TAIWANESE JOURNAL OF MATHEMATICS
Vol. 8, No. 1, pp. 1-14, March 2004
This paper is available online at http://www.math.nthu.edu.tw/tjm/

ENERGY TRANSPORT PROBLEM FOR SELF-GRAVITATING PARTICLES

Takashi Suzuki

Abstract. We study the stationary state of the energy transport problem for self-gravitating particles and show that radially symmetric negative energy solutions exist if and only if $3 \cdot \; n \cdot \; 9$, where n denotes the space dimension.

1. Introduction

Our purpose is to study the stationary state of an elliptic-parabolic system of partial differential equations modeling the motion of Brownian particles subject to the thermodynamical process and self-gravitations. It is proposed by Biler, Krzywicki, and Nadzieja [2], taking into account of the Poisson type coupling of the self-interaction in the Streater's model [10], and therefore, is regarded as the case that the temperature varies in the drift-diffusion model ([9]).

This system is concerned with the density of the mean field of the particle, the temperature, and the gravitational potential created by those particles, respectively, denoted by $u \geq 0$, $\theta \geq 0$, and φ. Letting φ_0 to be the potential of the outer force, it is actually given as

$$
\begin{aligned}
u_t &= \nabla \cdot \left[\Box \left(\nabla u + \tfrac{u}{\theta} \nabla \left(\varphi + \varphi_0 \right) \right) \right] \\
(\theta u)_t &= \nabla \cdot (\lambda \nabla \theta) + \nabla \cdot \left[\Box \left(\theta \nabla u + u \nabla \left(\varphi + \varphi_0 \right) \right) \right] \\
&\quad + \nabla (\varphi + \varphi_0) \cdot \left[\Box \left(\nabla u + \tfrac{u}{\theta} \nabla \left(\varphi + \varphi_0 \right) \right) \right] \\
\Delta \varphi &= u
\end{aligned}
$$

Received April 15, 2003.
Communicated by S. B. Hsu.
2000 *Mathematics Subject Classification*: 35Q, 35J60, 82C.
Key words and phrases: energy transport, self-gravitating particles, drift-diffusion model, Gel'fand equation.

in a bounded domain $\subset \mathbf{R}^n$, with the boundary condition

$$\begin{aligned} \frac{\partial u}{\partial \nu} + \frac{u}{\theta}\frac{\partial}{\partial \nu}(\varphi + \varphi_0) &= 0 \\ \frac{\partial \theta}{\partial \nu} &= 0 \\ \varphi &= 0 \end{aligned}$$

on ∂ , the smooth boundary, and with the initial condition

$$(u, \theta)|_{t=0} = (u_0, \theta_0) > 0,$$

where ν denotes the outer unit normal vector and the physical coefficients $\Box, \lambda \geq 0$ are the functions of x, u, θ, φ, which may be 0 only at $\theta = 0$.

Thus, the flux of mass is given as

$$j = -\Box\left[\nabla u + \frac{u}{\theta}\nabla(\varphi + \varphi_0)\right] \tag{1}$$

and the system involves the mass conservation of the particle in such a way as

$$\begin{aligned} u_t &= -\nabla j && \text{in} \quad \times (0, T) \\ \nu \cdot j &= 0 && \text{on} \quad \partial \quad \times (0, T), \end{aligned} \tag{2}$$

which implies $u > 0$ from the maximum principle, and also its total mass preserving as

$$\frac{dM}{dt} = 0 \qquad \text{for} \qquad M = \int u.$$

It also includes

$$(u\theta)_t = \nabla \cdot (\lambda \nabla \theta) - \nabla \cdot (\theta j) - \nabla(\varphi + \varphi_0) \cdot j, \tag{3}$$

which is equivalent to

$$u\theta_t = \nabla \cdot (\lambda \nabla \theta) - j \cdot \nabla \theta + \frac{\theta}{\Box u}(j + \Box \nabla u) \cdot j \tag{4}$$

by (2) and (1). Thus, first $\theta > 0$ follows from (4), and then (3) implies that

$$\begin{aligned} \frac{d}{dt}\int u\theta &= -\int \nabla(\varphi + \varphi_0) \cdot j = \int (\varphi + \varphi_0)\nabla \cdot j \\ &= -\frac{d}{dt}\int \varphi_0 u - \int \varphi u_t \end{aligned}$$

with

$$\begin{aligned}\int \varphi u_t = \int \varphi \Delta \varphi_t = -\int \nabla \varphi \cdot \nabla \varphi_t = -\frac{1}{2}\frac{d}{dt}\int |\nabla \varphi|^2 \\ = \frac{1}{2}\frac{d}{dt}\int u\varphi,\end{aligned}$$

and in this way, the total energy

$$E = \int u\left(\theta + \frac{1}{2}\varphi + \varphi_0\right)$$

is preserved as

$$\frac{dE}{dt} = 0.$$

Finally, from (4) we have

$$u(\log\theta)_t = \theta^{-1}\nabla\cdot(\lambda\nabla\theta) - j\cdot\nabla\log\theta + \frac{|j|^2}{\Box u} + \nabla\log u\cdot j,$$

and it holds that

$$\int u(\log\theta)_t = \frac{d}{dt}\int u\log\theta - \int u_t(\log\theta)$$

and

$$\begin{aligned}&\int\left(\theta^{-1}\nabla\cdot(\lambda\nabla\theta) - j\cdot\nabla\log\theta + \frac{|j|^2}{\Box u} + j\cdot\nabla\log u\right)\\ &= \int\left(\lambda\theta^{-2}|\nabla\theta|^2 + \frac{|j|^2}{\Box u}\right) + \int(\nabla\cdot j)\log\theta - \int(\nabla\cdot j)\log u.\end{aligned}$$

This implies from (2) that

$$\begin{aligned}\frac{d}{dt}\int u\log\theta &= \int\left(\lambda|\nabla\log\theta|^2 + \frac{|j|^2}{\Box u}\right) + \int u_t\log u\\ &= \int\left(\lambda|\nabla\log\theta|^2 + \frac{|j|^2}{\Box u}\right) + \frac{d}{dt}\int u(\log u - 1),\end{aligned}$$

and hence the increase of the entropy follows as

$$\frac{dW}{dt} + \int\left(\lambda|\nabla\log\theta|^2 + \frac{|j|^2}{\Box u}\right) = 0 \tag{5}$$

with

$$W = \int u \left(\log(u/\theta) - 1\right).$$

In the equilibrium state, it holds by (5) that $\theta =$ constant and

$$j = -\square \left[\nabla u + \frac{u}{\theta}\nabla(\varphi + \varphi_0)\right] = 0.$$

This implies

$$\nabla \left(\log u + (\varphi + \varphi_0)/\theta\right) = 0$$

and hence we have

$$u = \sigma \cdot e^{-(\varphi+\varphi_0)/\theta}$$

with the unknown constant $\sigma > 0$, which is prescribed by

$$M = \int u \qquad \text{as} \qquad \sigma = \frac{M}{\int e^{-(\varphi+\varphi_0)/\theta}}.$$

We obtain

$$u = \frac{M e^{-(\varphi+\varphi_0)/\theta}}{\int e^{-(\varphi+\varphi_0)/\theta}}$$

and hence

$$\Delta\varphi = \frac{M e^{-(\varphi+\varphi_0)/\theta}}{\int e^{-(\varphi+\varphi_0)/\theta}}$$

holds true.

Writing $v = -\varphi/\theta$, $\lambda = M/\theta$, and $K(x) = e^{-\varphi_0/\theta}$, we have

$$-\Delta v = \frac{\lambda K(x) e^v}{\int K(x) e^v} \quad \text{in} \quad , \qquad v = 0 \quad \text{on} \quad \partial \ ,$$

which is known as the Gel'fand equation ([11]), but the unknown constant $\theta > 0$ is prescribed by

$$\begin{aligned} E &= \int u \left(\theta + \varphi_0 + \frac{1}{2}\varphi\right) = \theta M + \int \Delta\varphi \left(\varphi_0 + \frac{1}{2}\varphi\right) \\ &= \theta M - \int \nabla\varphi \cdot \nabla \left(\varphi_0 + \frac{1}{2}\varphi\right) \\ &= \theta M - \frac{\theta^2}{2}\|\nabla v\|_2^2 - \frac{\lambda\theta^2 \int (K \log K)(x) e^v}{\int K(x) e^v}. \end{aligned}$$

This means, in the case of $\varphi_0 = 0$, that

$$\lambda - \frac{1}{2}\left\|\nabla v\right\|_2^2 = \frac{E}{M^2}\cdot\lambda^2$$

by $\theta = M/\lambda$. Thus, given (E, M), we get the problem to find (λ, v) satisfying

$$-\Delta v = \frac{\lambda e^v}{\int \ e^v} \quad \text{in} \quad \ , \qquad v = 0 \quad \text{on} \quad \partial \ , \tag{6}$$

and

$$\mathcal{E} = \frac{E}{M^2}\lambda^2 \qquad \text{for} \qquad \mathcal{E} = \lambda - \frac{1}{2}\left\|\nabla v\right\|_2^2 . \tag{7}$$

We shall study this problem for the case of $n \geq 3$.

2. Summary

First, we review the work [1] in short. In fact, putting

$$\sigma = \lambda / \int \ e^v \qquad \text{and} \qquad F = \sigma\left(e^v - 1\right),$$

we get from (6) the Pohozaev identity

$$n\int \ F + \frac{2-n}{2}\int \ |\nabla v|^2 = \frac{1}{2}\int_{\partial} \ \left(\frac{\partial v}{\partial \nu}\right)^2 (x\cdot\nu) \geq 0, \tag{8}$$

where it holds that

$$\begin{aligned} \int \ F &= \sigma\int \ e^v - \sigma|\ \ | = \lambda - \sigma|\ \ | \\ &= \lambda\left(1 - \tfrac{1}{X}\right) \end{aligned} \tag{9}$$

for

$$X = \frac{1}{|\ \ |}\int \ e^v .$$

Thus, we obtain

$$\frac{1}{2}\left\|\nabla v\right\|_2^2 \cdot \quad \frac{n}{n-2}\cdot\lambda\cdot\left(1 - \frac{1}{X}\right). \tag{10}$$

On the other hand, we have for

$$= \lambda e^v / \int \ e^v$$

that

$$-\Delta v = u \quad \text{in} \quad , \qquad v = 0 \quad \text{on} \quad \partial$$

and hence it follows that

$$v = \log u - \log \left(\lambda / \int e^v \right) = \log u - \log \left(\frac{\lambda}{X | \ |} \right)$$

and

$$\|\nabla v\|_2^2 = \int uv = \int u \log u - \lambda \log \left(\frac{\lambda}{X | \ |} \right).$$

Here, because $f(s) = s \log s$ is a convex function of $s > 0$, we have

$$\frac{1}{| \ |} \int f(u) \geq f \left(\frac{1}{| \ |} \int u \right) = f \left(\frac{\lambda}{| \ |} \right),$$

which means that

$$\int u \log u \geq \lambda \log \left(\frac{\lambda}{| \ |} \right). \tag{11}$$

Thus, we obtain

$$\|\nabla v\|_2^2 \geq \lambda \log X. \tag{12}$$

Inequalities (10) and (12) are now summarized as

$$\log X \cdot \ \frac{2n}{n-2} \left(1 - \frac{1}{X} \right)$$

for $X = | \ |^{-1} \int e^v$, or equivalently,

$$1 < X \cdot \ X_*(n),$$

where $X_* = X_*(n)$ denotes the unique solution to

$$\log X = \frac{2n}{n-2} \left(1 - \frac{1}{X} \right) \qquad (X > 1).$$

Thus, we obtain

$$\begin{aligned} \mathcal{E} = \ & \lambda - \frac{1}{2} \|\nabla v\|_2^2 \geq \lambda - \frac{n}{n-2} \lambda \left(1 - \frac{1}{X} \right) = \frac{1}{n-2} \left\{ \frac{n}{X} - 2 \right\} \cdot \lambda \\ \geq \ & \frac{1}{n} \left\{ \frac{n}{X_*} - 2 \right\} \cdot \lambda. \end{aligned}$$

In this way, we get the conclusion that the solution to (6) with (7) exists only if

$$\frac{\lambda}{M^2} \cdot E \geq \frac{1}{n-2} \left\{ \frac{n}{X_*} - 2 \right\}.$$

In particular, the non-positive energy solution does not exist in the case of $X_* \cdot \frac{n}{2}$, which is equivalent to

$$\log X \geq \frac{2n}{n-2} \left(1 - \frac{1}{X} \right)$$

for $X = n/2$, or $n \geq 15$.

On the other hand, the high-energy solution actually exists in the following way. First, from the theory of Crandall and Rabinowitz [3], there exists a family of solutions $\{(\sigma, v_\sigma)\}_{0<\sigma\ll 1}$ for

(13) $$-\Delta v = \sigma e^v \quad \text{in} \quad , \qquad v = 0 \quad \text{on} \quad \partial$$

satisfying

$$\lim_{\sigma\downarrow 0} \|v_\sigma\|_\infty = 0.$$

This family generates the one to (6) by

$$\lambda = \sigma \cdot \int e^{v_\sigma}.$$

Here, we have $\lambda \approx \sigma$ and $\|\nabla v_\sigma\|_2^2 \approx \sigma^2$, and it holds that

$$\frac{\mathcal{E}}{\lambda^2} = \frac{1}{\lambda} - \frac{\|\nabla v\|_2^2}{2\lambda^2} \quad \to +\infty$$

as $\sigma \downarrow 0$. Therefore, problem (6) with (7) has a solution if $E/M^2 \gg 1$.

Under those considerations, [1] conjectured that there exists $\ell_0 \in \mathbf{R}$ such that $E/M^2 > \ell_0$ and $E/M^2 < \ell_0$ imply the existence and the non-existence of the solution, respectively, in the case of $n \geq 3$, and it proved, among many other things, that this is actually the case if is a ball by the Emden transformation, and also proposed the problem of the existence of negative energy solutions.

Actually, from the known result on the Gel'fand equation it follows immediately that the set of solutions to (6) forms a smooth one-parameter family in $[0,\infty) \times C(\overline{\ })$ if is a ball, labeled by $\{(\lambda(s), v(s))\}_{s\in\mathbf{R}}$. It satisfies $\lambda(s) \to 0$ as $s \to -\infty$ and also

(14) $$v(s) \quad \to \quad \begin{cases} 0 & (s \to -\infty) \quad \text{in} \quad L^\infty(\ ) \\ 2\log\frac{1}{|x|} & (s \to +\infty) \quad \text{in} \quad W^{2,q}(\ ) \end{cases}$$

for $q \in [1, n/2)$. Therefore, the profile of the set of solutions to (13) with (14) for given ℓ is obtained from the study of

$$s \in \mathbf{R} \quad \mapsto \quad \ell(s) \equiv \frac{1}{\lambda(s)} - \frac{\|\nabla v(s)\|_2^2}{2\lambda(s)^2}.$$

Actually, refining the argument in [1], we shall show the following.

Theorem 1. *Let* $\ = B \equiv \{|x| < 1\} \subset \mathbf{R}^n$ *and* ℓ_0 *be the threshold value for the existence of the solution to* (6) *with* (7) *proven by* [1]. *Then, putting*

$$\ell_* = \frac{n-3}{2\omega_n(n-2)} \geq 0,$$

we have the following, where ω_n *denotes the surface area of* ∂B,

1. *If* $n \geq 10$, *then* $\ell_0 = \ell_*$. *For each* $\ell \equiv E/M^2 > \ell_0$, *the solution to* (6) *with* (7) *is unique, and there is no solution for* $\ell = \ell_0$. *In particular, no negative energy solution is admitted to* (6) *with* (7). *Actually, under the above parametrization* $\{(\lambda(s), v(s))\}_{s \in \mathbf{R}}$ *of the solution to* (6), *the mapping* $s \mapsto \ell(s)$ *is monotone decreasing and* $\lim_{s \to +\infty} \ell(s) = \ell_0$.

2. *If* $3 \cdot \ n \cdot \ 9$, *then* $\ell_0 < 0$ *and* $\ell = \ell_0$ *takes the slution. The solutions are infinitely many for* $\ell = \ell_*$ *and the number of solutions grows* $+\infty$ *as* $\ell \to \ell_* \pm 0$. *Actually the function* $s \mapsto \ell(s)$ *meets transversalily with the line* $\ell = \ell_*$ *infinitely many times in the above parametrization. Furthermore,* $\ell_0 = \inf_{s \in \mathbf{R}} \ell(s) < 0$ *is attained and it holds that* $\lim_{s \to +\infty} \ell(s) = \ell_*$. *In particular, negative energy solutions exist for each* $\ell \in [\ell_0, 0)$, *and in the case of* $n = 3$, *their number grows infinite as* $\ell \uparrow 0$. *Finally, the number of solutions to* (6) *with* (7) *is finite if* $n = 4$ *and* $0 < |\ell - \ell_*| \ll 1$.

From this theorem, we get the suggestion of the complex transient dynamics in low energy level to the non-stationary problem with $n = 3$.

3. Emden Transformation

First, we note that the solution to (6) or (13) is always radially symmetric as $v = v(|x|)$ in the case of $\ = B$, from the theorem of Gidas, Ni, and Nirenberg [5]. Next, the solution to (13) is classified by the Emden transformation, so that we take the orbit

$$\mathcal{O} = \{(w(t), \dot{w}(t)) \mid t \in \mathbf{R}\}$$

by

$$
(15) \qquad \begin{aligned} &\ddot{w} + (n-2)\dot{w} + 2(n-2)\left(e^{w} - 1\right) = 0 \qquad (-\infty < t < \infty) \\ &\lim_{t \to -\infty} \{w(t) - 2t\} = \lim_{t \to -\infty} e^{-t} \{\dot{w}(t) - 2\} = 0. \end{aligned}
$$

This orbit is in $\dot{w} < 2$ and is absorbed into the origin $(0,0)$ as $t \to +\infty$. It is spiral if $3 \cdot \ n \cdot \ 9$, and if $n \geq 10$, then $\mathcal{O}$ is expressed as a graph $\dot{w} = \dot{w}(w)$ with $w < 0$. In any case, the solution (σ, v) to (13) is bi-jectively associated with each point $(w(s), \dot{w}(s))$ on $\mathcal{O}$ by

$$
\begin{aligned} &\sigma = 2(n-2)e^{w(s)} \\ &v(r) = w(t) - 2t - w(s) + 2s \\ &r = e^{t-s}. \end{aligned}
$$

See Gel'fand [4], Joseph and Lundgren [6], Nagasaki and Suzuki [7], [8] for those facts and more details.

Then, the solution to (13) is recovered by

$$
\sigma = \frac{\lambda}{\int e^{v}},
$$

or

$$
\begin{aligned} \lambda(s) &= \int (-\Delta v) = -\omega_n v_r(1) = -\omega_n (\dot{w} - 2)\big|_{r=1} \\ &= \omega_n \left(2 - \dot{w}(s)\right). \end{aligned}
$$

It holds also that

$$
\begin{aligned} \|\nabla v\|_2^2 &= \omega_n \int_0^1 v_r^2 r^{n-1} dr = \omega_n \int_{-\infty}^{s} \left(\dot{w}(t) - 2\right)^2 e^{(n-2)(t-s)} dt \\ &= \omega_n \int_{-\infty}^{0} \left(\dot{w}(t+s) - 2\right)^2 e^{(n-2)t} dt = \frac{1}{\omega_n} \int_{-\infty}^{0} \lambda(t+s)^2 e^{(n-2)t} dt, \end{aligned}
$$

and hence we get that

$$
(16) \qquad \begin{aligned} \ell(s) &= \frac{1}{\lambda(s)} \left\{ 1 - \frac{1}{2\omega_n \lambda(s)} \int_{-\infty}^{0} \lambda(t+s)^2 e^{(n-2)t} dt \right\} \\ \lambda(s) &= \omega_n \left(2 - \dot{w}(s)\right). \end{aligned}
$$

From the second expression of (16) and the profile of $\mathcal{O}$, the bifurcation diagram of the solution set to (6) is taken clearly. It is actually very similar to the one for (6), including infinite and no bendings for $3 \cdot \ n \cdot \ 9$ and $n \geq 10$, respectively, with $\lambda(s) \to 0$ and $\|v(s)\|_\infty \to 0$ as $s \to -\infty$ and (14). The difference is the singular value and it holds actually that

$$\lim_{s\to+\infty} \lambda(s) = 2\omega_n \tag{17}$$

by $\lim_{s\to+\infty}(w(s), \dot{w}(s)) = (0,0)$. Henceforth, the first bending point of the above diagram is denoted by $(\lambda(s_0), v(s_0))$ for $s_0 \in \mathbf{R}$ in the case of $3 \leq n \leq 9$. If $n \geq 10$, we set $s_0 = +\infty$.

Then, putting $\lambda_0 = \lim_{s\to s_0} \lambda(s)$, we have

$$\lambda(s) \in (0, \lambda_0] \quad \text{for} \quad s \in \mathbf{R}, \qquad \lim_{s\to-\infty} \lambda(s) = 0,$$

and

$$\frac{\lambda(t+s)}{\lambda(s)} \leq 1 \qquad \text{for} \qquad s < s_0 \quad \text{and} \quad t \leq 0.$$

Those relations imply the existence of the high-energy solution to (6) with (7) again. In fact, it follows from the dominated convergence theorem that

$$\ell(s) \geq \frac{1}{\lambda(s)} \left\{ 1 - \frac{1}{2\omega_n} \int_{-\infty}^0 \lambda(t+s) e^{(n-2)t} dt \right\} \quad \to \quad +\infty \tag{18}$$

as $s \to -\infty$. Similarly, from (17) we obtain

$$\lim_{s\to+\infty} \ell(s) = \frac{1}{2\omega_n} \left\{ 1 - \frac{4\omega_n^2}{2\omega_n \cdot 2\omega_n} \cdot \int_{-\infty}^0 e^{(n-2)t} dt \right\} = \ell_*,$$

which implies that $\ell_0 \leq \ell_*$.

Letting

$$I(s) = \frac{1}{2\omega_n} \int_{-\infty}^0 \lambda(t+s)^2 e^{(n-2)t} dt,$$

we have

$$\lambda(s)^2 \ell(s) = \lambda(s) - I(s),$$

and then

$$\lambda^2 \dot{\ell} = (1 - 2\lambda\ell)\dot{\lambda} - \dot{I}$$

follows. If $n \geq 10$, it holds always that $\dot{\lambda}(s) > 0$, and hence $\dot{\ell} < 0$ if $\ell > 1/2\lambda$, or equivalently, $I < \lambda/2$. This is actually the case as

$$I(s) < \frac{1}{2\omega_n} \cdot \lambda(s) \cdot 2\omega_n \int_{-\infty}^0 e^{(n-2)t} dt = \frac{\lambda(s)}{n-2} < \frac{\lambda(s)}{2}.$$

Thus, the mapping $s \in \mathbf{R} \mapsto \ell(s)$ is monotone increasing in this case of $n \geq 10$, and the proof of the first part of Theorem 1 is complete.

4. Low Dimensional Case

Now, we show the second part of Theorem 1, where $3 \cdot \ n \cdot \ 9$. For this purpose, we make use of the Pohozaev identity (8). In fact, if $\ = B$, then $| \ | = \omega_n/n$ and it holds that

$$\int_{\partial} \left(\frac{\partial v}{\partial \nu}\right)^2 (x \cdot \nu) = \omega_n v_r(1)^2 = \frac{\lambda^2}{\omega_n}.$$

On the other hand, we have from (8) that

$$\frac{n-2}{2} \|\nabla v\|_2^2 = n\lambda - \sigma\omega_n - \frac{\lambda^2}{2\omega_n}.$$

Thus, we obtain

$$\begin{aligned} \mathcal{E} = \ & \lambda - \frac{\|\nabla v\|_2^2}{2} = \frac{\omega_n \sigma}{n-2} - \frac{2\lambda}{n-2} + \frac{\lambda^2}{2(n-2)\omega_n} \\ = \ & 2\omega_n e^{w(s)} - \frac{2\omega_n (2 - \dot{w}(s))}{n-2} + \frac{\lambda^2}{2(n-2)\omega_n}, \end{aligned}$$

or equivalently,

$$\begin{aligned} \frac{\mathcal{E}}{\lambda^2} - \frac{1}{2(n-2)\omega_n} &= \frac{2\omega_n}{n-2} \left\{ (n-2)e^{w(s)} - 2 + \dot{w}(s) \right\} / \omega_n^2 (2 - \dot{w}(s))^2 \\ &= \frac{2}{\omega_n (n-2)} \cdot G(s), \end{aligned}$$

where

$$G(s) = \frac{(n-2)e^{w(s)} - 2 + \dot{w}(s)}{(2 - \dot{w}(s))^2}.$$

Let us confirm that

$$\lim_{s \to +\infty} G(s) = \frac{n-4}{4}$$

and

$$\ell(s) = \frac{2}{(n-2)\omega_n} \left\{ G(s) + \frac{1}{4} \right\}.$$

Therefore, it follows from (18) that

$$\lim_{s \to -\infty} G(s) = +\infty.$$

Given $\mu \in \mathbf{R}$, we now study the equation $G(s) = \mu$. In fact, $G = \mu$ is equivalent to

$$(n-2)e^{w} - 2 + \dot{w} = \mu(2-\dot{w})^2, \tag{19}$$

which defines curves in $w - \dot{w}$ plane. If $\mu = 0$, it is realized as the graph $\dot{w} = 2 - (n-2)e^{w}$, denoted by $\mathcal{S}_0$, which is monotone decreasing in $w \in \mathbf{R}$ and approaches $\dot{w} = 2$ as $t \to -\infty$. If $\mu > 0$, it has two components and the one in $\dot{w} < 2$, denoted by $\mathcal{S}_\mu$ is given by

$$\dot{w} = 2 + \frac{1 - \sqrt{1 + 4\mu(n-2)e^{w}}}{2\mu}.$$

It is monotone decreasing in $w \in \mathbf{R}$ and approaches $\dot{w} = 2$ as $w \to -\infty$. More precisely, it holds that

$$\begin{aligned} \dot{w} \sim \quad & 2 + \frac{1}{2\mu} - \frac{1}{2\mu}\left(1 + 2\mu(n-2)e^{w}\right) \\ = \quad & 2 - (n-2)e^{w} \end{aligned} \tag{20}$$

as $w \to -\infty$. In the case of $\mu < 0$, it is connected and is expressed as

$$\dot{w} = 2 + \frac{1 \pm \sqrt{1 + 4\mu(n-2)e^{w}}}{2\mu}$$

with $w \cdot \ w_*(\mu) = \log\left(-\frac{1}{4\mu(n-2)}\right)$. It is again denoted by $\mathcal{S}_\mu$. Actually, it is in $2 + 1/\mu < \dot{w} < 2$ and $w \cdot \ w_*(\mu)$, and the upper part has the asymptotics (20) as $w \to -\infty$.

On the other hand, for $(w(t), \dot{w}(t)) \in \mathcal{O}$ we have from (15) that

$$w(t) = 2t - 2(n-2)\int_{-\infty}^{t} e^{-(n-2)\eta} d\eta \int_{-\infty}^{\eta} e^{(n-2)\xi}\left(e^{w(\xi)} - 1\right) d\xi,$$

which implies that

$$\dot{w} \sim 2 - \frac{2(n-2)}{n} e^{w}$$

as $w \to -\infty$ by $w \sim 2t$. Thus, $\mathcal{O}$ is in the above of $\mathcal{S}_\mu$ in $w - \dot{w}$ plane for $w \ll -1$.

The solution to (6) with (7) for

$$\frac{E}{M^2} = \frac{2}{(n-2)\omega_n}\left(\mu + \frac{1}{4}\right)$$

corresponds bi-jectively to $(w(s), \dot{w}(s)) \in \mathcal{O} \cap \mathcal{S}_\mu$ satisfying $G(s) = \mu$. To get the profile of $\mathcal{S}_\mu$ for $\mu \ll -1$, we note that $\lim_{\mu \to -\infty} w_*(\mu) = -\infty$ and

$\lim_{\mu \to -\infty} \left(2 + \frac{1}{2\mu}\right) = 2$ hold. Then, we see $\mathcal{O} \cap \mathcal{S}_\mu = \emptyset$ in this case of $\ell \ll -1$. We also see that $\ell_0 = \inf_{s \in \mathbf{R}} \ell(s)$ is attained and $\ell_0 < \ell_* = \lim_{s \to} \ell(s)$, and hence

(21) $$\ell_0 = \inf_{s \in \mathbf{R}} \ell(s) < \lim_{s \to +\infty} \ell(s) = \ell_* < \lim_{s \to -\infty} \ell(s) = +\infty$$

holds true. We see that $\ell = 0$ corresponds to $\mu = -1/4$, and then $\mathcal{S}_\mu$ takes $\dot{w} = \pm 2$ as the asymptotic lines for $w \to -\infty$ and $(\log(n-2), 0)$ is its vertex. Because $\mathcal{O}$ is in the above of $\mathcal{S}_\mu$ in $w - \dot{w}$ plane for $w \to -\infty$, they meet each other for $\mu = -1/4$. This is also the case with $\left|\mu + \frac{1}{4}\right| \ll 1$ and it holds that $\ell_0 < 0$.

On the other hand, the case $0 \in \mathcal{S}_\mu$ occurs if and only if $\mu = \frac{n-4}{4}$, and then $\sharp\,(\mathcal{S}_\mu \cap \mathcal{O}) = +\infty$ follows because $\mathcal{O}$ is absorbed into $(w, \dot{w}) = (0, 0)$ spirally. Therefore, there is $s_k \to +\infty$ such that $G(s_k) = \frac{n-4}{4}$, and furthermore, we can show that infinitely many of those crossing points are transversal. In fact, we have

$$\frac{d}{dt} \begin{pmatrix} w \\ \dot{w} \end{pmatrix} = \begin{pmatrix} \dot{w} \\ -(n-2)\left\{\dot{w} + 2\left(e^w - 1\right)\right\} \end{pmatrix}$$

on $\mathcal{O}$ and the tangential vector of $\mathcal{O}$ is positive and negative in w direction if and only if $\dot{w} > 0$ and $\dot{w} < 0$, respectively. Also, it is positive and negative in $\dot{w}$ direction if and only if $\dot{w} < 2 - 2e^w$ and $\dot{w} > 2 - 2e^w$, respectively. From those facts, we see that if $n = 3$, then the crossing points of $\mathcal{O} \cap \mathcal{S}_{(n-4)/4}$ in $\dot{w} < 0$ are infinitely many and transversal. In the case of $n = 4$, $\mathcal{S}_{(n-4)/4}$ coincides with $\dot{w} = 2 - 2e^w$ and any of those crossing points are transversal. Finally, if $n \geq 5$, then $\mathcal{S}_{(n-4)/4}$ is above in $w - \dot{w}$ plane near $(w, \dot{w}) = (0, 0)$ in $\dot{w} < 0$ and any crossing point in that region is transversal. Thus, $G'(s_k) \neq 0$ holds for infinitely many s_k's, and the number of the solution to $G(s) = \mu$ becomes infinite as $\mu \to (n-4)/4 \pm 0$. The same situation arises for $\ell(s) = \ell$ with $\ell \to \ell_* \pm 0$.

For the solutions to $\ell(s) = \ell_*$ to be at most finite for $\ell \neq \ell_*$, it must hold that $G'(s_k) \neq 0$ for any k sufficiently large in the above notation, which, is open except for $n = 4$. However, in this case of $n = 4$, $G'(s_k) \neq 0$ holds for any k, and this, together with (21), implies the finiteness of the solution for $0 < |\ell - \ell_*| \ll 1$. Then, the proof is complete.

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Takashi Suzuki
Department of Mathematical Science,
Graduate School of Engineering Science,
Osaka University,
Osaka, Japan