_i) = g^{-1}g(A_i) \supset A_i.$$

Therefore, by Theorem 2, we have $\bigcap_{i=0}^n X_i \neq \emptyset$. However,

$$\bigcap_{i=0}^{n} X_i = \bigcap_{i=0}^{n} g^{-1}(A_i) = g^{-1}(\bigcap_{i=0}^{n} A_i) = g^{-1}(\emptyset) = \emptyset.$$

This is a contradiction. ■

Consequently, we conclude that each of Theorems 1 and 2 is also equivalent to each of the mathematical trinity. More precisely, in this section, we proved or indicated the following implications:

The Brouwer theorem $\Longleftrightarrow$ the Sperner lemma $\Longleftrightarrow$ the KKM theorem (closed) $\Longleftrightarrow$ the KKM theorem (open) $\Longrightarrow$ Theorem 1 $\Longleftrightarrow$ Theorem 2 $\Longrightarrow$ the Brouwer theorem.

3. Fixed Point and Non-retract Theorems Equivalent to the Brouwer Theorem

In this section, we deduce various fixed point theorems, intermediate value theorems, various non-retract theorems, and the non-contractibility of a sphere. Those are shown to be all equivalent to the Brouwer theorem and, consequently, we would have our second classical circular tour which starts and ends with the Brouwer theorem.

Let $\mathbb{R}^n$ be the Euclidean n-space, $\mathbb{B}^n$ the n-ball $\|x\| \leq 1$, and $\mathbb{S}^{n-1}$ the $(n-1)$-sphere, where we always assume that $\mathbb{R}^n$ has the Euclidean norm $\|x\| = (\sum_{i=1}^n x_i^2)^{1/2}$ for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. These spaces appear in all of the theorems in this section, and the theorems are still valid whenever $\mathbb{R}^n$, $\mathbb{B}^n$, and $\mathbb{S}^{n-1}$ are replaced by their homeomorphic images, respectively.

We begin, in this section, with the following form of the Brouwer theorem:

Theorem 3. *A continuous map $f : \mathbb{B}^n \to \mathbb{B}^n$ has a fixed point $x_0 = f(x_0)$.*

Proof. Since $\mathbb{B}^n$ is homeomorphic to an n-simplex Δ_n, let $h : \mathbb{B}^n \to \Delta_n$ be a homeomorphism. Then $g = hfh^{-1} : \Delta_n \to \Delta_n$ is a continuous map and hence has a fixed point $y_0 = g(y_0) = (hfh^{-1})(y_0) \in \Delta_n$. Then $x_0 = h^{-1}(y_0) \in \mathbb{B}^n$ is a fixed point of $f : \mathbb{B}^n \to \mathbb{B}^n$, since $h^{-1}(y_0) = f(h^{-1}(y_0))$. ■

For a topological space X and $A \subset X$, a continuous map $r : X \to A$ with $r|_A = \mathrm{id}_A$ is called a *retraction* of X onto A, and A is called a *retract* of X.

From the Brouwer Theorem 3, we have the following consequence of a result of Bohl [4]:

Theorem 4 (Bohl). *Every continuous map $f : \mathbb{B}^n \to \mathbb{R}^n$ has at least one of the following properties*:

(a) *f has a fixed point*; *or*

(b) *there is an $x \in \mathbb{S}^{n-1}$ such that $x = \lambda f(x)$ for some $\lambda \in (0,1)$.*

Proof. Consider a retraction $r : \mathbb{R}^n \to \mathbb{B}^n$ defined by

$$r(x) = \begin{cases} x & \text{if } x \in \mathbb{B}^n, \\ x/\|x\| & \text{if } x \in \mathbb{R}^n \backslash \mathbb{B}^n. \end{cases}$$

Then $rf : \mathbb{B}^n \to \mathbb{B}^n$ has a fixed point $x_0 = (rf)(x_0) \in \mathbb{B}^n$ by Theorem 3.

If $f(x_0) \in \mathbb{B}^n$, then $x_0 = r(f(x_0)) = f(x_0)$ and hence we have case (a).

If $f(x_0) \in \mathbb{R}^n \backslash \mathbb{B}^n$, then $x_0 = f(x_0)/\|f(x_0)\| \in \mathbb{S}^{n-1}$ and $\|f(x_0)\| > 1$. By putting $\lambda = 1/\|f(x_0)\| \in (0,1)$, we have case (b). ■

Recall that Halpern [14-16] first introduced the outward and, later, inward sets as follows:

Let E be a topological vector space and $X \subset E$. The *inward* and *outward sets* of X at $x \in E$, $I_X(x)$ and $O_X(x)$, are defined as follows:

$$I_X(x) = x + \bigcup_{r>0} r(X - x), \quad O_X(x) = x + \bigcup_{r<0} r(X - x).$$

From now on, the closures of inward and outward sets of $\mathbb{B}^n$ at $x \in \mathbb{R}^n$ are denoted by $\overline{I(x)}$ and $\overline{O(x)}$, respectively.

From Theorem 4, we have the following particular form of a result of Halpern and Bergman [16]:

Theorem 5. *Any continuous map* $f : \mathbb{B}^n \to \mathbb{R}^n$ *such that* $f(x) \in \overline{I(x)}$ *for* $x \in \mathbb{S}^{n-1}$ *has a fixed point* $x_0 = f(x_0) \in \mathbb{B}^n$.

Proof. This follows from the fact that

$$\overline{I(x)} \cap \{\lambda x : \lambda > 1\} = \emptyset$$

for any $x \in \mathbb{S}^{n-1}$. ■

From Theorem 5, we have the following particular form of another result of Halpern and Bergman [16]:

Theorem 6. *Any continuous map* $f : \mathbb{B}^n \to \mathbb{R}^n$ *such that* $f(x) \in \overline{O(x)}$ *for* $x \in \mathbb{S}^{n-1}$ *has a fixed point. Moreover, we have* $\mathbb{B}^n \subset f(\mathbb{B}^n)$.

Proof. Let $g : \mathbb{B}^n \to \mathbb{R}^n$ be a map given by $g(x) = 2x - f(x)$ for $x \in \mathbb{B}^n$. Then $x - g(x) = -(x - f(x))$, so that f and g have the same fixed point. We note that $f(x) \in \overline{O(x)}$ if and only if $g(x) \in \overline{I(x)}$. Therefore, by Theorem 5, g has a fixed point $x_0 = g(x_0)$ and hence $x_0 = f(x_0)$.

To show $\mathbb{B}^n \subset f(\mathbb{B}^n)$, let us suppose the contrary. Clearly, we can assume that 0 is a point of $\mathbb{B}^n \backslash f(\mathbb{B}^n)$. The complement U of $f(\mathbb{B}^n)$ is a neighborhood of 0, so we can choose $c > 1$ such that $cU \supset \mathbb{B}^n$. Then $cf(\mathbb{B}^n)$ is disjoint from $\mathbb{B}^n$, and so the map cf can have no fixed point. However, it is clear that $cf(x) \in \overline{O(x)}$ for $x \in \mathbb{S}^{n-1}$. This is a contradiction. ■

Note that each of Theorems 4-6 is a generalized (but equivalent) form of the Brouwer fixed point theorem. The following is recently due to Lax [23]:

Theorem 7 (Intermediate value theorem). *Let* $f : \mathbb{B}^n \to \mathbb{R}^n$ *be a continuous map such that* $f|_{\mathbb{S}^{n-1}} = \mathrm{id}_{\mathbb{S}^{n-1}}$ (*that is*, $f(x) = x$ *for* $\|x\| = 1$). *Then* $\mathbb{B}^n \subset f(\mathbb{B}^n)$.

Since $x = f(x) \in \overline{O(x)}$ for $x \in \mathbb{S}^{n-1}$, Theorem 7 follows immediately from Theorem 6.

A well-known argument applies to deduce the Brouwer theorem from Theorem 7 as in [23].

There have appeared many forms of non-retract theorems, that is, $\mathbb{S}^{n-1}$ cannot be a retract of $\mathbb{B}^n$. The following is taken from Kulpa [20, 21]:

Theorem 8 (Borsuk's non-retract theorem). *Let $f : X \to \mathbb{R}^n$ be a continuous map from a compact set $X \subset \mathbb{R}^n$. If $f(x) = x$ for each $x \in \partial X$, then $X \subset f(X)$.*

Proof. We may assume that $X \subset \mathbb{B}^n$ and extend the map f to a continuous map $h : \mathbb{B}^n \to \mathbb{R}^n$ such that $h(x) = x$ for each $x \in \mathbb{B}^n \backslash X$. Then $h|_{\mathbb{S}^{n-1}} = \mathrm{id}_{\mathbb{S}^{n-1}}$ and hence $\mathbb{B}^n \subset h(\mathbb{B}^n)$ by Theorem 7. Since $X \subset \mathbb{B}^n \subset h(\mathbb{B}^n) = h(X \cup (\mathbb{B}^n \backslash X)) = h(X) \cup h(\mathbb{B}^n \backslash X) = h(X) \cup (\mathbb{B}^n \backslash X)$, we should have $X \subset h(X) = f(X)$. ■

For $X = \mathbb{B}^n$, Theorem 8 reduces to Theorem 7.

From Theorem 8, we have the following:

Theorem 9. *Let X be a compact subset of $\mathbb{R}^n$ with nonempty interior. Then ∂X is not a retract of X.*

Proof. If $f : X \to \partial X$ is a retraction, then $X \subset f(X)$ by Theorem 8. Note that $f(X) = \partial X \supset X$ which contradicts Int $X \neq \emptyset$.

The following immediate consequence of Theorem 9 is taken from Dugundji [7, p. 341]:

Theorem 10. *If U is a bounded open subset of $\mathbb{R}^n$, then ∂U is not a retract of $\overline{U}$.*

From Theorem 10, we have immediately the following which originated from Bohl [4]:

Theorem 11. *$\mathbb{S}^{n-1}$ is not a retract of $\mathbb{B}^n$.*

This can also follow immediately from Theorem 2 by considering a map $g : \Delta_n \to \partial\Delta_n$ as in the proof of the Brouwer theorem in Section 2.

The following two theorems are due to Borsuk [5, p. 12]:

Theorem 12 (Borsuk). *If X is a closed subset of $\mathbb{R}^n$ and G is one of the bounded components of $\mathbb{R}^n \backslash X$, then there is no continuous map $f : \overline{G} \to X$ such that $f(x) = x$ for every $x \in \overline{G} \backslash G$.*

Proof. We may assume that $0 \in G$ and $\operatorname{diam} G \leq 1$. Then $G \subset \mathbb{B}^n$. If there is a continuous map $f : \overline{G} \to X$ such that $f = \text{id}$ on $\overline{G} \backslash G$, then we can define a continuous map $\phi : \mathbb{B}^n \to \mathbb{S}^{n-1}$ by

$$\phi(x) = \begin{cases} x/\|x\| & \text{for} \quad x \in \mathbb{B}^n \backslash G, \\ f(x)/\|f(x)\| & \text{for} \quad x \in \overline{G}. \end{cases}$$

This contradicts Theorem 11. ■

Theorem 13 (Borsuk). *If G is a nonempty open bounded subset of $\mathbb{R}^n$, then the set $X = \mathbb{R}^n \backslash G$ is not a retract of $\mathbb{R}^n$.*

Proof. We may assume that G is one of the bounded components of $\mathbb{R}^n \backslash X$. Suppose the contrary that there is a continuous map $f : \mathbb{R}^n \to X$ such that $f|_X = \text{id}_X$. Since $\overline{G} \subset \mathbb{R}^n$ and $\overline{G} \backslash G \subset \mathbb{R}^n \backslash G = X$, we have a continuous map $f|_{\overline{G}} : \overline{G} \to X$ such that $f(x) = x$ for $x \in \overline{G} \backslash G$. This contradicts Theorem 12. ■

A topological space is said to be *contractible* if its identity map is homotopic to a constant map to a point in that space, or its identity map is inessential.

Theorem 14. *$\mathbb{S}^{n-1}$ is not contractible.*

Proof. Suppose the contrary that $\mathbb{S}^{n-1}$ is contractible, that is, there exists a homotopy $H : c \simeq \text{id}$ such that $H(x, 0) = c(x) = x_0 \in \mathbb{S}^{n-1}$ and $H(x, 1) = x$ for all $x \in \mathbb{S}^{n-1}$. Define a map $r : \mathbb{R}^n \to \mathbb{R}^n \backslash (\text{Int } \mathbb{B}^n)$ by

$$r(x) = \begin{cases} x_0 & \text{if} \quad \|x\| \leq 1/2, \\ H(x/\|x\|, 2\|x\| - 1) & \text{if} \quad 1/2 \leq \|x\| \leq 1, \\ x & \text{if} \quad \|x\| \geq 1. \end{cases}$$

Then r is continuous and $r = $id on $\mathbb{R}^n \backslash (\text{Int } \mathbb{B}^n)$. Therefore r is a retraction. This contradicts Theorem 13 with $G = $Int $\mathbb{B}^n$. ■

From Theorem 14, we can deduce the Brouwer theorem:

Proof of Theorem 3 *using Theorem* 14. Assume $f(x) \neq x$ for all $x \in \mathbb{B}^n$. Let $r(x)$ be the point of $\mathbb{S}^{n-1}$ such that x lies on the line segment from $f(x)$ to $r(x)$. The continuity of r can be shown by that of f and some elementary geometry. Since $r(x) = x$ for $x \in \mathbb{S}^{n-1}$, $r : \mathbb{B}^n \to \mathbb{S}^{n-1}$ is a retraction. [This contradicts Theorem 11 and hence we may stop here. However, our aim is to use Theorem 14.] Then

$$H(x, t) = r((1 - t)x) \quad \text{for } x \in \mathbb{S}^{n-1} \text{ and } t \in [0, 1]$$

yields a homotopy contracting $\mathbb{S}^{n-1}$ to a point. This contradicts Theorem 14. ■

We note that, in [9], Theorem 14 was proved using homology theory and applied to obtain the non-retract theorem, the Brouwer theorem, the invariance of dimension, Borsuk's separation criterion, Borsuk's theorem on connectedness of open subsets of $\mathbb{S}^n$, the invariance of domain, and some results on locally Euclidean spaces. All of these follow from Theorem 14 using some lemmas (the results of [9, Chap. XI, §2]) and other simple geometric arguments. See also [7].

Acknowledgements

This research is supported in part by the Non-directed Research Fund, Korea Research Foundation, 1997. Parts of this paper were given by the first author as invited talks at Kyungnam University, Masan, Korea, July 29, 1999; Ecole Normal Superieur, Zielona Góra, Poland, Sept. 2, 1999; TGRC, Kyungpook National University, Taegu, Korea, Nov. 3, 1999; and ICMAA 2000, National Sun Yat-sen University, Kaohsiung, Taiwan, Jan. 19, 2000.

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