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SELF-SIMILAR SOLUTIONS TO A NONLINEAR PARABOLIC-ELLIPTIC SYSTEM

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Abstract. We study the forward self-similar solutions to a parabolic-elliptic system

$$u_t = \Delta u - \nabla \cdot (u \nabla v), \quad 0 = \Delta v + u$$

in the whole space $\mathbf{R}^2$. First it is proved that self-similar solutions (u, v) must be radially symmetric about the origin. Then the structure of the set of self-similar solutions is investigated. As a consequence, it is shown that there exists a self-similar solution (u, v) if and only if $\|u\|_{L^1(\mathbf{R}^2)} < 8\pi$, and that the profile function of u forms a delta function singularity as $\|u\|_{L^1(\mathbf{R}^2)} \to 8\pi$.

1. Introduction

We are concerned with the parabolic-elliptic system of partial differential equations

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) & \text{in } \mathbf{R}^2 \times (0, \infty), \\ 0 = \Delta v + u & \text{in } \mathbf{R}^2 \times (0, \infty). \end{cases} \tag{1.1}$$

The system (1.1) has been used as a model for several biological and physical problems, as for instance chemotaxis [8, 14] and gravitational interaction of particles [16, 17, 3]. The convective term $-\nabla \cdot (u \nabla v)$ in the first equation describes aggregation directed towards the origin with a velocity proportional to ∇v. In the biological model, u and v stand for the density of the organisms and the concentration of the chemical substance, respectively. In the physical model, u is the density of particles interacting with themselves through the gravitational potential $-v$.

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The existence and uniqueness of local and global in time solutions to the Cauchy problem of (1.1) is studied by Biler [1] under minimal regularity assumptions on the initial conditions.

The system (1.1) is invariant under the similarity transformation

$$u_\lambda(x,t) = \lambda^2 u(\lambda x, \lambda^2 t) \quad \text{and} \quad v_\lambda(x,t) = v(\lambda x, \lambda^2 t)$$

for $\lambda > 0$, that is, if (u, v) is a solution of (1.1) then so is (u_λ, v_λ). Then it is natural to consider solutions which satisfy the scaling property

$$u(x,t) \equiv u_\lambda(x,t) \quad \text{and} \quad v(x,t) \equiv v_\lambda(x,t) \quad \text{for all } \lambda > 0. \tag{1.2}$$

They are called forward self-similar solutions. By the definition, they are global in time, and heuristically they describe large time behavior of general solutions to (1.1). Letting $\lambda = 1/\sqrt{t}$ in (1.2), we see that (u, v) has the special form

$$u(x,t) = \frac{1}{t}\phi\left(\frac{x}{\sqrt{t}}\right) \quad \text{and} \quad v(x,t) = \psi\left(\frac{x}{\sqrt{t}}\right) \tag{1.3}$$

for $x \in \mathbf{R}^2$ and $t > 0$. By a direct computation it is shown that (ϕ, ψ) satisfies

$$\begin{cases} \nabla \cdot (\nabla\phi - \phi\nabla\psi) + \phi &= 0 \qquad \text{in } \mathbf{R}^2, \\ \Delta\psi + \phi &= 0 \qquad \text{in } \mathbf{R}^2. \end{cases} \tag{1.4}$$

It follows from (1.3) that

$$\int_{\mathbf{R}^2} u(x,t)dx = \int_{\mathbf{R}^2} \phi(y)dy$$

for $\phi \in L^1(\mathbf{R}^N)$. Then the self-similar solution (u, v) preserves the mass $\|u(\cdot,t)\|_{L^1(\mathbf{R}^2)}$.

We see that if (ϕ, ψ) is a solution of (1.4) then $(\phi, \psi + c)$ is also a solution of (1.4) for any constant c. Then we may assume that $\psi(0) = 0$ if $\psi(0) < \infty$, in particular, if $\psi^+ \in L^\infty(\mathbf{R}^2)$, where $a^+ = \max\{a, 0\}$. We are concerned with the classical solutions $(\phi, \psi) \in C^2(\mathbf{R}^2) \times C^2(\mathbf{R}^2)$ of (1.4) satisfying

$$\phi \geq 0 \quad \text{in } \mathbf{R}^2, \quad \phi \in L^1(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2), \quad \psi^+ \in L^\infty(\mathbf{R}^2), \quad \text{and} \quad \psi(0) = 0. \tag{1.5}$$

Define the solution set $\mathcal{S}$ of (1.4) as

$$\mathcal{S} = \{(\phi, \psi) \in C^2(\mathbf{R}^2) \times C^2(\mathbf{R}^2) : (\phi, \psi) \text{ is a solution of (1.4) with (1.5)}\}. \tag{1.6}$$

Put $\phi(x) = ce^{-|x|^2/4}e^{\psi(x)}$, where c is a positive constant. Then we see that ϕ satisfies the first equation of (1.4), and so we can obtain the solutions $(\phi, \psi) \in \mathcal{S}$ if we find a solutions ψ of

$$\Delta\psi + ce^{-|x|^2/4}e^{\psi} = 0 \qquad \text{in } \mathbf{R}^2$$

satisfying $\psi(0) = 0$. By the ODE arguments in Section 3 below, we easily see that (1.4) has radial solutions $(\phi, \psi) \in \mathcal{S}$. In this paper we investigate the structure of the solution set $\mathcal{S}$.

Theorem 1. *If $(\phi, \psi) \in \mathcal{S}$ then we have*

$$\phi(x) = \sigma e^{-|x|^2/4} e^{\psi(x)} \tag{1.7}$$

for some constant $\sigma > 0$. Furthermore, ϕ and ψ are radially symmetric about the origin, and satisfy $\partial\phi/\partial r < 0$ and $\partial\psi/\partial r < 0$ for $r = |x| > 0$.

Theorem 2. *The solution set $\mathcal{S}$ is expressed as a one parameter family:*

$$\mathcal{S} = \{(\phi(\sigma), \psi(\sigma)) : \sigma > 0\}.$$

Put $\lambda(\sigma) = \|\phi(\sigma)\|_{L^1(\mathbf{R}^2)}$. Then $(\phi(\sigma), \psi(\sigma))$ and $\lambda(\sigma)$ satisfy the following properties:

(i) $\sigma \mapsto (\phi(\sigma), \psi(\sigma))$ *is continuous in $C^2(\mathbf{R}^2) \times C^2_{\mathrm{loc}}(\mathbf{R}^2)$;*

(ii) $\sigma \mapsto \lambda(\sigma) \in \mathbf{R}$ *is continuous, and $0 < \lambda(\sigma) < 8\pi$ for $\sigma > 0$;*

(iii) $\lambda(\sigma) \to 0$ *and $(\phi(\sigma), \psi(\sigma)) \to (0, 0)$ in $C^2(\mathbf{R}^2) \times C^2_{\mathrm{loc}}(\mathbf{R}^2)$ as $\sigma \to 0$;*

(iv) $\lambda(\sigma) \to 8\pi$ *and $\phi(\sigma)dx \rightharpoonup 8\pi\delta_0(dx)$ in the sense of measure as $\sigma \to \infty$, where $\delta_0(dx)$ denotes Dirac's delta function with the support in origin.*

Remark. (i) It has been shown by Biler [1, 2] that $\|\phi\|_{L^1(\mathbf{R}^2)} < 8\pi$ for $(\phi, \psi) \in \mathcal{S}$. We however give the proof in Appendix for the sake of convenience.

(ii) In [9] Kurokiba and Ogawa discussed the existence of the blow-up solutions for the drift-diffusion system in $\mathbf{R}^2$, which includes (1.1) in the special case. As a consequence of [9] it has been shown that $\mathcal{S}$ has no (ϕ, ψ) satisfying $\|\phi\|_{L^1(\mathbf{R}^2)} > 8\pi$ since each element of $\mathcal{S}$ is identified with a time global solution to (1.1) studied by them.

As a consequence of Theorem 2 we obtain the following result, which has been obtained by Biler [2, Proposition 3] in the radial case.

Corollary. *There exists a solution $(\phi, \psi) \in \mathcal{S}$ satisfying $\|\phi\|_{L^1(\mathbf{R}^2)} = \lambda$ if and only if $\lambda \in (0, 8\pi)$.*

The proof of Theorem 1 consists of two steps. First we show that (1.7) holds by employing the Liouville type result. Then we show the radial symmetry of solutions by the method of moving planes.

It follows from (1.7) that the system (1.4) is reduced to the equation

$$\Delta\psi + \sigma e^{-|x|^2/4} e^{\psi} = 0 \qquad \text{in } \mathbf{R}^2 \tag{1.8}$$

for some positive constant σ. Let $\lambda = \|\phi\|_{L^1(\mathbf{R}^2)}$. From (1.7) we see that

$$\lambda = \sigma \int_{\mathbf{R}^2} e^{-|y|^2/4} e^{\psi(y)} dy.$$

Then (1.8) is rewritten as the elliptic equation with nonlocal term,

$$\Delta\psi + \lambda e^{-|x|^2/4} e^{\psi} \Big/ \int_{\mathbf{R}^2} e^{-|y|^2/4} e^{\psi(y)} dy = 0 \qquad \text{in } \mathbf{R}^2. \tag{1.9}$$

The proof of Theorem 2 is based on the ODE arguments to (1.8) and the blow-up arguments to (1.9). In particular, we employ the results by Brezis and Merle [4] and Li-Shafrir [10] concerning the asymptotic behavior of sequences of solutions of $-\Delta u_k = V_k(x) e^{u_k}$ in , where $\subset \mathbf{R}^2$ is a bounded domain.

We organize this paper as follows. In Section 2 we prove Theorem 1 by employing the Liouville type result and the method of moving planes. In Section 3 we prove Theorem 2 by using the ODE arguments and the blow-up arguments. In Appendix, we show the upper bounds of $\|\phi\|_{L^1(\mathbf{R}^2)}$.

2. Proof of Theorm 1.

First we show the following lemma.

Lemma 2.1. *Let* $f \in L^\infty(\mathbf{R}^2) \cap L^1(\mathbf{R}^2)$. *Define* $w(x)$ *as*

$$w(x) = \frac{1}{2\pi} \int_{\mathbf{R}^2} (\log|x-y| - \log|y|) f(y) dy.$$

Then

(i) $|w(x)| = O(\log|x|)$ *as* $|x| \to \infty$*;*

(ii) $|\nabla w(x)| = o(|x|)$ *as* $|x| \to \infty$.

Proof. (i) Following the argument by [6], we easily obtain

$$\lim_{|x|\to\infty} \frac{w(x)}{\log|x|} = \frac{1}{2\pi} \int_{\mathbf{R}^2} f(y) dy.$$

(See also [12, Lemma 2.1].) This implies that (i) holds.

(ii) We see that

$$\nabla w(x) = \frac{1}{2\pi} \int_{\mathbf{R}^2} \frac{x-y}{|x-y|^2} f(y) dy.$$

For any $\varepsilon > 0$, we write

$$|\nabla w(x)| \cdot \quad \frac{1}{2\pi}\left(\int_{|x-y| \quad \varepsilon|x|} + \int_{|x-y|\geq\varepsilon|x|} \frac{|f(y)|}{|x-y|}dy\right) \equiv I_1 + I_2.$$

By $f \in L^\infty(\mathbf{R}^2)$ we obtain

(2.1)
$$I_1 \cdot \quad \frac{\|f\|_{L^\infty(\mathbf{R}^2)}}{2\pi}\int_{|x-y| \quad \varepsilon|x|} \frac{dy}{|x-y|} = \|f\|_{L^\infty(\mathbf{R}^2)}\int_0^{\varepsilon|x|} dr = \varepsilon|x|\|f\|_{L^\infty(\mathbf{R}^2)}.$$

By $f \in L^1(\mathbf{R}^2)$ we have

$$I_2 \cdot \quad \frac{1}{2\pi\varepsilon|x|}\int_{\mathbf{R}^2} |f(y)|dy = \frac{\|f\|_{L^1(\mathbf{R}^2)}}{2\pi\varepsilon|x|} \to 0 \quad \text{as } |x| \to \infty. \tag{2.2}$$

Then it follows from (2.1) and (2.2) that

$$\lim_{|x|\to\infty} \frac{|\nabla w(x)|}{|x|} \cdot \quad \varepsilon\|f\|_{L^\infty(\mathbf{R}^2)}.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that (ii) holds. ■

We prepare the Liouville type result [13, Lemma 2.1] for second order elliptic inequalities essentially due to Meyers and Serrin [11].

Lemma 2.2. *Let w satisfy $\Delta w + \nabla b \cdot \nabla w \geq 0$ in $\mathbf{R}^2$. Assume that $x \cdot \nabla b(x) \cdot \quad 0$ for large $|x|$. If $\sup_{x\in\mathbf{R}^2} w(x) < \infty$ then w must be a constant function.*

We consider the radial symmetry of solutions for the equation

$$\Delta u + V(|x|)e^u = 0 \qquad \text{in } \mathbf{R}^2. \tag{2.3}$$

The symmetry and monotonicity properties of the solutions for equation (2.3) have been studied by Chanillo and Kiessling [5], Cheng and Lin [6], and Naito [12]. The following result may be found in [12, Corollary 2].

Lemma 2.3. *Assume that $V(r)$ is nonnegative for $r \geq 0$ and that*

$$\limsup_{r\to\infty} r^\alpha V(r) < \infty$$

for some $\alpha > 2$. Let u be a solution of (2.3) satisfying $u^+ \in L^\infty(\mathbf{R}^2)$. Then u must be radially symmetric about the origin and $\partial u/\partial r < 0$ for $r > 0$.

Proof of Theorem 1. Let $(\phi, \psi) \in \mathcal{S}$. First we show that

$$|\nabla\psi(x)| = o(|x|) \quad \text{as} \quad |x| \to \infty. \tag{2.4}$$

Define $w(x)$ as

$$w(x) = \frac{1}{2\pi}\int_{\mathbf{R}^2} (\log|x-y| - \log|y|)\,\phi(y)dy.$$

Then we see that $w(x)$ is well defined and that $\Delta w = \phi$ in $\mathbf{R}^2$. Since $\phi \in L^\infty(\mathbf{R}^2) \cap L^1(\mathbf{R}^2)$, from (i) of Lemma 2.1 we have $|w(x)| = O(\log|x|)$ as $|x| \to \infty$. Put $z(x) = \psi(x) + w(x)$. Then $\Delta z = 0$ in $\mathbf{R}^2$ and $z(x) = O(\log|x|)$ as $|x| \to \infty$. By the Liouville theorem (see, e.g., [12, Lemma 2.2]), z must be a constant. Thus we obtain $\nabla\psi = -\nabla w$. From (ii) of Lemma 2.1 we obtain (2.4).

Now we verify that (1.7) holds for some constant $\sigma > 0$. Put $w(x) = -\phi(x)e^{|x|^2/4}e^{-\psi(x)} \cdot\ 0$. Then $e^{-|x|^2/4}e^{\psi}\nabla w = -\nabla\phi - x\phi/2 + \phi\nabla\psi$. From the first equation of (1.4) we have

$$\nabla\cdot(e^{-|x|^2/4}e^{\psi}\nabla w) = 0, \quad \text{or} \quad \Delta w + \nabla b\cdot\nabla w = 0 \quad \text{in } \mathbf{R}^2,$$

where $\nabla b(x) = -x/2 + \nabla\psi(x)$. From (2.4) we have

$$x\cdot\nabla b(x) = \left(-\frac{|x|^2}{2} + x\cdot\nabla\psi(x)\right) \cdot\ 0$$

for large $|x|$. As a consequence of Lemma 2.2, w must be a constant function. This implies that (1.7) holds for some constant $\sigma > 0$.

From (1.7) it follows that ψ solve the equation (1.8). By applying Lemma 2.3 we obtain $\psi = \psi(|x|)$ and $\partial\psi/\partial r < 0$ for $r > 0$. From (1.7) we also obtain $\phi = \phi(|x|)$ and $\partial\phi/\partial r < 0$ for $r > 0$. ■

3. Proof of Theorm 2

Let $(\phi, \psi) \in \mathcal{S}$. From Theorem 1 it follows that ϕ is given by (1.7) for some $\sigma > 0$ and ψ solves the equation (1.8). Moreover, ψ is radially symmetric about the origin. Then $\psi = \psi(r)$, $r = |x|$, satisfies

$$\begin{cases} \psi_{rr} + \dfrac{1}{r}\psi_r + \sigma e^{-r^2/4}e^{\psi} = 0, \quad r > 0, \\ \psi_r(0) = 0 \quad \text{and} \quad \psi(0) = 0, \end{cases} \tag{3.1$_\sigma$}$$

where $\sigma > 0$. We denote by $\psi(r;\sigma)$ the solution of the problem $(3.1)_\sigma$. We easily see that $\psi(r;\sigma)$ is defined on $[0,\infty)$ and satisfies

$$\psi(r;\sigma) = -\sigma\int_0^r \frac{1}{s}\left(\int_0^s te^{-t^2/4}e^{\psi(t;\sigma)}dt\right)ds \tag{3.2}$$

and

$$\psi_r(r;\sigma) = -\frac{\sigma}{r}\int_0^r se^{-s^2/4}e^{\psi(s;\sigma)}ds. \tag{3.3}$$

We show the following lemma.

Lemma 3.1. (i) *For any* $\sigma > 0$, $\psi(\cdot,\sigma) \in C^2[0,\infty)$;

(ii) $\sigma \mapsto \psi(\cdot,\sigma)$ *is continuous in* $C^2_{\mathrm{loc}}[0,\infty)$;

(iii) $\psi(\cdot,\sigma) \to 0$ *in* $C^2_{\mathrm{loc}}[0,\infty)$ *as* $\sigma \to 0$.

Proof. (i) By using the L'Hospital's rule we obtain

$$\lim_{r\to 0}\frac{\psi_r(r)}{r} = \lim_{r\to 0} -\frac{\sigma}{r^2}\int_0^r se^{-s^2/4}e^{\psi(s)}ds = -\frac{\sigma}{2},$$

which implies $\psi \in C^2[0,\infty)$.

(ii) We easily see that (ii) holds by a general ODE theory concerning the continuous dependence of solutions on the parameter, (see, e.g., [7]).

(iii) We note that $\psi(r,\sigma) \cdot\ 0$ for $r \geq 0$. Then, from (3.2) and (3.3) we have

$$|\psi_r(r,\sigma)| \cdot\ \frac{\sigma}{r}\int_0^r sds = \frac{\sigma}{2}r \quad \text{and} \quad |\psi(r,\sigma)| \cdot\ \sigma\int_0^r \frac{1}{s}\left(\int_0^s tdt\right)ds = \frac{\sigma}{4}r^2 \quad \text{for } r \geq 0.$$

From the equation in $(3.1)_\sigma$ we have $|\psi_{rr}(r,\sigma)| \cdot\ (3\sigma)/2$ for $r \geq 0$. Take any $r_0 > 0$. Then it follows that

$$\|\psi(\cdot,\sigma)\|_{C^2[0,r_0]} \cdot\ \sigma\left(\frac{r_0^2}{4} + \frac{r_0}{2} + \frac{3}{2}\right),$$

which implies that (iii) holds. ∎

Define $\phi(r;\sigma)$ and $\lambda(\sigma)$, respectively, by

$$\phi(r;\sigma) = \sigma e^{-r^2/4}e^{\psi(r:\sigma)} \quad \text{and} \quad \lambda(\sigma) = \int_{\mathbf{R}^2}\phi(|y|;\sigma)dy = 2\pi\int_0^\infty r\phi(r;\sigma)dr.$$

Let $\{\sigma_k\}$ be a sequence such that $\sigma_k \to \infty$ as $k \to \infty$. We investigate the asymptotic behavior of sequences $\{(\phi(r,\sigma_k),\psi(r,\sigma_k)\}$ and $\{\lambda(\sigma_k)\}$ as $k \to \infty$. For simplicity, one sets

$$\phi_k(x) = \phi(|x|,\sigma_k), \quad \psi_k(x) = \psi(|x|,\sigma_k), \quad \text{and} \quad \lambda_k = \lambda(\sigma_k).$$

We show the following:

Lemma 3.2. *Assume that* $\sigma_k \to \infty$ *as* $k \to \infty$*. Then there exists a subsequence (still denoted by* $\{\sigma_k\}$*) such that* $\lambda_k \to 8\pi$ *as* $k \to \infty$ *and*

$$\phi_k(x)dx \rightharpoonup 8\pi\delta_0(dx) \quad \text{as } k \to \infty \tag{3.4}$$

in the sense of measure, where $\delta_0(dx)$ *is Dirac's delta function with the support in origin.*

In order to prove Lemma 3.2 we essentially use the results by Brezis-Merle [4, Theorem 3] and Li-Shafrir [10, Theorem] concerning the asymptotic behavior of sequences of solutions to

$$-\Delta u_k = V_k(x)e^{u_k} \qquad \text{in } \Omega, \tag{3.5}$$

where $\Omega \subset \mathbf{R}^2$ is a bounded domain and V_k is a nonnegative continuous functions. A special case of their result is the following:

Theorem A ([4, 10]). *Suppose* $V_k \in C(\overline{\Omega})$*,* $V_k \geq 0$ *in* Ω*,* $V_k \to V_0$ *in* $C(\overline{\Omega})$*. Let* $\{v_k\}$ *be a sequence of solutions of* (3.5) *with* $\|e^{v_k}\|_{L^1(\Omega)} \le C$ *for some positive constant* C*. Then there exists a subsequence* (*still denoted by* $\{v_k\}$) *satisfying one of the following alternatives*:

(i) $\{v_k\}$ *is bounded in* $L^\infty_{\mathrm{loc}}(\Omega)$*;*

(ii) $v_k \to -\infty$ *uniformly on compact subset of* Ω*;*

(iii) *there exists a finite blow-up set* $\mathcal{B} = \{a_1, \dots, a_m\} \subset \Omega$ *such that, for any* $1 \le i \le m$*, there exists* $\{x_k\} \subset \Omega$*,* $x_k \to a_i$*,* $v_k(x_k) \to \infty$*, and* $v_k \to -\infty$ *uniformly on compact subsets of* $\Omega \setminus \mathcal{B}$*. Moreover,* $V_k e^{v_k} dx \rightharpoonup \sum_{i=1}^m \alpha_i \delta_{a_i}(dx)$ *in the sense of measure with* $\alpha_i = 8\pi m_i$*,* $m_i \in \mathbf{N}$*, where* $\delta_{a_i}(dx)$ *is Dirac's delta function with the support in* $x = a_i$*.*

Proof of Lemma 3.2. By the definition of λ_k and (1.7) we see that

$$\lambda_k = \sigma_k \int_{\mathbf{R}^2} e^{-|y|^2/x} e^{\psi_k(y)} dy. \tag{3.6}$$

Then ψ_k satisfies the equation

$$\Delta\psi_k + \lambda_k e^{-|x|^2/4} e^{\psi_k} \Big/ \int_{\mathbf{R}^2} e^{-|y|^2/4} e^{\psi_k(y)} dy = 0 \qquad \text{in } \mathbf{R}^2. \tag{3.7}$$

Now define v_k as

$$v_k(x) = \psi_k(x) - \log\left(\int_{\mathbf{R}^2} e^{-|y|^2/4} e^{\psi_k(y)} dy \right).$$

Take any $R > 0$ and put $B_R = \{x \in \mathbf{R}^2 : |x| < R\}$. It follows from (3.7) that

$$-\Delta v_k = -\Delta \psi_k = \lambda_k e^{-|x|^2/4} e^{v_k} \qquad \text{for } x \in B_R.$$

Then we have

$$-\Delta v_k = V_k(x) e^{v_k} \qquad \text{in } B_R, \tag{3.8}$$

where $V_k(x) = \lambda_k e^{-|x|^2/4}$. By Biler [1, 2] we have $\lambda_k \in (0, 8\pi)$. (See also Proposition A.1 in Appendix below.) Then there exists a subsequence (still denoted by $\{\lambda_k\}$) such that $\lambda_k \to \lambda_0 \in [0, 8\pi]$. Then $V_k(x) \to V_0(x) \equiv \lambda_0 e^{-|x|^2/4}$ in $C(\overline{B_R})$ as $k \to \infty$. We also note that

$$\int_{B_R} e^{v_k(y)} dy \cdot \int_{B_R} e^{\psi_k(y)} dy \Bigg/ \int_{\mathbf{R}^2} e^{-|y|^2/4 + \psi_k(y)} dy \cdot \; C \tag{3.9}$$

for some constant $C > 0$. Hence, by applying Theorem A, there exists a subsequence (still denoted by $\{v_k\}$) satisfying one of the alternatives (i), (ii), and (iii) in Theorem A.

Since $\{\lambda_k\} \in (0, 8\pi)$ and $\sigma_k \to \infty$, from (3.6) we obtain

$$\int_{\mathbf{R}^2} e^{-|y|^2/x} e^{\psi_k(y)} dy \to 0 \quad \text{as } k \to \infty.$$

It follows that

$$v_k(0) = \psi_k(0) - \log\left(\int_{\mathbf{R}^2} e^{-|y|^2/4} e^{\psi_k(y)} dy \right) \to \infty,$$

which implies that either (i) or (ii) does not hold. Thus the third alternative (iii) must hold. Since $v_k = v_k(r)$, $r = |x|$, is strictly decreasing and (3.9) holds, we find that the blow-up set $\mathcal{B} = \{0\}$. Then $v_k(0) \to \infty$ and $v_k \to -\infty$ uniformly on compact subset of $B_R \setminus \{0\}$. Moreover

$$\int_{B_R} V_k e^{v_k} dx \to 8m\pi \quad \text{as } k \to \infty \tag{3.10}$$

for some $m \in \mathbf{N}$.

From (3.8) and the second equation of (1.4) we have

$$V_k e^{v_k} = -\Delta v_k = -\Delta \psi_k = \phi_k.$$

Then from (3.10) we have

$$\lambda_k = \int_{\mathbf{R}^2} \phi_k(y) dy = \int_{\mathbf{R}^2} V_k e^{v_k} dy \geq \int_{B_R} V_k e^{v_k} dy \to 8m\pi \quad \text{as } k \to \infty.$$

From $\lambda_k \in (0, 8\pi)$, we obtain $m = 1$ in (3.10). Moreover, we have $\lambda_k = \int_{\mathbf{R}^2} \phi_k(y)dy \to 8\pi$ and

$$\int_{\mathbf{R}^2 \setminus B_R} V_k e^{v_k} dy \to 0 \quad \text{as } k \to \infty. \tag{3.11}$$

Since $\{\phi_k\}$ is bounded in $L^1(\mathbf{R}^2)$, we may extract a subsequence, which we call again $\{\phi_k\}$, such that ϕ_k converges in the sense of measures on $\mathbf{R}^2$ to some nonnegative bounded measure μ, i.e.,

$$\int_{\mathbf{R}^2} \eta\phi_k(x)dx \to \int_{\mathbf{R}^2} \eta d\mu \quad \text{as } k \to \infty$$

for every $\eta \in C(\mathbf{R}^2)$ with compact support. Since (3.11) holds for every $R > 0$, we have $\phi_k \to 0$ in $L^1_{\text{loc}}(\mathbf{R}^2 \setminus \{0\})$. Then μ is supported on $\{0\}$. Therefore we obtain $d\mu = \alpha\delta_0(dx)$ with $\alpha = 8\pi$, which implies that (3.4) holds. This completes the proof of Lemma 3.2. ■

Proof of Theorem 2. By Theorem 1 we see that if $(\phi, \psi) \in \mathcal{S}$ then ϕ is given by (1.7) for some $\sigma > 0$ and ψ is radially symmetric and solves the problem $(3.1)_\sigma$. Conversely, it is clear that if $\psi = \psi(|x|)$ is a solution of the problem $(3.1)_\sigma$, and if ϕ is given by (1.7), then $(\phi, \psi) \in \mathcal{S}$. Therefore, $(\phi, \psi) \in \mathcal{S}$ if and only if $\psi = \psi(r)$, $r = |y|$, solves $(3.1)_\sigma$ for some $\sigma > 0$ and ϕ is given by (1.7). Then $\mathcal{S}$ is written by one parameter families $(\phi(r, \sigma), \psi(r, \sigma))$ on $\sigma > 0$. From (ii) and (iii) of Lemma 3.1 we have $\sigma \mapsto (\phi(\cdot; \sigma), \psi(\cdot; \sigma))$ is continuous in $C^2[0, \infty) \times C^2_{\text{loc}}[0, \infty)$ and $(\phi(\cdot; \sigma), \psi(\cdot; \sigma)) \to (0, 0)$ in $C^2[0, \infty) \times C^2_{\text{loc}}[0, \infty)$ as $\sigma \to 0$. By the definition of $\lambda(\sigma)$ and (1.7) we see that

$$\lambda(\sigma) = \sigma \int_{\mathbf{R}^2} e^{-|y|^2/4} e^{\psi(|y|:\sigma)} dy \cdot \ \sigma \int_{\mathbf{R}^2} e^{-|y|^2/4} dy.$$

Then $\lambda(\sigma)$ is continuous and satisfies $\lambda(\sigma) \to 0$ as $\sigma \to 0$. By Biler [1, 2] we have $\lambda(\sigma) \in (0, 8\pi)$. Hence, (i), (ii), and (iii) holds.

Let $\{\sigma_k\}$ be a sequence satisfying $\sigma_k \to \infty$ as $k \to \infty$. Then, by Lemma 3.2, there exists a subsequence (still denoted by $\{\sigma_k\}$) such that $\lambda(\sigma_k) \to 8\pi$ and $\phi(|x|, \sigma_k)dx \rightharpoonup 8\pi\delta_0(dx)$ as $k \to \infty$. This implies that (iv) holds. This completes the proof of Theorem 2. ■

Appendix

We investigate the upper bounds of $\|\phi\|_{L^1(\mathbf{R}^2)}$ for $(\phi, \psi) \in \mathcal{S}$. The following result has been obtained by Biller [1, 2]. However, we give here a proof for the sake of convenience.

Proposition A.1. *If* $(\phi, \psi) \in \mathcal{S}$ *then* $\|\phi\|_{L^1(\mathbf{R}^2)} < 8\pi$.

By Theorem 1 the solution $(\phi, \psi) \in \mathcal{S}$ must be radially symmetric about the origin. Define u and v, respectively, as

$$u(r,t) = \frac{1}{t}\phi\left(\frac{r}{\sqrt{t}}\right) \quad \text{and} \quad v(r,t) = \psi\left(\frac{r}{\sqrt{t}}\right).$$

First we show the following lemma.

Lemma A.1. *Put* U *as*

$$U(r,t) = \int_0^r su(s,t)ds.$$

Then U *satisfies*

$$U_t = r\left(r^{-1}U_r\right)_r + \frac{UU_r}{r} \tag{A.1}$$

for $(r,t) \in [0,\infty) \times (0,\infty)$.

Proof. Since (u, v) solves (1.1), we see that

$$ru_t = (ru_r)_r - ru_r v_r - u(rv_r)_r \quad \text{and} \quad 0 = (rv_r)_r + ru.$$

Then we obtain

$$\int_0^r su_t(s,t)ds = ru_r - ruv_r \quad \text{and} \quad -rv_r = \int_0^r su(s,t)ds.$$

It follows that

$$\frac{\partial}{\partial t}\left(\int_0^r su(s,t)ds\right) = ru_r + u\int_0^r su(s,t)ds.$$

From $U_r = ru$ and $(r^{-1}U_r)_r = u_r$, we obtain (A.1). ■

Proof of Proposition A.1. Define Φ as

$$\Phi(s) = \frac{1}{2}\int_0^s \phi(\sqrt{t})dt = \int_0^{\sqrt{s}} r\phi(r)dr.$$

Then we have

$$\int_{\mathbf{R}^2} \phi(|y|)dy = 2\pi\int_0^\infty r\phi(r)dr = 2\pi\lim_{s\to\infty}\Phi(s). \tag{A.2}$$

By the definition of U and u we have

$$U(r,t)=\int_0^r \frac{s}{t}\phi\left(\frac{s}{\sqrt{t}}\right)ds=\frac{1}{2}\int_0^{r^2/t}\phi(\sqrt{s})ds=\Phi\left(\frac{r^2}{t}\right).$$

Then from (A.1) we obtain

$$\Phi_{ss}+\frac{1}{4}\Phi_s+\frac{\Phi\Phi_s}{2s}=0 \quad \text{for } s>0.$$

By the change of variables $\tau=\log s$ and $z(\tau)=\Phi(s)$ it follows that

(A.3) $$z_{\tau\tau}-z_\tau+\frac{1}{4}e^\tau z_\tau+\frac{1}{2}zz_\tau=0 \quad \text{for } \tau>-\infty.$$

We note here that

(A.4) $z_\tau(\tau)=\frac{1}{2}s\phi(\sqrt{s})>0$ for $\tau>-\infty$, and $z(\tau),\ z_\tau(\tau)\to 0$ as $\tau\to-\infty$.

Put $\ell(\tau)=z_\tau(\tau)-z(\tau)+z(\tau)^2/4$. Then it follows from (A.3) and (A.4) that

$$\ell_\tau(\tau)=z_{\tau\tau}-z_\tau+\frac{1}{2}zz_\tau<0$$

for $\tau>-\infty$ and $\lim_{\tau\to-\infty}\ell(\tau)=0$. Therefore, we have $\lim_{\tau\to\infty}\ell(\tau)<0$. Since

$$\ell(\tau)=z_\tau(\tau)-1+\frac{(z(\tau)-2)^2}{4}>-1+\frac{(z(\tau)-2)^2}{4},$$

we obtain $\lim_{\tau\to\infty}(z(\tau)-2)^2<4$. This implies that $\lim_{\tau\to\infty}z(\tau)<4$, that is, $\lim_{s\to\infty}\Phi(s)<4$. From (A.2) we obtain $\|\phi\|_{L^1(\mathbf{R}^2)}<8\pi$. ∎

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