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GLOBAL DYNAMICAL BEHAVIORS FOR AN SIR EPIDEMIC MODEL WITH TIME DELAY AND PULSE VACCINATION

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Abstract. Since the investigation of impulsive delay differential equations are inchoate, the literature on delay epidemic models with pulse vaccination is not extensive. In this paper, we propose a delay SIR epidemic model with impulsive effect and analyze the global dynamics behaviors of the model. Using the discrete dynamical system determined by the stroboscopic map, we show that there exists an 'infection-free' periodic solution in the model. Further, we prove that the 'infection-free' periodic solution is globally attractive when the period of impulsive effect is less than some critical value. The condition for the permanence of population for epidemic model with pulse vaccination is given, which means the epidemic disease to spread around. We conclude that time delay and pulse bring great effects on the dynamics of the model. It illustrated theoretically that we should look carefully after the infectious patients to shorten the convalescence period (or infections period). Lastly, the biologic discussion and strategy for the elimination of infectious diseases are also given. In this paper, the main feature is that we introduce time delay together with pulse into epidemic model, and we investigate their effects on the dynamics of model.

1. Introduction

The SIR infections disease model is an important biologic model and has been studied by many authors [1-16]. It is well-known that one of strategies to control infectious diseases is vaccination. Then a number of epidemic models in ecology can be formulated as dynamical systems of differential equations with vaccination [17-22]. It is very important that one investigates under what conditions a given agent

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can invade partially vaccinated population, i.e., how large a fraction of the population do we have to keep vaccinated in order to prevent the agent from establishing. Pulse vaccination seems more reasonable than traditional continuous constant vaccination in real world. Recently, pulse vaccination, the repeated application of vaccine over a defined age range, is gaining prominence as a strategy for the elimination of childhood viral infectious such as measles hepatitis, parotitis, smallpox and phthisis. Pulse vaccination strategy (PVS) [4-8, 21], consists of periodical repetitions of impulsive vaccinations in a population, on all the age cohorts, differently from the traditional constant vaccination.

Generally speaking, after infectious individuals lived through a infection period, he recovers completely and transfer to the 'removed' class, so, the number of the death of the infectious should be considered during convalescence, which is called the phenomena of 'time delay'. Then time delay has very important biologic meaning in epidemic models. The investigation of impulsive delay differential equations is inchoate, and most of impulsive delay differential equations are analyzed in theory (see [23-25]), but the good results on global qualitative analysis for delay biologic models with impulse effect are not extensive. Now most of the research literature on SIR epidemic models are established by ODE (see [3, 15, 16]), delayed ODE (see [1, 2, 9-14]) or impulsive ODE (see [4-8, 20-22]). However, epidemic models with time delay and impulsive effects are not extensive. Time delay and pulse are introduced into epidemic disease models, which greatly enriches biologic background. The most basic and important questions to ask for epidemic disease models in the theory of mathematical ecology are global asymptotic behaviors, the persistence, and extinctions of the population. Therefore, in present paper, we consider a new SIR epidemic model with impulsive vaccination and time delay and study their dynamic behaviors (the 'infection-free' periodic solution, the permanence, global attractive behavior) under pulse vaccination. The main aim of the present paper is to introduce time delay, pulse vaccination in epidemic model and to obtain some important qualitative properties and valid pulse vaccination strategy.

The organization of this paper is as follows. In the next section, we introduce the SIR epidemic model with time delay and pulse vaccination. In Section 3, we investigate the dynamic behaviors of the SIR epidemic model, obtain the sufficient condition of global attractivity of 'infection-free' periodic solution. In Section 4, we investigate the permanence of the population of the model. Lastly, biologic discussion and pulse vaccination strategy for the elimination of infectious diseases are given.

2. SIR Epidemic Model and Preliminary Information

SIR models with time delay were investigated by a number of authors (see[12-

16,23,24]). For example,

$$(1)\qquad \begin{cases} \dot{S}(t) = \mu - \beta S(t)I(t) - \mu S(t), \\ \dot{I}(t) = \beta S(t)I(t) - \beta S(t-\tau)I(t-\tau) - \mu I(t), \\ \dot{R}(t) = \beta S(t-\tau)I(t-\tau) - \mu R(t), \end{cases}$$

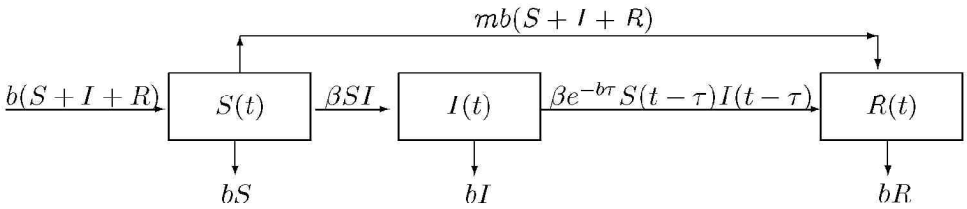
which represents an SIR model with epidemics spread via a vector with an convalescence period τ (or infections period). For the detailed biological meanings, we refer to [1, 2, 13].

Based on system (1), we further give the following assumptions:

(i) All newborns transfer to the 'susceptible' class (S) as soon as who are born, and the treatment with pulse vaccination is taken successfully to a proportion of all newborns;

(ii) Any infectious individual who recovers completely and transfer to the 'removed' class (R) as soon as lived through the infections period τ, thus, the number of death of the infectious during convalescence (with τ) should be considered, which is called the phenomena of 'time delay', that is the $\beta e^{-b\tau}S(t-\tau)I(t-\tau)$ term.

Therefore, under assumptions above, we consider a new SIR model with a time delay and pulse vaccination as follows:



The compartmental diagram for the SIR model with pulse vaccination.

$$(2)\qquad \begin{cases} \left.\begin{aligned} &\dot{S}(t) = -\beta S(t)I(t) - bS(t) + b(S(t) + I(t) + R(t)), \\ &\dot{I}(t) = \beta S(t)I(t) - \beta e^{-b\tau}S(t-\tau)I(t-\tau) - bI(t), \\ &\dot{R}(t) = \beta e^{-b\tau}S(t-\tau)I(t-\tau) - bR(t), \end{aligned}\right\} \ t \neq nT, n \in N, \\ \left.\begin{aligned} &S(t^+) = S(t) - mb(S(t) + I(t) + R(t)), \\ &I(t^+) = I(t), \\ &R(t^+) = R(t) + mb(S(t) + I(t) + R(t)), \end{aligned}\right\} \ t = nT, n \in N, \\ S(t) + I(t) + R(t) = 1, \end{cases}$$

where $m($ with $0 < m < 1)$ is the proportion of those vaccinated successfully to all of newborns.

Pulse vaccination can be defined as the repeated application of vaccine among an age range. Assume the pulse scheme proposes to vaccinate successfully a fraction m of the newborn population in a single pulse, applied every T years. Pulse vaccination gives lifelong immunity to a proportion of the population (with $mb(S(t)+I(t)+R(t))$), as a consequence, transferred to the 'recovered' class (R) of the population. Immediately following each vaccination pulse, system (2) evolves from its new initial state without being further affected by the vaccination scheme until the next pulse is applied. The underlying principle is to apply vaccination pulses frequently enough so as to prevent the infectious population from ever growing.

To study (2), we only need consider the following system:

$$\left\{\begin{array}{l} \left.\begin{array}{l} \dot{S}(t) = -\beta S(t)I(t) - bS(t) + b, \\ I(t) = \beta \displaystyle\int_{t-\tau}^{t} S(\theta)I(\theta)e^{-b(t-\theta)}d\theta, \\ \dot{R}(t) = \beta e^{-b\tau}S(t-\tau)I(t-\tau) - bR(t), \end{array}\right\} \ t \neq nT, n \in N, \\ \left.\begin{array}{l} S(t^+) = S(t) - mb, \\ I(t^+) = I(t), \\ R(t^+) = R(t) + mb, \end{array}\right\} \ t = nT, n \in N, \end{array}\right. \tag{3}$$

where $\beta \int_{t-\tau}^{t} S(\theta)I(\theta)e^{-b(t-\theta)}d\theta$ represents the number of the infectious at time t, which includes the increasing infectious who come from the susceptible, the losing infectious who die and transfer to the 'removed' class except for the death from $t-\tau$ to t. Note that the variable R does not appear in the first two equations of system (3), hence we only need to consider the subsystem of (3) as follows:

$$\left\{\begin{array}{l} \left.\begin{array}{l} \dot{S} = -\beta S(t)I(t) - bS(t) + b, \\ I(t) = \beta \displaystyle\int_{t-\tau}^{t} S(\theta)I(\theta)e^{-b(t-\theta)}d\theta, \end{array}\right\} \ t \neq nT, n \in N, \\ \left.\begin{array}{l} S(t^+) = S(t) - mb, \\ I(t^+) = I(t), \end{array}\right\} \ t = nT, n \in N. \end{array}\right. \tag{4}$$

We also note that time delay together with pulse vaccination is introduced into model (4), which is the feature of this paper.

In the following, we will give some definitions, notations and lemmas which will be useful for our main results.

Let

$$C_h^+ = \{\phi = (\phi_1(s), \phi_2(s)) \in C_h : \phi_i(0) > 0 \ (i = 1, 2)\},$$

where $\phi_i(s)$ is positive, bounded and continuous function for $s \in [-\tau, 0]$. Motivated by the application of systems (4) to population dynamics (see [26]), we assume that

solutions of systems (4) satisfy the initial conditions

$$\phi \in C_h^+, \quad \phi(0) > 0. \tag{5}$$

Lemma 2.1. (see[27]) *Consider the following impulse differential inequalities:*

$$\begin{aligned} w'(t) &\le p(t)w(t) + q(t), \qquad t \ne t_k, \\ w(t_k^+) &\le d_k w(t_k) + b_k, \qquad t = t_k, k \in N, \end{aligned} \tag{6}$$

where $p(t), q(t) \in C(R_+, R), d_k \ge 0$, *and* b_k *are constants.*

Assume

(A_0) *the sequence* $\{t_k\}$ *satisfies* $0 \le t_0 < t_1 < t_2 < \cdots$, *with* $\lim_{t\to\infty} t_k = \infty$;

$(A_1) w \in PC'(R_+, R)$ *and* $w(t)$ *is left-continuous at* $t_k, k \in N$.

Then

$$\begin{aligned} w(t) \le\ & w(t_0) \prod_{t_0<t_k<t} d_k \exp\left(\int_{t_0}^{t} p(s)ds\right) \\ & + \sum_{t_0<t_k<t} \left(\prod_{t_k<t_j<t} d_j \exp\left(\int_{t_k}^{t} p(s)ds\right)\right) b_k \\ & + \int_{t_0}^{t} \prod_{s<t_k<t} d_k \exp\left(\int_{s}^{t} p(\theta)d\theta\right) q(s)ds, t \ge t_0. \end{aligned} \tag{7}$$

Similarly, if the signs of inequality of (6) are reverse, then that of (7) is also reverse.

When pulse vaccination and delays are incorporated into SIR models, the systems become nonautonomous, which cause us a lot of trouble in studying the models. Therefore, the literature on this one is not extensive. Now we study the global attractivity of 'infection-free' solution (infection-eradication) and the permanence (infection-epidemic) of the systems (4) with initial conditions (5) under the influence of pulse and time delay.

3. Global Attractivity of 'Infection-free' Periodic Solution of (4)

We begin the analysis of (4) by first demonstrating the existence of an 'infection-free' solution, in which infectious individuals are entirely absent from the population permanently,i.e.,

$$I(t) = 0, \quad t \ge 0. \tag{8}$$

This is motivated by the fact that $I^* = 0$ is an equilibrium solution for the variable $I(t)$, as it leaves $I'(t) = 0$. Under these conditions, we show below

that the susceptible population oscillates with period T in synchronization with the periodic pulse vaccination.

Assuming (8), we know that the growth of the susceptible in the time-interval $nT < t \leq (n+1)T$ and give some basic properties of the following subsystem of (4)

$$\begin{cases} \dot{S}(t) = -bS(t) + b, & t \neq nT, n \in N \\ S(t^+) = S(t) - mb, & t = nT, n \in N. \end{cases} \tag{9}$$

For integrating and solving equation (9) between pulses, using the discrete dynamical system determined by the stroboscopic map (see [27]), yields

$$\begin{cases} S^*(t) = 1 - \dfrac{mb}{1-e^{-bT}} e^{-b(t-nT)}, & t \in (nT, (n+1)T], n \in N \\ S^*(0^+) = S^*(nT^+) = 1 - \dfrac{mb}{1-e^{-bT}}, \end{cases} \tag{10}$$

which is unique a globally asymptotically stable positive periodic solution of system (9).

Since the solution of (9) is

$$\begin{cases} S(t) = \left(S(0^+) - \left(1 - \dfrac{mb}{1-e^{-bT}}\right)\right) e^{-bt} + S^*(t),\ t \in (nT, (n+1)T], n \in N \\ S^*(0^+) = 1 - \dfrac{mb}{1-e^{-bT}}, \end{cases} \tag{11}$$

we have the following Lemma (3.1).

Lemma 3.1. *System (9) has unique a globally asymptotically stable positive periodic solution $S^*(t)$, that is, the system (4) has a 'infection-free' periodic solution $(S^*(t), 0)$ for $t \in (nT, (n+1)T], n \in N$, $S^*(nT^+) = S^*(0^+) = 1 - \frac{mb}{1-e^{-bT}} > 0$ and for every solution $S(t)$ of (9) we have $S(t) \to S^*(t)$ as $t \to \infty$.*

According to biologic meanings, we always assume initial condition $S^*(0^+) > 0$ and $S(0^+) > 0$, that is , $1 - \frac{m}{1-e^{-bT}} > 0$ through this paper.

Theorem 3.1. *If $R_1 < 1$, then $(S^*(t), 0)$ is globally attractive, where*

$$R_1 = \frac{T\beta}{m\beta + bT}. \tag{12}$$

Proof. From $R_1 < 1$, we can choose a $\varepsilon > 0$ small enough such that

$$\delta = \exp\left(\int_0^T (\beta(S^*(t) + \varepsilon) - b)\, dt\right) < 1.$$

Note that $dS(t)/dt \le b - bS(t), S(t^+) = S(t) - mb$, consider the following impulsive differential equation

$$\begin{cases} \dfrac{du(t)}{dt} = b - bu(t), & t \ne nT, n \in N, \\ u(t^+) = u(t) - mb, & t = nT, n \in N, \\ u(0^+) = S(0^+). \end{cases} \tag{13}$$

By using comparison theorem of impulsive equation (see Theorem 3.1.1 in [27]), we have $S(t) \le u(t)$ and $u(t) \to S^*(t)$ as $t \to \infty$, where $S^*(t) = 1 - \frac{mb}{1-e^{-bT}} e^{-b(t-nT)}$ is unique a globally asymptotically stable positive periodic solution of system (13).

Hence

$$S(t) \le u(t) < S^*(t) + \varepsilon, \tag{14}$$

for all t large enough. For convenience, we may assume (14) holds for all $t \ge 0$. From the second equation of (4) and (14), we get

$$\begin{cases} \dot{I}(t) \le [\beta(S^*(t) + \varepsilon) - b]I(t), t \ne nT, n \in N, \\ I(nT^+) = I(nT), \qquad\qquad t = nT, n \in N, \end{cases} \tag{15}$$

which leads to

$$\begin{aligned} I((n+1)T) &\le I(nT^+) \exp\left(\int_{nT}^{(n+1)T} (\beta(S^*(t) + \varepsilon) - b)\, dt \right) \\ &= I(nT) \exp\left(\int_0^T (\beta(S^*(t) + \varepsilon) - b)\, dt \right) \\ &= I(nT)\delta. \end{aligned} \tag{16}$$

Hence $I(nT) \le I(0^+)\delta^n$ and $I(nT) \to 0$ as $t \to \infty$. Therefore $I(t) \to 0$ as $t \to \infty$.

Without loss of generality, we may assume that $0 < I(t) < \varepsilon$ for all $t \ge 0$, then from system(4) we have

$$\frac{dS}{dt} \ge -(\beta\varepsilon + b)S(t) + b.$$

Then we have $z_1^*(t) \le S(t)$ and $z_1^*(t) \to S^*(t)$, as $t \to \infty$, where $z_1^*(t)$ is unique a globally asymptotically stable positive periodic solution of

$$\begin{cases} \dfrac{dz_1(t)}{dt} = -(\beta\varepsilon + b)z_1(t) + b, & t \ne nT, n \in N, \\ z_1(t^+) = z_1(t) - mb, & t = nT, n \in N, \\ z_1(0^+) = S(0^+) > 0. \end{cases} \tag{17}$$

From (17), for $nT < t \leq (n+1)T$ we have

$$z_1^*(t) = \frac{b}{\beta\varepsilon + b} - \frac{mb}{1 - \exp\{-(\beta\varepsilon + b)T\}} \exp\{-(\beta\varepsilon + b)(t - nT)\}.$$

Therefore, for any $\varepsilon_1 > 0$ there exists a $T_1 > 0$ such that

$$S(t) > z_1^*(t) - \varepsilon_1, \quad \text{for } t > T_1. \tag{18}$$

Let $\varepsilon \to 0$, from (14) and (18), we have

$$S^*(t) - \varepsilon_1 < S(t) < S^*(t) + \varepsilon_1$$

for t large enough, which implies $S(t) \to S^*(t)$ as $t \to \infty$. This completes the proof.

Set

$$T_* = \frac{m\beta}{\beta - b}, \quad m^* = \frac{T(\beta - b)}{\beta}.$$

Corollary 3.1.

(*i*) *If $\beta \leq b$, then 'infection-free' periodic solution $(S^*(t), 0)$ is globally attractive.*

(*ii*) *If $\beta > b$ and $T < T_*$ or $m > m^*$, then 'infection-free' periodic solution $(S^*(t), 0)$ is globally attractive.*

Remark 3.1. By (12), we can easily see that a short period of pulsing (with T) or a large pulse vaccination rate (with m) is sufficient condition for the eradication of the disease.

The permanence of the population of epidemic model implies that the disease spreads around and generates an endemic ultimately, so, we need to investigate the permanence of the population of model (4) now.

4. The Permanence of the Population of (4)

Before starting our theorem, we give the following definition.

Definition 4.1. (see [28]) System(4) is said to be uniformly persistent if there are positive constants $m_i, i = 1, 2$ and a finite time T_0 such that for all solutions $(S(t), I(t))$ with initial values $S(0^+) > 0, I(0^+) > 0, m_1 \leq S(t), m_2 \leq I(t)$ holds for all $t \geq T_0$.

Definition 4.2. (see [28]) System(4) is said to be permanent if system (4) is uniformly persistent and bounded, that is, there are positive constants $m_i, M_i, i =$

$1, 2$ and a finite time T_0 such that for all solutions $(S(t), I(t))$ with initial values $S(0^+) > 0, I(0^+) > 0, m_1 \le S(t) \le M_1, m_2 \le I(t) \le M_2$ holds for all $t \ge T_0$.

Theorem 4.1. *If $R_2 > 1$, then the system (4) is uniformly persistent, that is, there exist two positive constants m_1 and m_2 such that $S(t) \ge m_1$, $I(t) \ge m_2$ for t large enough, where*

$$R_2 = \frac{\beta(1 - e^{-b\tau})(1 - e^{-bT})}{b[1 - e^{-bT} + m\beta(1 - e^{-b\tau})]}. \tag{19}$$

Proof. Suppose that $X(t) = (S(t), I(t))$ is any positive solution of system (4) with initial conditions (5). The second equation of system (4) may be rewritten as follow:

$$\dot{I}(t) = [\beta(1 - e^{-b\tau})S(t) - b]I(t) + \beta e^{-b\tau} \frac{d}{dt} \int_{t-\tau}^{t} S(\theta)I(\theta)d\theta. \tag{20}$$

Define

$$V(t) = I(t) - \beta e^{-b\tau} \int_{t-\tau}^{t} S(\theta)I(\theta)d\theta.$$

Calculating the derivative of $V(t)$ along the solution of (4) and (20), it follows that

$$\frac{dV(t)}{dt} = b\left[\frac{\beta(1 - e^{-b\tau})}{b} S(t) - 1\right] I(t). \tag{21}$$

Set

$$m_2^* = \frac{(1 - e^{-b\tau})(1 - e^{-bT})}{1 - e^{-bT} + m\beta(1 - e^{-b\tau})} - \frac{b}{\beta}.$$

Since $R_2 > 1$, then $m_2^* > 0$ and there exists a positive constant ε_1 small enough such that

$$\frac{\beta(1 - e^{-b\tau})\varrho}{b} > 1, \tag{22}$$

where

$$\varrho = \frac{b}{m_2^*\beta + b} - \frac{mb}{1 - \exp\{-(m_2^*\beta + b)T\}} - \varepsilon_1 > 0.$$

For any positive constant t_0, we claim that the inequality $I(t) < m_2^*$ cannot hold for all $t \ge t_0$. Otherwise, there is a positive constant t_0, such that $I(t) < m_2^*$ for all $t \ge t_0$. From the first equation of system (4), we have

$$\begin{cases} \dfrac{dS(t)}{dt} \ge -(m_2^*\beta + b)S(t) + b, \ t \ne nT, \\ S(t^+) = S(t) - mb, \qquad\qquad t = nT. \end{cases}$$

So we have $S(t) \geq z_2(t)$ and $z_2(t) \to z_2^*(t), t \to \infty$, where $z_2(t)$ is the solution of

$$\begin{cases} \dfrac{dz}{dt} = -(m_2^*\beta + b)z + b, & t \neq nT, \\ z(t^+) = z(t) - mb, & t = nT, \\ S(0^+) = z(0^+) \end{cases} \tag{23}$$

and

$$z_2^*(t) = \frac{b}{m_2^*\beta + b} - \frac{mb}{1 - \exp\{-(m_2^*\beta + b)T\}} \exp\{-(\beta m_2^* + b)(t - nT)\},\ t \in (nT, (n+1)T]$$

is unique a globally asymptotically stable positive periodic solution of system (23). Therefore, there exists such $T_1 \geq t_0 + \tau$, for $t \geq T_1$ that

$$S(t) \geq z_2(t) > z_2^*(t) - \varepsilon_1 > z_2^*(0^+) - \varepsilon_1, \tag{24}$$

where

$$z_2^*(0^+) - \varepsilon_1 = \frac{b}{m_2^*\beta + b} - \frac{mb}{1 - \exp\{-(m_2^*\beta + b)T\}} - \varepsilon_1 =: \varrho. \tag{25}$$

Namely, we have that

$$S(t) > \varrho, \tag{26}$$

By (21) and (26), we see that

$$\frac{dV(t)}{dt} > b\left[\frac{\beta(1 - e^{-b\tau})}{b}\varrho - 1\right] I(t), \quad t \geq T_1. \tag{27}$$

Let

$$I^l = \min_{t \in [T_1, T_1 + \tau]} I(t).$$

We claim $I(t) \geq I^l$ for all $t \geq T_1$. Otherwise, there exists a nonnegative constant T_2 such that $I(t) \geq I^l$ for $t \in [T_1, T_1 + \tau + T_2]$ and $I(T_1 + \tau + T_2) = I^l$. From the second equation of (4) and (26), we have

$$\begin{aligned} I(T_1 + \tau + T_2) &= \beta \int_{T_1 + T_2}^{T_1 + \tau + T_2} S(\theta) I(\theta) \exp(-b(T_1 + \tau + T_2 - \theta)) d\theta \\ &> \frac{\beta(1 - e^{-b\tau})\varrho}{b} I^l \\ &> I^l, \end{aligned}$$

which is a contradiction. Hence we get that $I(t) \geq I^l > 0$ for all $t \geq T_1$. From (27), we have

$$\frac{dV(t)}{dt} > b\left[\frac{\beta(1-e^{-b\tau})}{b}\varrho - 1\right] I^l > 0.$$

Then there exists such $T_1^* \geq T_1$ that $V(t) > 0$ for all $t \geq T_1^*$. By (27) and the definition of $V(t)$, we obtain that for $t \geq T_1^*$,

$$\frac{dV(t)}{dt} > b\left[\frac{\beta(1-e^{-b\tau})}{b}\varrho - 1\right] V(t),$$

which implies $V(t) \to +\infty$ as $t \to +\infty$. This is a contradiction to the inequality $V(t) \leq 1 + \beta\tau \exp(-b\tau)$. Therefore, for any positive constant t_0, the inequality $I(t) < m_2^*$ cannot hold for all $t \geq t_0$.

If $I(t) \geq m_2^*$ holds true for all t large enough, then our aim is obtained. Otherwise, $I(t)$ is oscillatory about m_2^*. Then there exist two positive constants $\bar{t}, \delta$ such that

$$I(\bar{t}) = I(\bar{t}+\delta) = m_2^*$$

and

$$I(t) < m_2^*, \quad \text{for } \bar{t} < t < \bar{t}+\delta.$$

When $\bar{t}$ is large enough, the inequality $S(t) > \varrho$ holds true for $\bar{t} < t < \bar{t}+\delta$. Since the positive solutions of (4) are ultimately bounded and $I(t)$ is not effected by impulses, $I(t)$ is uniformly continuous. Hence there is a constant T_3 (with $0 < T_3 < \tau$ and T_3 is independent of $\bar{t}$) such that $I(t) > \frac{m_2^*}{2}$ for all $\bar{t} \leq t \leq \bar{t}+T_3$. If $\delta \leq T_3$, there is nothing to prove. If $T_3 < \delta \leq \tau$, for $\bar{t}+T_3 < t \leq \bar{t}+\delta$,

$$\begin{aligned} I(t) &= \beta \int_{t-\tau}^{t} S(\theta)I(\theta)\exp(-b(t-\theta))d\theta \\ &\geq \beta \int_{\bar{t}}^{\bar{t}+T_3} S(\theta)I(\theta)\exp(-b(t-\theta))d\theta \\ &> \frac{\beta e^{-b\tau} m_2^* T_3 \varrho}{2} =: m_2^{**}. \end{aligned}$$

Set

$$m_2 = \min\left\{\frac{m_2^*}{2}, m_2^{**}\right\}.$$

Hence we get that $I(t) \geq m_2$ for all $\bar{t} \leq t \leq \bar{t}+\delta$. If $\delta > \tau$, we can obtain that $I(t) \geq m_2$ for $\bar{t} \leq t \leq \bar{t}+\tau$. Then, proceeding exactly as the proof for above claim, we can obtain that $I(t) \geq m_2$ for $\bar{t}+\tau \leq t \leq \bar{t}+\delta$. Since this kind of interval

$[\bar{t}, \bar{t}+\delta]$ is chosen in an arbitrary way when $\bar{t}$ is large enough, we conclude that $I(t) \geq m_2$ for t large enough.

From the first equation of (4), we have that

$$\frac{dS}{dt} \geq -(\beta + b)S(t) + b. \tag{28}$$

Then we have $z_3(t) \leq S(t)$, where $z_3(t)$ is a solution of

$$\begin{cases} \dfrac{dz_3(t)}{dt} = -(\beta + b)z_3(t) + b, & t \neq nT, n \in N, \\ z_3(t^+) = z_3(t) - mb, & t = nT, n \in N, \\ z_3(0^+) = S(0^+) > 0. \end{cases} \tag{29}$$

Suppose $z_3^*(t)$ is the unique globally attractive positive periodic solution of (29). Similar to the proof of Theorem 3.1, there exists a $T_4 > 0$ for $\varepsilon > 0$ small enough such that

$$S(t) \geq z_3(t) > z_3^*(t) - \varepsilon \geq z_3^*(0^+) - \varepsilon =: m_1, \quad \text{for } t > T_4. \tag{30}$$

In view of our above discussions, the choices of m_1 and m_2 are independent of positive solutions. This complete this proof.

Theorem 4.2. *If $R_2 > 1$, then system (4) is permanent.*

Proof. Suppose that $X(t) = (S(t), I(t))$ is any positive solution of system (4) with initial conditions (5). By Theorem 4.1, there exist positive constants m_1, m_2 and T^* such that $S(t) \geq m_1, I(t) \geq m_2$ for $t \geq T^*$.

Set

$$D = \{(S, I) \in R^2 | S(t) \geq m_1, I(t) \geq m_2, S(t) + I(t) \leq 1\}.$$

Then D is a bounded compact region which has positive distance from the coordinate axes. By Theorem 4.1, one obtains that every positive solution of system (4) with the initial condition (5) eventually enters and remains in the region D. The proof is completed.

Set

$$m_* = \frac{(1 - e^{-bT})[\beta(1 - e^{-b\tau}) - b]}{b\beta(1 - e^{-b\tau})}, \ \tau^* = -\frac{1}{b}\ln\left(1 - \frac{b(1 - e^{-bT})}{\beta(1 - e^{-bT} - mb)}\right),$$

$$T^* = -\frac{1}{b}\ln\left(1 - \frac{mb\beta(e^{b\tau} - 1)}{(\beta - b)e^{b\tau} - \beta}\right).$$

Corollary 4.1. *If $m < m_*$ or $\tau > \tau^*$ or $T > T^*$, then system (4) is permanent, that is, the disease can generate an epidemic.*

Remark 4.1. By (19), we can easily see that a long period of pulsing (with T) or a small pulse vaccination rate (with m) or a short latent period of the disease (with τ) is sufficient condition for the permanence of the model. Therefore, it illustrated theoretically that we should look carefully after the infectious patients to shorten the convalescence period τ (or infections period).

Remark 4.2. Since

$$R_2 = \frac{\beta(1-e^{-b\tau})(1-e^{-bT})}{b[1-e^{-bT}+m\beta(1-e^{-b\tau})]} = \frac{\beta T}{\frac{bT}{1-e^{-b\tau}}+\frac{m\beta bT}{1-e^{-bT}}} \leq R_1 = \frac{T\beta}{m\beta+bT},$$

when $R_1 < 1$, then $R_2 < 1$, when $R_2 > 1$, then $R_1 > 1$. We only give the 'relatively exact' sufficient and unnecessary conditions for global attractivity of 'infection-eradication' periodic solution (with $R_1 < 1$) and permanence of the epidemic disease (with $R_2 > 1$). It is worthwhile for us to study the case for $R_1 > 1$ and $R_2 < 1$ in the future work. However, we are unable to do so here due to the difficulty involved.

5. Discussion

According to the arguments above, our results are summarized as follows:

(i) By (19), we can see that R_2 directs proportion τ value(with time delay), which implies that time delay (with τ) measures the inhibition effect from the behavioral change of the infectious when they transfer to the 'removed' class (R);

(ii) By (12) and (19), we can find that R_1 and R_2 direct proportion T value, inverse proportion m value, which implies that pulse vaccination measures the inhibition effect from the behavioral change of the susceptible when they transfer to the 'infectious' class (R). Therefore, Our results indicate that a long period of pulsing or a small proportion of vaccinated newborns will not lead to eradication of the disease.

Firstly, we introduce a new SIR models with time delay and impulsive effect. Further, we show the existence of 'infection-free' periodic solution. Furthermore, we obtain the 'relatively exact' conditions for global attractivity of the 'infection-free' periodic solution. Lastly, we prove that system (4) is uniformly permanent under a appropriate condition, which implies that the epidemic disease is endemic.

From Theorem 3.1, 4.1 and 4.2, we know that if $m > m^*$ or $T < T_*$ or $R_1 < 1$, then 'infection-free' periodic solution $(S^*(t), 0)$ is globally attractive, and

that if $m < m_*$ or $T > T^*$ or $\tau > \tau^*$ or $R_2 > 1$, then systems (4) is permanent. Therefore, we can manipulate the vaccination period (with T) and the number (with m)of newborns vaccinated successfully such that $m > m^*$ or $T < T_*$ or $R_1 < 1$ in order to prevent the epidemic disease from spreading.

Since time delay together with pulse appears in (4), the system becomes nonautonomous, it is very difficult to obtain R_1 with τ and find the threshold $\tilde{m} = m^* = m_*$ or $\tilde{R} = R_1 = R_2$. Therefore, we only give the 'relatively exact' sufficient and unnecessary conditions for global attractivity of 'infection-eradication' periodic solution (with $R_1 < 1$) and uniform permanence of the epidemic disease (with $R_2 > 1$). Then we mention some future directions of work extending the present paper as fellows: (i) One can find the threshold $\tilde{m} = m^* = m_*$ or $\tilde{R} = R_1 = R_2 = R_3$, that is sufficient and necessary conditions for global attractivity of 'infection-eradication' periodic solution (with $\tilde{R} < 1$) and permanence of the epidemic disease (with $\tilde{R} > 1$).(ii) One can study whether there is a stable positive periodic solution and time delay has effect on the stability of periodic solution; (iii) Since the existence of time delay delays the time that the susceptible transfer to the 'infectious' class, one can obtain a condition (with $\tilde{R} < 1$) for global attractivity of 'infection-eradication' periodic solution and the condition depend on τ and should be less than R_1 (with $\tilde{R} \leq R_1$).

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