

COLORING SIERPIŃSKI GRAPHS AND SIERPIŃSKI GASKET GRAPHS

Sandi Klavžar

Abstract. Sierpiński graphs $S(n, 3)$ are the graphs of the Tower of Hanoi puzzle with n disks, while Sierpiński gasket graphs S_n are the graphs naturally defined by the finite number of iterations that lead to the Sierpiński gasket. An explicit labeling of the vertices of S_n is introduced. It is proved that S_n is uniquely 3-colorable, that $S(n, 3)$ is uniquely 3-edge-colorable, and that $\chi'(S_n) = 4$, thus answering a question from [15]. It is also shown that S_n contains a 1-perfect code only for $n = 1$ or $n = 3$ and that every $S(n, 3)$ contains a unique Hamiltonian cycle.

1. INTRODUCTION

Topological studies of the Lipscomb's space [11, 12] led in [8] to the definition of Sierpiński graphs $S(n, k)$. Another motivation for the introduction of these graphs is the fact that the graph $S(n, 3)$ is isomorphic to the graph of the Tower of Hanoi puzzle with n disks [8], see also [5]. Sierpiński graphs were also independently studied in [14], where it is shown that they arise in a natural way from regular graphs.

The graphs $S(n, k)$ have many appealing properties and were studied from different points of view. They possess (essentially) unique 1-perfect codes [9], a result proved before for $S(n, 3)$ in [2]. Alternative arguments for the uniqueness of 1-perfect codes in $S(n, k)$ were recently presented in [3] in order to determine their optimal $L(2, 1)$ -labelings. Moreover, (regularizations of) Sierpiński graphs are the first nontrivial families of graphs of "fractal" type for which the crossing number is known [10], while in [13] several metric invariants of these graphs are determined.

Received January 11, 2006, accepted November 14, 2006.

Communicated by Xuding Zhu.

2000 *Mathematics Subject Classification*: 05C15, 05C69, 05C45.

Key words and phrases: Sierpiński graphs, Sierpiński gasket graphs, Chromatic number, Chromatic index, 1-perfect code, Hamiltonian cycle.

Supported by the Ministry of Science of Slovenia under the grant P1-0297. The author is also with the Institute of Mathematics, Physics and Mechanics, Jadranska 19, 1000 Ljubljana.

Hinz and Schief [7] used the connection between the graphs $S(n, 3)$ and the Sierpiński gasket to compute the average distance of the latter, see also [1]. The graphs that are obtained in a natural way by n iterations of the process that leads to the Sierpiński gasket are called Sierpiński gasket graphs and denoted S_n . Teguia and Godbole [15] studied several properties of these graphs, in particular the chromatic number, the domination number, and the pebbling number.

In this paper we continue studies of the Sierpiński graphs $S(n, 3)$ and the Sierpiński gasket graphs S_n . We first introduce an explicit labeling of the vertices of S_n that is obtained by “contracting” the Sierpiński labeling of $S(n, 3)$. In the central part of the paper—Section 3—vertex- and edge-colorings of $S(n, 3)$ and S_n are treated. It is in particular shown that S_n is uniquely 3-colorable (hence strengthening a result from [15]), that $S(n, 3)$ is uniquely 3-edge-colorable, and that the chromatic index of S_n is 4 (hence answering a question from [15]). We conclude the paper by observing that S_n contains a 1-perfect code only for $n = 1$ and $n = 3$ and that every $S(n, 3)$ contains a unique Hamiltonian cycle.

As usual, $\chi(G)$, $\chi'(G)$, and $\gamma(G)$ denote the chromatic number of G , the chromatic index of G , and the domination number of G , respectively. A 1-perfect code (also known as an efficient dominating set) in a graph G is a vertex subset of G such that the closed neighborhoods of its elements form a partition of $V(G)$. For any other graph theoretic concept not defined here we refer to [16].

2. SIERPIŃSKI GRAPHS AND SIERPIŃSKI GASKET GRAPHS

The *Sierpiński graphs* $S(n, 3)$, $n \geq 1$, are defined in the following way:

$$V(S(n, 3)) = \{1, 2, 3\}^n,$$

two different vertices $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ being adjacent if and only if there exists an $h \in \{1, \dots, n\}$ such that

- (i) $u_t = v_t$, for $t = 1, \dots, h - 1$;
- (ii) $u_h \neq v_h$; and
- (iii) $u_t = v_h$ and $v_t = u_h$ for $t = h + 1, \dots, n$.

We will shortly write $\langle u_1 \dots u_n \rangle$ for (u_1, \dots, u_n) . (On figures this convention is further shortened to $u_1 \dots u_n$.) The graph $S(4, 3)$ is shown in Fig. 1.

The vertices $\langle 1 \dots 1 \rangle$, $\langle 2 \dots 2 \rangle$, and $\langle 3 \dots 3 \rangle$ are called the *extreme vertices* of $S(n, 3)$. For $i = 1, 2, 3$ let $S(n + 1, 3)_i$ be the subgraph of $S(n + 1, 3)$ induced by the vertices of the form $\langle i \dots \rangle$. Clearly, $S(n + 1, 3)_i$ is isomorphic to $S(n, 3)$.

The *Sierpiński gasket graphs* S_n , $n \geq 1$, are defined geometrically as the graphs whose vertices are the intersection points of the line segments of the finite Sierpiński

gasket σ_n and line segments of the gasket as edges, see [15]. The Sierpiński gasket graph S_4 is shown in Fig. 2.

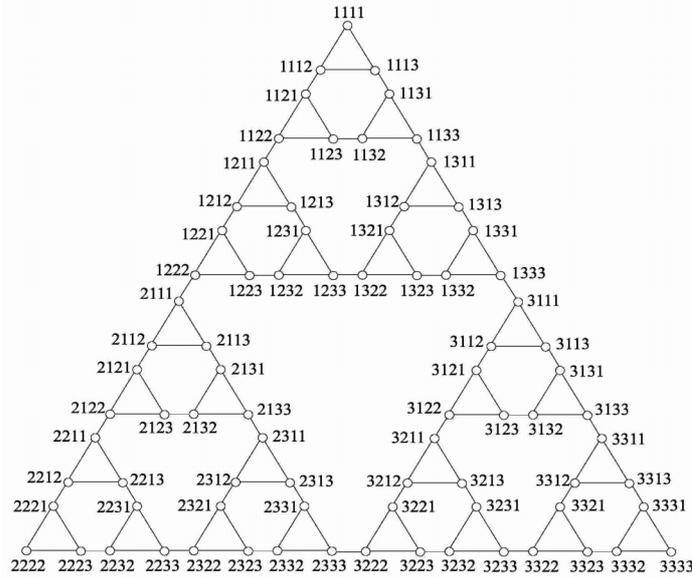


Fig. 1. $S(4, 3)$.

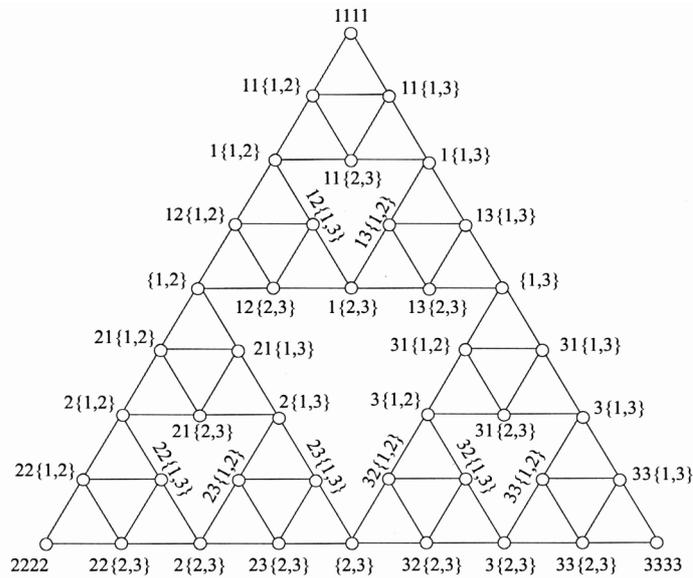


Fig. 2. S_4 with its quotient labeling.

We now give an alternative description of the graphs S_n that yields an explicit labeling of the vertices of S_n . For this sake observe that S_n can be obtained from $S(n, 3)$ by contracting all of its edges that lie in no triangle. Let $\langle u_1 \dots u_r i j \dots j \rangle$ and $\langle u_1 \dots u_r j i \dots i \rangle$ be endvertices of such an edge, then we will denote the corresponding vertex of S_n with $\langle u_1 \dots u_r \rangle \{i, j\}$. Note that $r \leq n - 2$ for such an edge. Then S_n is the graph with three special vertices $\langle 1 \dots 1 \rangle$, $\langle 2 \dots 2 \rangle$, and $\langle 3 \dots 3 \rangle$, called *extreme vertices of S_n* , together with the vertices of the form

$$\langle u_1 \dots u_r \rangle \{i, j\},$$

where $0 \leq r \leq n - 2$, and all the u_k 's, i and j are from $\{1, 2, 3\}$. Let us call this labeling the *quotient labeling* of S_n . (See Fig. 2 where the quotient labeling of S_4 is shown.) For a vertex $u = \langle u_1 \dots u_r \rangle \{i, j\}$ of S_n we will also say that $\langle u_1 \dots u_r \rangle$ is the *prefix* of u .

For $i = 1, 2, 3$ let $S_{n,i}$ be the subgraph of S_{n+1} induced by $\langle i \dots i \rangle$, $\{i, j\}$, $\{i, k\}$, where $\{i, j, k\} = \{1, 2, 3\}$, and all the vertices whose prefix starts with i . Note that $S_{n,i}$ is isomorphic to S_n .

To explicitly describe adjacencies with respect to the quotient labeling of S_n , $n \geq 2$, note first that an extreme vertex $\langle i \dots i \rangle$ of S_n is adjacent to vertices $\langle i \dots i \rangle \{i, j\}$ and $\langle i \dots i \rangle \{i, k\}$, where $\{i, j, k\} = \{1, 2, 3\}$ and the prefixes are of length $n - 2$. In particular, in S_2 , an extreme vertex $\langle ii \rangle$ is adjacent to $\{i, j\}$ and $\{i, k\}$. To describe neighbors of other vertices we need the following notations. For a vertex $u = \langle u_1 \dots u_n \rangle$ of $S(n, 3)$ and $s \leq n - 2$ let $u_{(s)} = \langle u_1 \dots u_s \rangle$. Let $u = \langle u_1 \dots u_r \rangle \{i, j\}$ be a vertex of S_n . Then let

$$\bar{u} = \langle u_1 \dots u_r i j \dots j \rangle \quad \text{and} \quad \bar{\bar{u}} = \langle u_1 \dots u_r j i \dots i \rangle$$

be the endvertices of the edge of $S(n, 3)$ contracted to u .

Proposition 2.1. *Let $n \geq 2$, let $u = \langle u_1 \dots u_r \rangle \{i, j\}$ be a vertex of S_n and let $\{i, j, k\} = \{1, 2, 3\}$.*

(i) *If $0 \leq r \leq n - 3$ then u is adjacent to*

$$\bar{u}_{(n-2)} \{i, j\}, \bar{u}_{(n-2)} \{j, k\}, \bar{\bar{u}}_{(n-2)} \{i, j\}, \text{ and } \bar{\bar{u}}_{(n-2)} \{i, k\}.$$

(ii) *If $r = n - 2$ then u is adjacent to $\bar{u}_{(n-2)} \{i, k\}$, $\bar{\bar{u}}_{(n-2)} \{j, k\}$, to*

$$\begin{cases} \bar{u}_{(t-1)} \{i, u_t\}, & t \text{ is the largest index with } u_t \neq i, 1 \leq t \leq n - 2; \\ \langle i \dots i \rangle, & \text{no such } t \text{ exists;} \end{cases}$$

and to

$$\begin{cases} \bar{\bar{u}}_{(s-1)} \{j, u_s\}, & s \text{ is the largest index with } u_s \neq j, 1 \leq s \leq n - 2; \\ \langle j \dots j \rangle, & \text{no such } s \text{ exists;} \end{cases}$$

Proof.

- (i) Since $r \leq n - 3$, $\bar{u} = \langle u_1 \dots u_r i j \dots j j \rangle$ and $\overline{\bar{u}} = \langle u_1 \dots u_r j i \dots i i \rangle$, where \bar{u} ends with at least two j 's and $\overline{\bar{u}}$ ends with at least two i 's. Hence \bar{u} is in $S(n, 3)$ adjacent to $\langle u_1 \dots u_r i j \dots j i \rangle$ and to $\langle u_1 \dots u_r i j \dots j k \rangle$. These two vertices contract to $\bar{u}_{(n-2)}\{i, j\}$ and $\bar{u}_{(n-2)}\{j, k\}$, respectively. Similarly, $\overline{\bar{u}}$ is adjacent to $\langle u_1 \dots u_r j i \dots i j \rangle$ and to $\langle u_1 \dots u_r j i \dots i k \rangle$ that contract to $\overline{\bar{u}}_{(n-2)}\{i, j\}$ and $\overline{\bar{u}}_{(n-2)}\{i, k\}$, the two remaining vertices of S_n adjacent to u . (Note that the argument also holds for $r = 0$).
- (ii) Let $r = n - 2$. Then $\bar{u} = \langle u_1 \dots u_{n-2} i j \rangle$ and $\overline{\bar{u}} = \langle u_1 \dots u_{n-2} j i \rangle$. The vertex \bar{u} is in $S(n, 3)$ adjacent to $\langle u_1 \dots u_{n-2} i k \rangle$ that contracts to $\bar{u}_{(n-2)}\{i, k\}$ and is also adjacent to $x = \langle u_1 \dots u_{n-2} i i \rangle$. If $u_1 = \dots = u_{n-2} = i$, then \bar{u} is adjacent to the extreme vertex $x = \langle i \dots i \rangle$ of $S(n, 3)$, therefore u is adjacent to the extreme vertex $\langle i \dots i \rangle$ of S_n . Suppose not all u_i 's are equal to i , and let t be the largest index such that $u_t \neq i$. Then $x = \langle u_1 \dots u_t i \dots i \rangle$, where $1 \leq t \leq n - 2$. In this case u is also adjacent to $\bar{u}_{(t-1)}\{i, u_t\}$.

The other two neighbors of u that arise from the neighbors of $\overline{\bar{u}}$ are obtained analogously as the neighbors induced by \bar{u} . The details are left to the reader. ■

We point out that in Proposition 2.1 the case $n = 2$ is treated in case (ii).

To conclude the section note that from the quotient labeling we can immediately infer that S_n contains $3 + \sum_{i=0}^{n-2} 3 \cdot 3^i = \frac{3}{2}(3^{n-1} + 1)$ vertices.

3. VERTEX- AND EDGE-COLORINGS

In this section we determine the chromatic number and the chromatic index of the Sierpiński graphs and the Sierpiński gasket graphs.

It is easy to see that for any $n \geq 1$, $\chi(S(n, 3)) = 3$. As observed by Parisse [13], a natural 3-coloring of $S(n, 3)$ can be obtained by setting

$$c(\langle u_1 \dots u_n \rangle) = u_n$$

for any vertex $\langle u_1 \dots u_n \rangle$ of $S(n, 3)$. Teguia and Godbole [15, Proposition 2] showed an analogous result for the graphs S_n , namely $\chi(S_n) = 3$, $n \geq 1$. We can strengthen the latter result as follows.

Theorem 3.1. *S_n is uniquely 3-colorable for any $n \geq 1$.*

Proof. We prove the theorem by induction on n and pose the following stronger induction assumption: S_n is uniquely 3-colorable and in every 3-coloring the extreme vertices receive different colors.

The claim is clearly true for $S_1 = K_3$. Suppose it holds for S_n , $n \geq 2$, and consider an arbitrary 3-coloring c of S_{n+1} . By the induction assumption, $S_{n+1,1}$ is uniquely 3-colorable and we may without loss of generality assume that $c(\langle 1 \dots 1 \rangle) = 1$, $c(\{1, 2\}) = 2$, and $c(\{1, 3\}) = 3$. Then, considering $S_{n+1,2}$, the induction assumption implies $c(\{2, 3\}) \neq 2$. Similarly because of $S_{n+1,3}$, we infer $c(\{2, 3\}) \neq 3$. So $c(\{2, 3\}) = 1$ and therefore $c(\langle 2 \dots 2 \rangle) = 3$ and $c(\langle 3 \dots 3 \rangle) = 2$. Induction completes the argument. ■

In the rest of this section we consider edge-colorings. To show that Sierpiński graphs are uniquely 3-edge-colorable we prove the following.

Theorem 3.2. *Let $n \geq 1$. The 3-colorings of S_n are in a 1-1 correspondence with the 3-edge-colorings of $S(n, 3)$.*

Proof. For $i, j \in \{1, 2, 3\}$, $i \neq j$, let $\overline{\{i, j\}} = \{1, 2, 3\} \setminus \{i, j\}$. In addition, for a vertex $u \in S(n, 3)$ let \tilde{u} be the vertex of S_n to which u is mapped while contracted $S(n, 3)$ to S_n .

Let c be a 3-coloring of S_n . Then for an arbitrary edge uv of $S(n, 3)$ set

$$c'(uv) = \begin{cases} \overline{\{c(\tilde{u}), c(\tilde{v})\}}, & \tilde{u} \neq \tilde{v}; \\ c(\tilde{u}), & \tilde{u} = \tilde{v}; \end{cases}$$

We claim that c' is an edge-coloring of $S(n, 3)$. Let uv and vw be adjacent edges of $S(n, 3)$. Suppose first that they belong to a common triangle. Then \tilde{u} , \tilde{v} , and \tilde{w} are pairwise different vertices of S_n which implies that $c'(uv) \neq c'(vw)$. The other case to consider is when, without loss of generality, vw belongs to a triangle while uv belongs to no triangle of $S(n, 3)$. Then $c'(uv) = c(\tilde{u}) = c(\tilde{v})$, while $c'(vw) = \overline{\{c(\tilde{v}), c(\tilde{w})\}} \neq c(\tilde{u}) = c'(uv)$.

Let now c' be a 3-edge-coloring of $S(n, 3)$. For a vertex $\langle u_1 \dots u_r \rangle \{i, j\}$ of S_n set

$$c(\langle u_1 \dots u_r \rangle \{i, j\}) = c'(\langle u_1 \dots u_r i j \dots j \rangle \langle u_1 \dots u_r j i \dots i \rangle).$$

Suppose $c(\tilde{u}) = c(\tilde{v})$ holds for adjacent vertices \tilde{u} and \tilde{v} of S_n . Since \tilde{u} and \tilde{v} are adjacent they lie in a common triangle $\tilde{T} = \tilde{u}\tilde{v}\tilde{w}$ that in turn corresponds to a triangle $T = uvw$ of $S(n, 3)$. Then $c(\tilde{u}) = c(\tilde{v})$ implies that the two edges of $S(n, 3)$ that are adjacent to u and v , and do not lie in T , receive the same c' -color. But then none of the edges of T can be colored with this color, a contradiction.

We have thus shown that c properly colors the subgraph of S_n induced by all but the three extreme vertices. Clearly, c can be uniquely extended to a 3-coloring of S_n . ■

Combining Theorems 3.1 and 3.2 we immediately get:

Corollary 3.3. $S(n, 3)$ is uniquely 3-edge-colorable for any $n \geq 1$.

We next determine the chromatic index of the Sierpiński gasket graphs, a question posed in [15].

Clearly, $\chi'(S_n) \geq 4$ for $n \geq 2$. Consider a 4-edge coloring of S_2 . The edges of its middle triangle receive different colors and their color classes contain at most two edges each. Since any color class contains at most 3 edges and S_2 has 9 edges, we infer that three color classes contain two edges while the remaining color class has 3 edges. But then the later edges alternate on the outer 6-cycle and we conclude that a 4-edge-coloring of S_2 is unique modulo permutations of the colors and the shift of the color class with three elements on the outer cycle. In particular, a 4-edge-coloring of S_2 is uniquely defined with the colors of its outer 6-cycle, hence in our figures we will color only such edges.

Note that $\chi'(S_1) = 3$, while for the other Sierpiński gasket graphs we have the following result.

Theorem 3.4. For any $n \geq 2$, $\chi'(S_n) = 4$.

Proof. It suffices to construct an edge-coloring with four colors for any $n \geq 2$.

Let c be an edge-coloring of S_n , then let C_1, C_2 , and C_3 be the sets of colors assigned by c to the two edges incident with $\langle 1 \dots 1 \rangle, \langle 2 \dots 2 \rangle$, and $\langle 3 \dots 3 \rangle$, respectively. To prove the theorem we pose the following stronger claim.

Claim. If n is even, then there exists a 4-edge-coloring c of S_n such that $C_1 = \{1, 2\}, C_2 = \{1, 3\}$, and $C_3 = \{1, 4\}$. If n is odd, then there exists a 4-edge-coloring c of S_n such that $C_1 = \{1, 2\}, C_2 = \{1, 3\}$, and $C_3 = \{2, 3\}$.

The claim is true for $n = 2, 3$ as demonstrated in Fig. 3.

Let n be even. Then color S_{n+1} as follows. Let c' be a coloring of $S_{n+1,1}$ such that $C'_1 = \{1, 2\}, C'_2 = \{1, 3\}$, and $C'_3 = \{1, 4\}$. Let c'' be a coloring of $S_{n+1,2}$ such

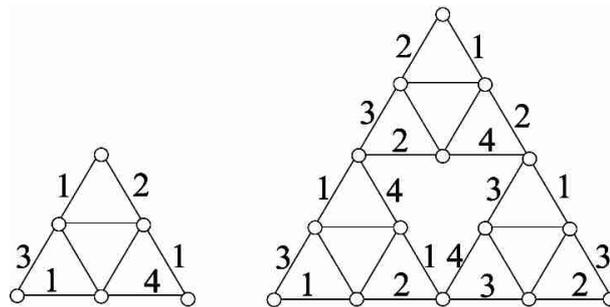


Fig. 3. 4-edge-colorings of S_2 and S_3 .

that $C_1'' = \{4, 2\}$, $C_2'' = \{4, 1\}$, and $C_3'' = \{4, 3\}$. Finally, let c''' be a coloring of $S_{n+1,3}$ such that $C_1''' = \{2, 3\}$, $C_2''' = \{2, 1\}$, and $C_3''' = \{2, 4\}$. See the left-hand side of Fig. 4. All these colorings exist by the induction assumption and by appropriate permutations of colors. Combine c' , c'' , and c''' to obtain a 4-edge-coloring of S_{n+1} . Finally exchange colors 3 and 4 to obtain a desired coloring.

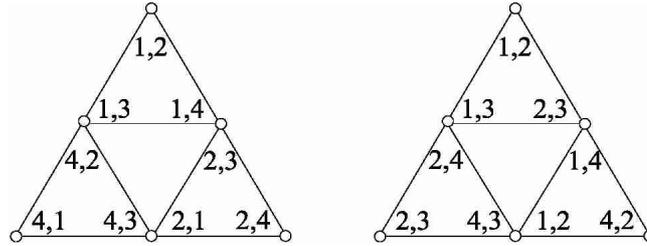


Fig. 4. Even to odd case, and odd to even case.

For n odd we proceed similarly. Let c' be a coloring of $S_{n+1,1}$ such that $C_1' = \{1, 2\}$, $C_2' = \{1, 3\}$, and $C_4' = \{2, 3\}$, let c'' be a coloring of $S_{n+1,2}$ such that $C_1'' = \{2, 4\}$, $C_2'' = \{2, 3\}$, and $C_3'' = \{4, 3\}$, and c''' a coloring of $S_{n+1,3}$ such that $C_1''' = \{1, 4\}$, $C_2''' = \{1, 2\}$, and $C_3''' = \{4, 2\}$, see the right-hand side of Fig. 4. Again, these colorings exist by the induction assumption and by appropriate permutations of colors. Combine c' , c'' , and c''' to color S_{n+1} and exchange colors 1 and 2 to obtain a desired coloring of S_{n+1} . ■

4. ON CODES AND HAMILTONICITY

In the final section we present two additional aspects of Sierpiński (gasket) graphs.

It is proved in [15] that for every $n \geq 4$, $\gamma(S_n) = 3\gamma(S_{n-1})$. This enables us to quickly prove the following result that is in a strike contrast to the fact already mentioned in the introduction that every Sierpiński graph $S(n, k)$ contains essentially a unique 1-perfect code. Recall that if C is a 1-perfect code of a graph G , then $|C| = \gamma(G)$, see [4, Theorem 4.2].

Proposition 4.1. S_n contains a 1-perfect code if and only if $n = 1$ or $n = 3$.

Proof. It is straightforward to verify the result for $n \leq 3$.

Let $n \geq 4$ and suppose that C is a 1-perfect code of S_n . Assume $\langle 1 \dots 1 \rangle \in C$ and consider the vertex $\langle 1 \dots 1 \rangle \{2, 3\}$ with the prefix of length $n - 2$. Then it can be dominated either with itself, with $\langle 1 \dots 1 \rangle \{1, 2\}$, or with $\langle 1 \dots 1 \rangle \{1, 3\}$, where

the last two vertices have prefixes of length $n - 3$. However, none of these vertices qualifies for C , so $\langle 1 \dots 1 \rangle \notin C$. Analogously $\langle 2 \dots 2 \rangle \notin C$ and $\langle 3 \dots 3 \rangle \notin C$. Hence every vertex of C is of degree 4 and as S_n contains $3(3^{n-1} + 1)/2$ vertices, we infer that $|C| = 3(3^{n-1} + 1)/10$. Since $|C| = \gamma(S_n)$, the above result of Teguia and Godbole implies that $\gamma(S_n) = |C| = 3^{n-2}$ for $n \geq 3$. But then $3(3^{n-1} + 1)/10 = 3^{n-2}$, which reduces to $3^{n-2} = 3$ with $n = 3$ as the solution. ■

We conclude the paper by proving that the Sierpiński graphs $S(n, 3)$ contain unique Hamiltonian cycles. For this sake we first show:

Lemma 4.2. *Let $n \geq 1$ and let u, v be extreme vertices of $S(n, 3)$. Then there exists a unique Hamiltonian u, v -path.*

Proof. The statement is clearly true for $n = 1$. Suppose it holds for $n \geq 2$ and without loss of generality consider the extreme vertices $\langle 1 \dots 1 \rangle$ and $\langle 2 \dots 2 \rangle$ of $S(n + 1, 3)$. By the induction assumption, there exists a unique Hamiltonian path P between $\langle 1 \dots 1 \rangle$ and $\langle 13 \dots 3 \rangle$ in $S(n + 1, 3)_1$, a unique Hamiltonian path Q between $\langle 31 \dots 1 \rangle$ and $\langle 32 \dots 2 \rangle$ in $S(n + 1, 3)_3$, and a unique Hamiltonian path S between $\langle 23 \dots 3 \rangle$ and $\langle 2 \dots 2 \rangle$ in $S(n + 1, 3)_2$. Then

$$\langle 1 \dots 1 \rangle P \langle 13 \dots 3 \rangle \langle 31 \dots 1 \rangle Q \langle 32 \dots 2 \rangle \langle 23 \dots 3 \rangle S \langle 2 \dots 2 \rangle$$

is a Hamiltonian path in $S(n + 1, 3)$. To see that it is unique, observe that $\langle 12 \dots 2 \rangle$ must appear before $\langle 13 \dots 3 \rangle$ on any Hamiltonian $\langle 1 \dots 1 \rangle, \langle 2 \dots 2 \rangle$ -path. Indeed, suppose this is not the case. Then if we proceed from $\langle 13 \dots 3 \rangle$ to $\langle 31 \dots 1 \rangle$, the vertex $\langle 12 \dots 2 \rangle$ would appear on the Hamiltonian path just after $\langle 21 \dots 1 \rangle$ which is clearly not possible. And if we proceed from $\langle 13 \dots 3 \rangle$ to a vertex of $S(n + 1, 3)_1$, then the vertex $\langle 32 \dots 2 \rangle$ would appear on the Hamiltonian path just after $\langle 23 \dots 3 \rangle$, which is also not possible. Similarly, $\langle 3 \dots 3 \rangle$ must appear before $\langle 32 \dots 2 \rangle$ on any Hamiltonian $\langle 1 \dots 1 \rangle, \langle 2 \dots 2 \rangle$ -path. Induction completes the argument. ■

Theorem 4.3. *$S(n, 3)$, $n \geq 1$, contains a unique Hamiltonian cycle.*

Proof. The case $n = 1$ is trivial. For $n > 1$ construct a Hamiltonian cycle of $S(n, 3)$ by combining a Hamiltonian $\langle 12 \dots 2 \rangle, \langle 13 \dots 3 \rangle$ -path in $S(n, 3)_1$, a Hamiltonian $\langle 31 \dots 1 \rangle, \langle 32 \dots 2 \rangle$ -path in $S(n, 3)_3$, and a Hamiltonian $\langle 23 \dots 3 \rangle, \langle 21 \dots 1 \rangle$ -path in $S(n, 3)_2$. By Lemma 4.2, this Hamiltonian cycle is unique. ■

ACKNOWLEDGMENT

I thank Marko Jakovac for a careful reading of the manuscript.

REFERENCES

1. T.-H. Chan, A statistical analysis of the Towers of Hanoi problem, *Int. J. Comput. Math.*, **28** (1989), 57-65.
2. P. Cull and I. Nelson, Error-correcting codes on the Towers of Hanoi graphs, *Discrete Math.*, **208/209** (1999), 157-175.
3. S. Gravier, S. Klavžar and M. Mollard, Codes and $L(2, 1)$ -labelings in Sierpiński graphs, *Taiwan. J. Math.*, **9** (2005), 671-681.
4. T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.
5. A. M. Hinz, Pascal's triangle and the Tower of Hanoi, *Amer. Math. Monthly*, **99** (1992), 538-544.
6. A. M. Hinz and D. Parisse, On the planarity of Hanoi graphs, *Expo. Math.*, **20** (2002), 263-268.
7. A. M. Hinz and A. Schief, The average distance on the Sierpiński gasket, *Probab. Theory Related Fields*, **87** (1990), 129-138.
8. S. Klavžar and U. Milutinović, Graphs $S(n, k)$ and a variant of the Tower of Hanoi problem, *Czechoslovak Math. J.*, **47(122)** (1997), 95-104.
9. S. Klavžar, U. Milutinović and C. Petr, 1-perfect codes in Sierpiński graphs, *Bull. Austral. Math. Soc.*, **66** (2002), 369-384.
10. S. Klavžar and B. Mohar, Crossing numbers of Sierpiński-like graphs, *J. Graph Theory*, **50** (2005), 186-198.
11. S. L. Lipscomb and J. C. Perry, Lipscomb's $L(A)$ space fractalized in Hilbert's $\ell^2(A)$ space, *Proc. Amer. Math. Soc.*, **115** (1992), 1157-1165.
12. U. Milutinović, Completeness of the Lipscomb space, *Glas. Mat. Ser. III*, **27(47)** (1992), 343-364.
13. D. Parisse, On some metric properties of the Sierpiński graphs $S(n, k)$, *Ars Combin.*, in press.
14. T. Pisanski and T. W. Tucker, Growth in repeated truncations of maps, *Atti Sem. Mat. Fis. Univ. Modena*, **49** (2001), 167-176.
15. A. M. Teguia and A. P. Godbole, Sierpiński gasket graphs and some of their properties, *Australas. J. Combin.*, **35** (2006), 181-192.
16. D. B. West, *Introduction to Graph Theory*, Second Edition, Prentice Hall, Inc., Upper Saddle River, NJ, 2001.

Sandi Klavžar

Department of Mathematics and Computer Science,

FNM, University of Maribor,

Koroška cesta 160,

2000 Maribor,

Slovenia

E-mail: sandi.klavzar@uni-mb.si