TAIWANESE JOURNAL OF MATHEMATICS
Vol. 8, No. 1, pp. 71-83, March 2004
This paper is available online at http://www.math.nthu.edu.tw/tjm/

THE EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF INDEFINITE WEIGHT SEMILINEAR ELLIPTIC PROBLEMS WITH CRITICAL SOBOLEV EXPONENT

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Abstract. We prove the existence of classical positive solutions for a class of indefinite weight semilinear elliptic partial defferential equations on the homogeneous Dirichlet boundary conditions and with that the growth of the perturbation is critical Soboler exponent.

1. Introduction

In this paper we discuss the existence of positive solutions of the following boundary value problems:

$$(I_\lambda)\begin{cases} -\Delta u = \lambda g(x) f(u) \quad \text{in} \quad , \\ u = 0 \quad \text{on} \quad \partial \quad , \end{cases}$$

where λ is a real parameter, is an open bounded domain in $\mathbb{R}^N$, $N \geq 3$, with the smooth boundary ∂ .

We shall consider the critical exponent case $f(u) = u(1 + |u|^p)$ with $p = 4/(N-2)$. The function $g : \overline{} \longrightarrow \mathbb{R}^1$ is smooth and changes sign.

We proved the existence of positive solutions of the following problems:

$$\begin{cases} -\Delta u = \lambda g(x) f(u) \quad \text{in} \quad , \\ (1-\alpha)\dfrac{\partial u}{\partial n} + \alpha u = 0 \quad \text{on} \quad \partial \quad , \end{cases}$$

Received April 15, 2003.
Communicated by S. B. Hsu.
2000 *Mathematics Subject Classification*: 35J20, 35J25, 35J65.
Key words and phrases: Indefinite weight semilinearproblems, positive solutions, critical Soboler exponent, Nehari manifold, implicit function theorem, Ekeland's variational principle.

in the case $0 < p < \frac{4}{N-2}$ (see [1]). Here $\alpha \in (0,1)$ or $\int g(x)dx \neq 0$ and $\alpha \in (\alpha_0, 0]$ for some constant $\alpha_0 < 0$. We used the constrained minimization method of the functional

$$E_\lambda(u) = \int |\nabla u|^2 - \lambda \int gu^2 + \frac{\alpha}{(1-\alpha)} \int_{\partial} u^2 dS_x$$

on the constrained set

$$\{u \in W^{1,2}(\quad) : \lambda \int g|u|^{p+2} = 1\}$$

to prove the existence. If $p = \frac{4}{N-2}$, the above set may not be weakly closed, and so we should find a different method to get a positive solution.

In Section 2, we show that a minimizing sequence of the functional which is induced by the weighted problem (I_λ) with $f(u) = u(1+|u|^p)$:

$$J_\lambda(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{\lambda}{2}\int gu^2 - \frac{\lambda}{p+2}\int g|u|^{p+2}$$

on the Nehari manifold:

$$M_\lambda = \left\{u \in W_0^{1,2}(\quad) : u \neq 0, < J'_\lambda(u), u >= 0\right\},$$

where

$$< J'_\lambda(u), u >= \int |\nabla u|^2 - \lambda \int gu^2(1+|u|^p),$$

converges to a positive function in $W_0^{1,2}(\quad)$ which is a classical positive solution of the problem (I_λ) if $\lambda^- < \lambda < \lambda^+$, and λ is near to either λ^- or λ^+, where λ^- and λ^+ are the principal eigenvalues of the following problem (See [3]):

$$(L)\begin{cases} -\Delta u = \lambda g(x)u \text{ in } \quad, \\ u = 0 \text{ on } \partial \quad. \end{cases}$$

Furthermore, we estimate the length of the intervals about λ in which the existence is guaranteed.

In the end of Section 2, we can show that, if $g(x) = 0$ on some open subset of , then (I_λ) has a positive solution for all $\lambda \in (\lambda^-, \lambda^+)$, except $\lambda \neq 0$. However, we note that if is a ball, $g = 1$, and $N = 3$, then (I_λ) has a positive solution if and only if $\frac{1}{4}\lambda_1 < \lambda < \lambda_1$, where λ_1 is the principal eigenvalue of $-\Delta$ (See [5]). As the application of the result, we can prove the existence of a positive solution of the following problem:

$$\begin{cases} -\Delta u = g(x)u^{\frac{N+2}{N-2}} \text{ in } \quad, \\ u = 0 \text{ on } \partial \quad, \end{cases}$$

if g satisfies the above same special condition. On the other hand, we note that, if $g = 1$ in $\bar{}$, we have had the nonexistence result of any positive solution (See [2]).

2. The Maian Results

We first recall some facts about how the method of eigencurves can be used to prove the convergence of a minimizing sequence of J_λ on some subset of the Nehari manifold. We define $\mu(\lambda)$ by

$$\mu(\lambda) = \inf\left\{\int (|\nabla u|^2 - \lambda g u^2) : u \in W_0^{1,2}(\ ), \int u^2 = 1\right\}$$

It can be shown that $\mu(0) > 0$ and the function $\lambda \longrightarrow \mu(\lambda)$ is a concave function such that $\mu(\lambda) \to -\infty$ as $\lambda \to \pm\infty$. So it follows that $\lambda \to \mu(\lambda)$ has exactly one negative zero λ^- and one positive zero λ^+, and those are principal eigenvalues for (L). Furthermore, the eigencurves $\lambda \to \mu(\lambda)$ can be used to produce an equivalent norm for $W_0^{1,2}(\ )$. In fact, it can be shown that, if $\lambda \in (\lambda^-, \lambda^+)$,

$$||u||_\lambda = \left\{\int \left[|\nabla u|^2 - \lambda g u^2\right]\right\}^{\frac{1}{2}}$$

defines a norm in $W_0^{1,2}(\ )$ which is equivalent to the usual norm for $W_0^{1,2}(\ )$ (See [1]).

Lemma 2.1 *Let $\lambda \in (\lambda^-, \lambda^+)$, $\lambda \neq 0$ and let*

$$M_\lambda = \left\{u \in W_0^{1,2}(\ ) : \ u \neq 0, < J'_\lambda(u), u >= 0\right\},$$

Then M_λ is a nonempty subset of $W_0^{1,2}(\ )$.

Proof. Since g changes sign, we can choose a nonzero function $u_0 \in W_0^{1,2}(\ )$ so that

$$\int g|u_0|^{p+2} > 0.$$

Let

$$t^p = \frac{\int |\nabla u_0|^2 - \lambda \int g u_0^2}{\lambda \int g|u_0|^{p+2}}.$$

Then $u = tu_0 \in M_\lambda$.

Definitions: We define the following functions: for $\lambda > 0$,

$$K_\lambda = \inf\left\{\int [|\nabla u|^2 - \lambda g u^2] : u \in W_0^{1,2}(\quad), \int g|u|^{p+2} = 1\right\}$$

and

$$K_0 = \inf\left\{\int |\nabla u|^2 : u \in W_0^{1,2}(\quad), \int g|u|^{p+2} = 1\right\}.$$

Lemma 2.2 $K_0 > 0$.

Proof. We show that $K_0 > 0$. If not, there is a sequence $u_n \in W_0^{1,2}(\quad)$ so that

$$\lim_{n\to\infty} \int |\nabla u_n|^2 = 0$$

and

$$\int g|u_n|^{p+2} = 1.$$

By the Sobolev embedding : $W_0^{1,2}(\quad) \hookrightarrow L^{\frac{2N}{N-2}}(\quad)$, it is impossible.

Remark: We note that K_λ is a concave continuous curve on the interval $[0, \lambda^+]$. Hence, $K_\lambda \cdot\; K_0$ for all $\lambda \geq 0$. Furthermore, by the Sobolev embedding, the equivalent norm, and the relations between the principal eigenvalues and the function g (See [1]), the following properties hold: (i) $K_{\lambda^+} = 0$, (ii) $K_\lambda > 0$ if $0 \cdot\; \lambda < \lambda^+$.

Definitions and Remarks: Let $\lambda \in (0, \lambda^+)$. We define the following sets:

$$H_\lambda = \left\{u \in W_0^{1,2}(\quad) : \lambda \int g|u|^{p+2} = 1\right\}.$$

Let $u \in H_\lambda$. Then $||u||_\lambda^{\frac{2}{p}} u \in M_\lambda$. If $u \in M_\lambda$, then $||u||_\lambda^{-\frac{2}{p+2}} u \in H_\lambda$. We define the functional $E_\lambda : H_\lambda \longrightarrow \mathbb{R}^1$ by

$$E_\lambda(u) = \int |\nabla u|^2 - \lambda \int g u^2.$$

Then we obtain

$$E_\lambda(u) = \square\frac{2(p+2)}{p} J_\lambda\left(||u||_\lambda^{\frac{2}{p}} u\right)\Bigg]^{\frac{p}{p+2}}$$

and

$$J_\lambda(u) = \frac{p}{2(p+2)} E_\lambda\left(||u||_\lambda^{-\frac{2}{p+2}} u\right)^{\frac{p+2}{p}}.$$

If we let

$$Q_\lambda = \inf E_\lambda(H_\lambda) \text{ and } C_\lambda = \inf J_\lambda(M_\lambda),$$

then by the simple calculation it follows that

$$Q_\lambda = \left[\frac{2(p+2)}{p} C_\lambda\right]^{\frac{p}{p+2}}.$$

This implies that if $\{u_n\}$ is a minimizing sequence of E_λ on H_λ, then $\{||u_n||_\lambda^{\frac{2}{p}} u_n\}$ is also a minimizing sequence of J_λ on M_λ and vice versa.

Remark: We can prove that $u = 0$ is not a limit point of M_λ if $0 < \lambda < \lambda^+$. To show that, we assume there is a sequence $\{u_n\}$ in M_λ so that $\|u_n\|_\lambda \to 0$ as $n \to \infty$. From the Sobolev embedding: $W_0^{1,2}(\ ) \hookrightarrow L^{\frac{2N}{N-2}}(\ )$, the sequence $\{w_n\}$ which is defined by $w_n = \frac{u_n}{\|u_n\|_\lambda}$ is a bounded sequence in $L^{\frac{2N}{N-2}}(\ )$. We hence have the following result:

$$\begin{aligned} 0 &= \frac{< J'_\lambda(u_n), u_n >}{||u_n||_\lambda^2} \\ &= \frac{\int |\nabla u_n|^2 - \lambda \int g u_n^2}{||u_n||_\lambda^2} + (||u_n||_\lambda)^{p+2} \int g|w_n|^{p+2} \to 1 \text{ as } n \to \infty, \end{aligned}$$

which leads to a contradiction. This implies that $Q_\lambda > 0$, and so $K_\lambda > 0$. In fact, we can show that if $u_n \in K_\lambda$, then $v_n = \lambda^{-\frac{1}{p+2}} u_n \in H_\lambda$. Hence, $K_\lambda > 0$.

Lemma 2.3 *There are two positive numbers δ_1 and δ_2 such that for any $\lambda \in (\lambda^-, \lambda^- + \delta_1) \cup (\lambda^+ - \delta_2, \lambda^+)$, if $\{u_n\}$ be a minimizing sequence of J_λ on M_λ. Then*

$$\liminf_{n\to\infty} \left| \int g u_n^2 \right| > 0.$$

Proof. Let φ^- and φ^+ be the corresponding eigenfunctions to the principal eigenvalues λ^- and λ^+, respectively. We can assume that

$$\int g|\varphi^-|^{p+2} = -1, \quad \int g|\varphi^+|^{p+2} = 1.$$

(See Lemma 3.1 in [1]). We also note that

$$\int g(\varphi^-)^2 < 0, \quad \int g(\varphi^+)^2 > 0.$$

Let

$$\delta_2 = \lambda^+ - \frac{\int |\nabla\varphi^+|^2 - K_0}{\int g|\varphi^+|^2} = \frac{K_0}{\int g|\varphi^+|^2}.$$

Then for $\lambda \in (\lambda^+ - \delta_2, \lambda^+)$ and if $\{u_n\}$ is a minimizing sequence of J_λ on M_λ, it is bounded in $W_0^{1,2}(\ \ )$, and then $u_n \to u$ weakly in $W_0^{1,2}(\ \ )$ and $u_n \to u$ strongly in $L^2(\ \ )$. By the previous equality about minimums we know that $\{\|u_n\|_\lambda^{-\frac{2}{p+2}} u_n\}$ is a minimizing sequence of E_λ on H_λ, and so there is a positive number q such that

$$\lim_{n\to\infty} \|u_n\|_\lambda^{-\frac{4}{p+2}} \Big[\int |\nabla u_n|^2 - \lambda \int g(u_n)^2 \Big] < q < K_0.$$

Since $\|u_n\|_\lambda \nrightarrow 0$ as $n \to \infty$, if $\int g(u_n)^2 \to 0$ as $n \to \infty$, we get

$$K_0 \cdot\ q < K_0,$$

which leads to a contradiction. Therefore,

$$\lim_{n\to\infty} \int g u_n^2 \neq 0.$$

Let

$$\delta_1 = \frac{\int |\nabla\varphi^-|^2 - K_0'}{\int g|\varphi^-|^2} - \lambda^- = -\frac{K_0'}{\int g|\phi^-|^2},$$

where

$$K_0' = \inf\Big\{ \int |\nabla u|^2 \ :\ u \in W_0^{1,2}(\ \ ),\ \int g|u|^{p+2} = -1 \Big\}.$$

For the value $\lambda \in (\lambda^-, \lambda^- + \delta_1)$, we can get the same results by the above methods.

This completes the proof.

We denote by $B_\varepsilon(X)$ the ball in a Hilbert space X centered at 0 and of radius ε. We state the following:

Proposition (See [4]) *Let J be a C^1-functional on a Hilbert space X and let M be a closed subset of X verifying the following property:*

For any $u \in M$ with $J'(u) \neq 0$, there exists, for a small enough $\varepsilon > 0$, a Fréchet differentiable function $s_u : B_\varepsilon(X) \longrightarrow \mathbb{R}^1$ such that, by setting $t_u(\delta) = s_u\left(\delta \frac{J'(u)}{||J'(u)||}\right)$ for $0 \cdot\ \delta \cdot\ \varepsilon$, we have

$$t_u(0) = 1 \text{ and } t_u(\delta)\left(u - \delta \frac{J'(u)}{||J'(u)||}\right) \in M.$$

If J is bounded below on M, then for any minimizing sequence $\{v_n\}$ in M for J, there exists another minimizing sequence $\{u_n\}$ in M of J such that

$$J(u_n) \cdot \; J(v_n), \; \lim_{n\to\infty} ||u_n - v_n|| = 0$$

and

$$||J'(u_n)|| \cdot \; \frac{1}{n}\left(1 + ||u_n|||t'_{u_n}(0)|\right) + |t'_{u_n}(0)|| < J'(u_n), u_n > |,$$

where $<\,,\,>$ is the inner product in X.

Proof. Let $C = \inf J(M)$. Use Ekeland's variational principle (see [4]) to get a minimizing sequence $\{u_n\}$ in M with the following properties:

(i) $J(u_n) \cdot \; J(v_n) < C + \frac{1}{n}$,

(ii) $\lim_{n\to\infty} ||u_n - v_n|| = 0$,

(iii) $J(w) \geq J(u_n) - \frac{1}{n}||w - u_n||$ for all $w \in M$.

Let us assume $||J'(u_n)|| > 0$ for n large, since otherwise we are done. Apply the hypothesis on the set M with $u = u_n$ to find $t_n(\delta) = s_{u_n}\left(\delta \frac{J'(u_n)}{||J'(u_n)||}\right)$ such that $w_\delta = t_n(\delta)\left(u_n - \delta \frac{J'(u_n)}{||J'(u_n)||}\right) \in M$ for all small enough $\delta \geq 0$.

Use now the mean value theorem to get

$$\begin{aligned} \frac{1}{n}||w_\delta - u_n|| \geq\; & J(u_n) - J(w_\delta) \\ =\; & (1 - t_n(\delta)) < J'(w_\delta), u_n > + \delta t_n(\delta) < J'(w_\delta), \frac{J'(u_n)}{||J'(u_n)||} > + o(\delta) \end{aligned}$$

where $\frac{o(\delta)}{\delta} \to 0$ as $\delta \to 0$. Dividing by $\delta > 0$ and passing to the limit as $\delta \to 0$ we derive

$$\frac{1}{n}\left(1 + |t'_n(0)|||u_n||\right) \geq -t'_n(0) < J'(u_n), u_n > + ||J'(u_n)||,$$

which is our claim.

Lemma 2.5 *Given $\lambda \in (\lambda^-, \lambda^+), \lambda \neq 0$, J_λ is bounded below on M_λ and there exists a minimizing sequence $\{u_n\}$ of J_λ on M_λ so that*

$$\lim_{n\to\infty} ||J'_\lambda(u_n)||_\lambda = 0$$

and

$$\lim_{n\to\infty} J_\lambda(u_n) = \inf J_\lambda(M_\lambda)$$

Proof. Let $\lambda > 0$. We show that J_λ is bounded below on M_λ. In fact, the following can be checked easily: if $u \in M_\lambda$, then

$$\lambda \int g|u|^{p+2} > 0$$

and

$$J_\lambda(u) = \frac{p\lambda}{2(p+2)} \int g|u|^{p+2}.$$

Let $u \in M_\lambda$. Define $G : \mathbb{R}^1 \times W_0^{1,2}(\ \ ) \longrightarrow \mathbb{R}^1$ by $G(s, w) = \Phi_\lambda(s(u - w))$, where $\Phi_\lambda : W_0^{1,2}(\ \ ) \longrightarrow \mathbb{R}^1$ is a functional defined by

$$\Phi_\lambda(u) = \int |\nabla u|^2 - \lambda \int gu^2 - \lambda \int g|u|^{p+2}.$$

Since $G(1, 0) = 0$ and

$$\begin{aligned} \frac{d}{ds}G(1,0) &= 2\int |\nabla u|^2 - 2\lambda \int gu^2 - \lambda(p+2) \int g|u|^{p+2} \\ &= -p\left(\int \left[|\nabla u|^2 - \lambda g u^2\right]\right) \neq 0. \end{aligned}$$

Hence, we can apply the Implicit Function Theorem at $(1, 0)$ and get that for $\delta > 0$ small enough, there exists a differentiable function $s_u : B_\delta(W_0^{1,2}(\ \ )) \longrightarrow \mathbb{R}^1$ such that $s_u(0) = 1, s_u(w)(u - w) \in M_\lambda$, and

$$< s_u'(0), w >= \frac{< \Phi_\lambda'(u), w >}{< \Phi_\lambda'(u), u >}$$

for all $w \in B_\delta(W_0^{1,2}(\ \ ))$. From the identification of duality to the Hilbert space $W_0^{1,2}(\ \ )$, we let

$$w_u = \frac{J_\lambda'(u)}{||J_\lambda'(u)||_\lambda} \text{ and } t_u(\rho) = s_u(\rho w_u)$$

for all $0 \cdot \ \rho \cdot \ \delta$. Then $t_u(0) = 1$ and

$$t_u(\rho)(u - \rho w_u) = s_u(\rho w_u)(u - \rho w_u) \in M_\lambda.$$

From Proposition 2.4, there is a minimizing sequence $\{u_n\}$ of J_λ on M_λ so that

$$J_\lambda(u_n) \cdot \ J_\lambda(v_n) < \inf J_\lambda(M_\lambda) + \frac{1}{n}, \ \lim_{n\to\infty} ||u_n - v_n||_\lambda = 0,$$

and

$$||J_\lambda'(u_n)||_\lambda \cdot \ \frac{1}{n}\left(1+|t_{u_n}'(0)|||u_n||_\lambda\right)+|t_{u_n}'(0)||<J_\lambda'(u_n),u_n>|.$$

Since $J_\lambda(u_n)=\frac{\lambda p}{2(p+2)}||u_n||_\lambda^2$, so the sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\ )$. Let $||u_n||_\lambda \cdot \ C_1$ for all n. Then

$$||J_\lambda'(u_n)||_\lambda \cdot \ \frac{1}{n}\left(1+|t_{u_n}'(0)|C_1\right).$$

Since

$$|t_{u_n}'(0)|=\frac{|<\Phi_\lambda'(u_n),w_n>|}{p||u_n||_\lambda^2},$$

where $w_n=w_{u_n}$, and $\lim_{n\to\infty}\inf||u_n||_\lambda>0$, if we show that $|t_{u_n}'(0)|$ is uniformly bounded on n, we are done. In fact,

$$\left|<\Phi_\lambda'(u_n),w_n>\right| \cdot \ 2\int|\nabla u_n\cdot\nabla w_n|+2\lambda\int|u_nw_n|+\lambda(p+2)\int|g||u_n|^{p+1}|w_n|,$$

the well-known Sobolev embedding theorem, $||w_n||_\lambda=1$ for all n, and Hölder inequality, we have two positive constants C_2 and C_3 so that

$$\left|<\Phi_\lambda'(u_n),w_n>\right| \cdot \ C_2||u_n||_\lambda+C_3.$$

Since $\{u_n\}$ is a bounded sequence in $W_0^{1,2}(\ )$, so is $<\Phi_\lambda'(u_n),w_n>$ on n. Therefore, we can conclude that

$$\lim_{n\to\infty}||J_\lambda'(u_n)||_\lambda=0.$$

Clearly, we note that

$$\lim_{n\to\infty}J_\lambda(u_n)=\inf J_\lambda(M_\lambda).$$

For $\lambda<0$, we can get the same result by the above methods.

Theorem 2.6 *For any* $\lambda\in(\lambda^-,\lambda^-+\delta_1)\cup(\lambda^+-\delta_2,\lambda^+)$, $\lambda\neq0$, *the problem* (I_λ) *has a positive solution.*

Proof. Let

$$c=\inf J_\lambda(M_\lambda)$$

and let $\{u_n\}$ be a sequence in M_λ such that

$$\lim_{n\to\infty}J_\lambda(u_n)=c.$$

By Lemma 2.5, we can assume that

$$\lim_{n\to\infty} ||J'_\lambda(u_n)||_\lambda = 0.$$

Then $\{u_n\}$ is bounded and we can find a weak limit point u of the sequence in $W_0^{1,2}(\ )$. We can also assume that $\{u_n\}$ converges weakly to u and, by the Rellich-Kondrakov Theorem(see [4]), that $u_n \to u$ strongly in $L^q(\ )$ for all $q < \frac{2N}{N-2}$. In particular, for any $v \in C_0^\infty(\ )$,

$$< J'_\lambda(u_n), v >= \int \nabla u_n \cdot \nabla v - \lambda \int g u_n v - \lambda \int g u_n |u_n|^p v,$$

which converges as $n \to \infty$ to

$$\int (\nabla u \cdot \nabla v - \lambda g u v - \lambda g u |u|^p v) dx =< J'_\lambda(u), v > .$$

Hence, $< J'_\lambda(u), v >= 0$ for all $v \in W_0^{1,2}(\ )$ which means that u is a weak solution for (I_λ). In particular, $< J'_\lambda(u), u >= 0$. Since $\liminf_{n\to\infty} \left| \int g u_n^2 \right| > 0$ by Lemma 2.3, we have that $u \neq 0$. Therefore, $u \in M_\lambda$.

Since J_λ is weakly lower semi-continuous, we get

$$c \cdot \ J_\lambda(u) \cdot \ \lim_{n\to\infty} J_\lambda(u_n) = c.$$

It follows that $J_\lambda(u) = c$ and that $||u_n||_\lambda \to ||u||_\lambda$ which implies that $u_n \to u$ strongly in $W_0^{1,2}(\ )$. Since J'_λ is continuous at u, we get $J'_\lambda(u) = 0$.

The positivity of u is clear from the equality $J_\lambda(u) = J_\lambda(|u|)$.

This completes the proof.

Theorem 2.7 *If there is a open subset $_g$ of so that $g(x) = 0$ for all $x \in\ _g$. Then for any $\lambda \in (\lambda^-, \lambda^+), \lambda \neq 0$, the problem (I_λ) has a positive solution.*

Proof. By Theorem 2.6, we have a positive solution u_λ of the problem:

$$\begin{cases} -\Delta u = \lambda g(x) u + g(x) u |u|^p \text{ in } \ , \\ u = 0 \text{ on } \partial\ , \end{cases}$$

for $\lambda \in (\lambda^- + \delta_1, \lambda^-) \cup (\lambda^+ - \delta_2, \lambda^+)$.

Let $0 < \lambda < \lambda^+$ and let $\{u_n\}$ be a minimizing sequence of J_λ in M_λ.

If $K_\lambda < K_0$ on $(0, \lambda^+)$, we note that, since $K_\lambda = \lambda^{\frac{2}{p+2}} Q_\lambda$, so

$$\lim_{n\to\infty} \lambda^{\frac{2}{p+2}} ||u_n||^{-\frac{4}{p+2}} \Box \int |\nabla u_n|^2 - \lambda \int g u_n^2 \Big] = K_\lambda,$$

by the same method in Lemma 2.3, we can have the same inequality about the limit $\liminf_{n\to\infty}\left|\int gu_n^2\right| > 0$, and so Theorem 2.6 implies the existence of a positive solution of (I_λ). We note that $K_\lambda \cdot\ K_0$ for all $\lambda \in (0, \lambda^+)$.

Suppose that there is λ_0 so that $0 < \lambda_0 < \lambda^+$ and $K_\lambda = K_0$ for the value $\lambda \in (0, \lambda_0]$ and $K_\lambda < K_0$ on (λ_0, λ^+). Let u_λ be the positive minimizer of the functional J_λ on M_λ for $\lambda \in (\lambda_0, \lambda^+)$. Let

$$t_\lambda^p = \frac{\lambda \int gu_\lambda^{p+2} + (\lambda - \lambda_0)\int gu_\lambda^2}{\lambda_0 \int gu_\lambda^{p+2}}.$$

Then $t_\lambda u_\lambda \in M_{\lambda_0}$, and

$$J_{\lambda_0}(t_\lambda u_\lambda) = t_\lambda^2 \ J_\lambda(u_\lambda) + \frac{p}{2(p+2)}(\lambda - \lambda_0)\int gu_\lambda^2\Big].$$

As the previous calculation in Remark, we note that

$$\inf_{\lambda\to\lambda_0}\int gu_\lambda^{p+2} > 0,$$

and we also note that

$$\liminf_{\lambda\to\lambda_0} J_\lambda(u_\lambda) < \infty$$

implies that

$$\liminf_{\lambda\to\lambda_0}\left|\int gu_\lambda^2\right| \neq \infty,$$

and hence, $t_\lambda \to 1$ as $\lambda \to \lambda_0$. Since $\{t_\lambda u_\lambda\}$ is a minimizing sequence of J_{λ_0} as $\lambda \to \lambda_0$, we get the weak limit u_{λ_0} of u_λ so that

$$\lim_{\lambda\to\lambda_0} t_\lambda u_\lambda = u_{\lambda_0} \quad \text{in} \quad L^2(\ \ ).$$

If $u_{\lambda_0} \neq 0$, we know that it is the minimizer of J_{λ_0} and is the positive solution of the above boundary value problem with respect to λ_0. Let

$$v_\lambda = \lambda^{-\frac{1}{p+2}}\|u_\lambda\|_\lambda^{-\frac{2}{p+2}} u_\lambda.$$

Then

$$K_\lambda = \int |\nabla v_\lambda|^2 - \lambda\int g(v_\lambda)^2$$

and

$$K_{\lambda_0} = \int |\nabla v_{\lambda_0}|^2 - \lambda_0\int g(v_{\lambda_0})^2.$$

Then

$$-\int g(v_\lambda)^2 \cdot \ \frac{K_\lambda - K_{\lambda_0}}{\lambda - \lambda_0} \ \cdot \ -\int g(v_{\lambda_0})^2.$$

Taking the limit on the both side as $\lambda \to \lambda_0$, we get

$$\frac{dK_\lambda}{d\lambda}(\lambda_0) = -\int g(v_{\lambda_0})^2.$$

Hence, K_λ is differentiable at $\lambda = \lambda_0$, and so

$$\int g(v_{\lambda_0})^2 = 0.$$

Since

$$K_{\lambda_0} = K_0 = \int |\nabla v_{\lambda_0}|^2,$$

so v_{λ_0} is also a positive solution of the problem:

$$\begin{cases} -\Delta u = g(x)u|u|^p \text{ in } \quad , \\ u = 0 \text{ on } \partial \quad , \end{cases}$$

which leads to a contradiction.

Let $u_{\lambda_0} = 0$. The $u_\lambda \to 0$ a.e. in as $\lambda \to \lambda_0$. Since $\Delta u_\lambda(x) = 0$ in $_g$, by the Maximum Principle and the Harnack Inequality (See [6]), we can argue that $u_\lambda \to 0$ uniformly in , and then by the Lebesgue dominated convergence

$$1 = \lim_{\lambda \to \lambda_0} \|u_\lambda\|_\lambda^{-2} \int g|u_\lambda|^{p+2} = 0,$$

since $\|u_\lambda\|_\lambda \nrightarrow 0$ as $\lambda \to \lambda_0$, which also leads to a contradiction.

We can get the same result for the value $\lambda < 0$.

This completes the proof.

Theorem 2.8 *Let $g(x) = 0$ on some open subset of . Then the following problem:*

$$\begin{cases} -\Delta u = g(x)u|u|^{\frac{4}{N-2}} \text{ in } \quad , \\ u = 0 \text{ on } \partial \quad , \end{cases}$$

has a positive solution.

Proof. With the result of Theorem 2.7 if we let $\lambda_0 = 0$, the proof for the convergence of a minimizing sequence of the functional:

$$J(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{p+2}\int g|u|^{p+2}dx$$

on the Nehari manifold

$$\{u \in W_0^{1,2}(\ ) : u \neq 0, \int |\nabla u|^2 - \int g|u|^{p+2} = 0\}$$

can be produced by the proof of Theorem 2.7.

ACKNOWLEDGEMENT

This work was supported by grant No. R05-2002-000-00013-0 from the Basic Research Program of the Korea Science and Engineering Foundation. This work was also studied on the research professor support project of Cheju National University Development Foundation.

REFERENCES

1. B. Ko and K. Brown, The existence of positive solutions for a class of indefinite weight semilinear elliptic boundary value problems, *Nonlinear Analysis*, **39** (2000), 587-597.
2. R. Pohožaev, Eigenfunctions on the equation $\Delta u + \lambda f(u) = 0$, *Soviet Math. Dokl.*, **6** (1965), 1408-1411.
3. G. Afrouzi and K. Brown, On principal eigenvalues for boundary value problems with indefinite weight and Robin boundary conditions, *Proc. AMS* **127(1)** (1999), 125-130.
4. N. Ghoussoub, *Duality and perturbation methods in critical point theory*, Cambridge University Press, Cambridge, 1993.
5. H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.*, **36** (1983), 437-477.
6. D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, 1983.

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