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UNBOUNDED FATOU COMPONENTS OF COMPOSITE TRANSCENDENTAL MEROMORPHIC FUNCTIONS WITH FINITELY MANY POLES

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Abstract. Let $f_i, i = 1, 2, \ldots, m$ be transcendental meromorphic functions of order less than $\frac{1}{2}$ with at most finitely many poles and at least one of them has positive lower order. Let $g = f_m \circ f_{m-1} \circ \cdots \circ f_1$. Then either g has no unbounded Fatou components or at least one unbounded Fatou component g is multiply connected.

1. Introduction

Let f be a transcendental meromorphic function in the complex plane $\mathbb{C}$. The n - iteration of $f(z)$ is denoted by $f^n(z) = f(f^{n-1}(z)), n = 1, 2, \ldots$. Then $f^n(z)$ is well defined for all $z \in \mathbb{C}$ outside a (possible) countable set consisting of the poles of $f^k(z), k = 1, 2, \ldots, n-1$. Define the Fatou set $F(f)$ of $f(z)$ as $F(f)$ $=\{z \in \bar{\mathbb{C}}$: $\{f^n(z)\}$ is well defined and normal in a neighborhood of z $\}$ and $J(f) = \bar{\mathbb{C}} - F(f)$ is the Julia set of $f(z)$. $F(f)$ is open and $J(f)$ is closed and perfect. It is well-known that $F(f)$ is completely invariant under f, that is, $z \in F(f)$ if and only if $f(z) \in F(f)$. Let U be a connected component of $F(f)$. For each $n \geq 1, f^n(U) \subseteq U$, then U is called a periodic component and such the smallest integer n is the period of periodic component U. In particular, a periodic component of periodic one is also called invariant. If for some n, U_n is periodic, but U is not periodic, then U is called preperiodic; U is called a Baker domain of period p, if U is periodic, $f^{np}(z) \to a \in \partial U \cup \{\infty\}$ in U as $n \to \infty$ and $f^p(z)$ is not defined at $z = a$; U is called a wandering domain if $U_m \cap U_n = \emptyset$ for all $m \neq n$.

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Let $f(z)$ be a meromorphic function in $\mathbb{C}$. Let $T(r, f)$ denote the Nevanlinna characteristic function of $f(z)$. The order and lower order of $f(z)$ are defined respectively by

$$\rho(f) = \limsup_{r\to+\infty} \frac{\log\ T(r, f)}{\log\ r}$$

and

$$\mu(f) = \liminf_{r\to+\infty} \frac{\log\ T(r, f)}{\log\ r}.$$

In [10], Zheng and Wang studied the non-existence of unbounded Fatou components of the composition of finitely many entire and meromorphic functions under some suitable conditions.

C. Cao and Y. Wang [6] generalized the result in [10] by studying the boundedness of Fatou components of composition of finitely many transcendental holomorphic functions with small growth. Their main result is the following.

Theorem 1.1. *Let $h(z) = f_N \circ f_{N-1} \circ \cdots \circ f_1(z)$ where $f_i(z)$, $i = 1,\ 2, \ldots,\ N$, are non-constant holomorphic functions in the plane, each having order less than $\frac{1}{2}$. If there is a number $j \in \{1,\ 2, \ldots,\ N\}$ such that the lower order of f_j is greater than 0, then every Fatou component of h is bounded.*

For more details on boundedness of components of $F(f)$ of transcendental entire function $f(z)$, we refer to Baker [2], Stallard [7], Wang [8], Zheng [9], and references cited therein.

In this paper, we discuss the boundedness of Fatou components of composition of finitely many transcendental meromorphic functions of order less than $\frac{1}{2}$ with finitely many poles and obtain a generalization of Theorem 1.1.

2. Main Results

In this paper, we mainly prove the following result.

Theorem 2.1. *Let $f_j(z), j = 1, 2, ..., m$, be transcendental meromorphic functions of order less than $\frac{1}{2}$ with at most finitely many poles and at least one of them has positive lower order. Let $g(z) = f_m \circ f_{m-1} \circ \ldots \circ f_1(z)$. Then either $g(z)$ has no unbounded Fatou components or at least one unbounded Fatou component is multiply connected.*

In order to prove Theorem 2.1, we need the following two lemmas and the basic knowledge of the hyperbolic metric.

Lemma 2.1. *Let $f(z)$ be a meromorphic function of order less than $\frac{1}{2}$ with finitely many poles. There exist $d > 1$ and $R > 0$ such that for all $r > R$, there exists $\tilde{r} \in (r, r^d)$ satisfying*

$$|f(z)| \geq m(\tilde{r}, f) = M(r, f)$$

for all $z \in \{z : |z| = \tilde{r}\}$.

Lemma 2.1 follows directly from [4], satz 1. Actually, $f(z)$ in Lemma 2.1 can be written into the form $f(z) = g(z) + R(z)$ where $g(z)$ is entire with order $\rho(g) = \rho(f) < 1/2$ and $R(z)$ is a rational function such that $R(z) \to 0$ as $z \to \infty$. It is well-known that Lemma 2.1 is true for g, and hence it is easy to see that Lemma 2.1 holds for f.

Lemma 2.2. *Let $f(z)$ be a transcendental meromorphic function with only finitely many poles, finite order ρ and positive lower order μ. Then for any $d > 1$ such that $d\mu > \rho$, we have*

$$\lim_{r\to\infty} \frac{\log M(r^d, f)}{\log M(r, f)} = \infty.$$

Lemma 2.2 follows immediately from the proof of Corollary 2 of Zheng [10]. In what follows, we provide some basic knowledge in hyperbolic geometry; for more details see [1], or [5]. An open set W in $\mathbb{C}$ is called hyperbolic if $\mathbb{C} - W$ contains at least two points. Let U be a hyperbolic domain in $\mathbb{C}$. Let $\lambda_U(z)$ be the density of the hyperbolic metric on U and let $\rho_U(z_1, z_2)$ be the hyperbolic distance between z_1 and z_2 in U, namely

$$\rho_U(z_1, z_2) = \inf_{\gamma \in U} \int_\gamma \lambda_U(z)|dz|,$$

where γ is a Jordan curve connecting z_1 and z_2 in U. If U is simply-connected and $d(z, \partial U)$ is the Euclidean distance between $z \in U$ and ∂U, then for any $z \in U$,

$$\frac{1}{2d(z, \partial U)} \leq \lambda_U(z) \leq \frac{2}{d(z, \partial U)}. \tag{1}$$

Let $f : U \to V$ be an analytic function, where U and V are hyperbolic domains. By the principle of hyperbolic metric, we have

$$\rho_V(f(z_1), f(z_2)) \leq \rho_U(z_1, z_2) \tag{2}$$

for any $z_1, z_2 \in U$.

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. Suppose that $F(g)$ has an unbounded component U and every unbounded component of $F(g)$ is simply-connected. Then by our assumption, U is simply connected. Take a point $z_0 \in U$. Then there exists a sufficiently large $R_0 > |z_0|$ so that each $f_j(z)$ has no poles in $\{z : |z| > R_0\}$.

We first prove the following result: there exists $h > 1$ such that for all sufficiently large r and for an arbitrary curve γ which intersects $\{z : |z| < r\}$ and $\{z : |z| > r^h\}$, we have

$$g(\gamma) \cap \{z : |z| < R\} \neq \emptyset \text{ and } g(\gamma) \cap \{z : |z| > R^h\} \neq \emptyset$$

where $R = M_m(r, g),\ M_1(r, g) = M(r, f_1), ...,\ M_m(r, g) = M(M_{m-1}(r, g), f_m)$. Assume that $f_k(z)$ has positive lower order, $k \in \{1, 2, \dots, m\}$. By Lemma 2.1, for each j, we have $t > 0$ such that for any $r > t$, there exists $\tilde{r_j} \in (r, r^d)$ such that

$$|f_j(z)| > M(r, f_j), \text{ on } \Gamma_j := \{z : |z| = \tilde{r_j}\}, j = 1, 2, \dots, m$$

where each $f_j(z)$ has no poles in $\{z : |z| > t\}$ and $M(r, f_j)$ is increasing for $r > t$. Assume that γ is a curve under our consideration for $h = d^{2k}$, where d is as in Lemma 2.2 for f_k, namely, $\gamma \cap \{z : |z| < r\} \neq \emptyset$ and $\gamma \cap \{z : |z| > r^h\} \neq \emptyset$. From Lemma 2.1, there exists $\tilde{r}_1 \in (r^{d^{2k-1}}, r^{d^{2k}})$ such that

$$|f_1(z)| > M(r^{d^{2k-1}}, f_1) > M(r, f_1)^{d^{2k-2}}, \text{ on } \Gamma_1 := \{z : |z| = \tilde{r}_1\}.$$

Let $R_1 = M(r, f_1)$. Then, $f_1(\gamma) \cap \{z : |z| > R_1^{d^{2k-2}}\} \neq \emptyset$ and from the maximum modulus principle, we have $f_1(\gamma) \cap \{z : |z| < R_1\} \neq \emptyset$.

Thus, there exists $\tilde{R}_1 \in (R_1^{d^{2k-3}}, R_1^{d^{2k-2}})$ such that

$$|f_2(z)| > M(R_1^{d^{2k-3}}, f_2) > M(R_1, f_2)^{d^{2k-4}}, \text{ on} \Gamma_2 := \{z : |z| = \tilde{R}_1\}.$$

Let $R_2 = M(R_1, f_2)$. Then,

$$f_2 \circ f_1(\gamma) \cap \{z : |z| < R_2\} \neq \emptyset \text{ and } f_2 \circ f_1(\gamma) \cap \{z : |z| > R_2^{d^{2k-4}}\} \neq \emptyset.$$

Thus, there exists $\tilde{R}_2 \in (R_2^{d^{2k-5}}, R_2^{d^{2k-4}})$ such that

$$|f_3(z)| > M(R_2^{d^{2k-5}}, f_3) > M(R_2, f_3)^{d^{2k-6}}, \text{ on} \Gamma_3 := \{z : |z| = \tilde{R}_2\}.$$

Inductively, we set $R_{k-2} = M(R_{k-3}, f_{k-2})$. Then,

$$f_{k-2} \circ f_{k-3} \circ \cdots \circ f_1(\gamma) \cap \{z : |z| > R_{k-2}^{d^4}\} \neq \emptyset$$

and

$$f_{k-2} \circ f_{k-3} \circ \cdots \circ f_1(\gamma) \cap \{z : |z| < R_{k-2}\} \neq \emptyset.$$

Thus, there exists $\tilde{R}_{k-2} \in (R_{k-2}^{d^3}, R_{k-2}^{d^4})$ such that

$$|f_{k-1}(z)| > M(R_{k-2}^{d^3}, f_{k-1}) > M(R_{k-2}, f_{k-1})^{d^2}, \text{ on } \Gamma_{k-1} := \{z : |z| = \tilde{R}_{k-2}\}.$$

Set $R_{k-1} = M(R_{k-2}, f_{k-1})$. Then,

$$f_{k-1} \circ f_{k-2} \circ \cdots \circ f_1(\gamma) \cap \{z : |z| > R_{k-1}^{d^2}\} \neq \emptyset$$

and

$$f_{k-1} \circ f_{k-2} \circ \cdots \circ f_1(\gamma) \cap \{z : |z| < R_{k-1}\} \neq \emptyset.$$

Thus, there exists $\tilde{R}_{k-1} \in (R_{k-1}^{d}, R_{k-1}^{d^2})$ such that

$$|f_k(z)| > M(R_{k-1}^{d}, f_k) > M(R_{k-1}, f_k)^{d^{2m}}, \text{ on } \Gamma_k := \{z : |z| = \tilde{R}_{k-1}\},$$

where the last inequality follows from 2.2.
Set $R_k = M(R_{k-1}, f_k)$. Then,

$$f_k \circ f_{k-1} \circ \cdots \circ f_1(\gamma) \cap \{z : |z| > R_k^{d^{2m}}\} \neq \emptyset$$

and

$$f_k \circ f_{k-1} \circ \cdots \circ f_1(\gamma) \cap \{z : |z| < R_k\} \neq \emptyset.$$

Thus, there exists $\tilde{R}_k \in (R_k^{d^{2m-1}}, R_k^{d^{2m}})$ such that

$$|f_{k+1}(z)| > M(R_k^{d^{2m-1}}, f_{k+1}) > M(R_k, f_{k+1})^{d^{2m-2}}, \text{ on } \Gamma_{k+1} := \{z : |z| = \tilde{R}_k\}.$$

Inductively, we set $R_{m-1} = M(R_{m-2}, f_{m-1})$. Then, we have

$$f_{m-1} \circ f_{m-2} \circ \cdots \circ f_1(\gamma) \cap \{z : |z| > R_{m-1}^{d^{2k+2}}\} \neq \emptyset$$

and

$$f_{m-1} \circ f_{m-2} \circ \cdots \circ f_1(\gamma) \cap \{z : |z| < R_{m-1}\} \neq \emptyset.$$

Thus, there exists $\tilde{R}_{m-1} \in (R_{m-1}^{d^{2k+1}}, R_{m-1}^{d^{2k+2}})$ such that

$$|f_m(z)| > M(R_{m-1}^{d^{2k+1}}, f_m) > M(R_{m-1}, f_m)^{d^{2k}} = M_m(r, g)^h,$$

on $\Gamma_m := \{z : |z| = \tilde{R}_{m-1}\}$.
Moreover, there exists a point $z_{m1} \in \gamma$ such that

$$|f_{m-1} \circ f_{m-2} \circ \cdots \circ f_1(z_{m1})| = \tilde{R}_{m-1}.$$

Thus, $|g(z_{m1})| > M_m(r,g)^h > M(R_0,g)^h > |g(z_0)|^h$. By setting $R_{m1} = M_m(r,g)$, we obtain

$$g(\gamma) \cap \{z : |z| = R_{m1}^h\} \neq \emptyset \text{ and } g(\gamma) \cap \{z : |z| = R_{m1}\} \neq \emptyset.$$

Repeating the previous process above inductively, there is a point $z_{mn} \in \gamma$ such that

$$|g^n(z_{mn})| > M(R_{mn},g)^h \geq M(R_0,g)^h > |g^n(z_0)|^h, \tag{3}$$

where $R_{mn} = M_m(R_{n-1},g)$. Since $g^n(U) \subseteq U_n$ and U is unbounded, so U_n is an unbounded component of $F(g)$ and by our assumption U_n is simply-connected. For an arbitrary point $a \in J(g)$, we obtain, by (1), that

$$\lambda_{U_n}(z) \geq \frac{1}{2d(z,\partial U_n)} \geq \frac{1}{2|z-a|} \geq \frac{1}{2(|z|+|a|)}. \tag{4}$$

It follows from (4) that

$$\begin{aligned}\rho_{U_n}(g^n(z_0), g^n(z_{mn})) &\geq \int_{|g^n(z_0)|}^{|g^n(z_{mn})|} \frac{dr}{2(r+|a|)} \\ &= \frac{1}{2}\log\frac{|g^n(z_{mn})|+|a|}{|g^n(z_0)|+|a|}.\end{aligned} \tag{5}$$

Set $A = \max\{\lambda_U(z_0,z) : z \in \gamma\}$. Clearly $A \in (0,+\infty)$. From (2), noting that $z_{mn} \in \gamma \subset U$, we have

$$\rho_{U_n}(g^n(z_0), g^n(z_{mn})) \leq \rho_U(z_0, z_{mn}) \leq A. \tag{6}$$

Therefore, by combining (3), (5) and (6) we obtain

$$|g^n(z_0)|^h < M(R_0,g^n)^h < |g^n(z_{mn})| + |a| \leq (|g^n(z_0)|+|a|)e^{2A}.$$

This is impossible, since a and e^{2A} are constants, $h > 1$ and $|g^n(z_0)| \to +\infty$ as $n \to +\infty$. Therefore, if $F(g)$ has an unbounded Fatou component, then at least one of them is multiply connected. This completes the proof. ■

Since all unbounded Fatou components of a transcendental entire function are simply connected [3], we obtain Theorem 1.1 as a corollary of Theorem 2.1.

Corollary 2.1. *Let $f_j(z), j = 1,2,...,m$, be transcendental entire functions with order less than $\frac{1}{2}$ and at least one of them has positive lower order. Let $g(z) = f_m \circ f_{m-1} \circ \ldots \circ f_1(z)$. Then $g(z)$ has no unbounded Fatou components.*

Remark 2.1. One may find another proof of Corollary 2.1 in [6].

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