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NON-CENTRAL MATRIX-VARIATE DIRICHLET DISTRIBUTION

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Abstract. Let $X_i \sim W_p(n_i, \Sigma, \Theta)$ where $\Theta = \text{diag}(\theta_i^2, 0, \ldots, 0)$, $i = 1, \ldots, r+1$. In this article the authors have derived the joint distribution of $U_i = C^{-1} X_i C'^{-1}$, $i = 1, \ldots, r$ where $\sum_{i=1}^{r+1} X_i = CC'$ and C is a lower triangular matrix. The joint distribution of $U_1, \ldots, U_r$ is a non-central matrix-variate Dirichlet distribution. Several properties of this distribution such as marginal and conditional distributions, distribution of partial sums, moments and asymptotic results have also been studied.

1. Introduction

The multivariate statistical analysis heavily depends upon multivariate normal distribution. Therefore, the distribution of sample sum of squares and crossproducts matrix, which has a Wishart distribution, plays an important role in almost all inferential procedures. A distribution closely connected to the Wishart, known as 'matrix-variate beta' was introduced by Prof. P. L. Hsu while studying distribution of roots of certain determinantal equation. The matrix-variate beta distribution arises in various problems in multivariate statistical analysis. Several test statistics in multivariate analysis of variance and covariance are functions of beta matrix. In Bayesian analysis, this distribution and some of its properties are utilized in preposterior analysis of parameters of normal multivariate regression models.

An extension of the matrix-variate beta distribution is the "matrix-variate Dirichlet distribution", which is useful in several testing problems in multivariate statistical analysis (Troskie [20]). For example, the likelihood ratio test statistic for testing homogeneity of several multivariate normal distributions is a function of Dirichlet matrices.

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In this article, we derive a non-central matrix-variate Dirichlet distribution. In Section 2, we give certain known definitions and results that are used to derive the main results. The non-central matrix-variate Dirichlet distribution has been derived in Section 3. Section 4 deals with certain properties and asymptotic expansion of this distribution.

2. Some Useful Results

In this section we give definitions and results that will be used in the subsequent sections. The generalized hypergeometric functions of one and several variables will be used to derive the density function, marginal and conditional distributions and several moment expressions of random matrices which are jointly distributed as non-central matrix-variate Dirichlet. Throughout this work we will use the Pochammer symbol $(a)_n$ defined by $(a)_n = a(a+1)\cdots(a+n-1) = (a)_{n-1}(a+n-1)$ for $n = 1, 2, \ldots$, and $(a)_0 = 1$.

The generalized hypergeometric function of scalar argument is defined by

$$
{}_pF_q\left(a_1, \ldots, a_p; b_1, \ldots, b_q; z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}, \tag{2.1}
$$

where a_i, $i = 1, \ldots, p$; b_j, $j = 1, \ldots, q$ are complex numbers with suitable restrictions and z is a complex variable. Conditions for the convergence of the series in (2.1) are available in the literature, see Luke [13]. From (2.1) it is easy to see that ${}_0F_1(b; x) = \sum_{k=0}^{\infty} \frac{x^k}{(b)_k k!}$ and ${}_1F_1(a; b; x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{x^k}{k!}$.

Next, we will define confluent hypergeometric and generalized Kampé de Fériet's functions of several variables. For further results and properties of these functions the reader is referred to Srivastava and Kashyap [17, Section II.7] and Srivastava and Karlsson [16, Section 1.4].

The confluent hypergeometric function in m variables $z_1, \ldots, z_m$ is defined by

$$
\mathbf{\Psi}_2^{(m)}[a; c_1, \ldots, c_m; z_1, \ldots, z_m] = \sum_{j_1, \cdots, j_m = 0}^{\infty} \frac{(a)_{j_1 + \cdots + j_m} z_1^{j_1} \cdots z_m^{j_m}}{(c_1)_{j_1} \cdots (c_m)_{j_m} j_1! \cdots j_m!} \tag{2.2}
$$

where the series expansion is valid for all $z_i \in \mathbb{R}$. Using the results

$$
(a)_j = \frac{\Gamma(a+j)}{\Gamma(a)} = \frac{1}{\Gamma(a)} \int_0^{\infty} \exp(-t) t^{a+j-1} \, dt, \mathrm{Re}(a) > 0,
$$

for $j = 0, 1, 2, \ldots$, and $\sum_{j_i=0}^{\infty} \frac{(tz_i)^{j_i}}{(c_i)_{j_i} j_i!} = {}_0F_1(c_i; tz_i)$ in (2.2), one obtains

$$
\mathbf{\Psi}_2^{(m)}[a; c_1, \ldots, c_m; z_1, \ldots, z_m] = \frac{1}{\Gamma(a)} \int_0^{\infty} \exp(-t) t^{a-1} \prod_{i=1}^{m} {}_0F_1(c_i; tz_i) \, dt. \tag{2.3}
$$

For $m = 1$, the function $\mathbf{\Psi}_2^{(m)}$ reduces to the confluent hypergeometric function ${}_1F_1$. For $m = 2$, $\mathbf{\Psi}_2^{(m)} = \mathbf{\Psi}_2$ is the Humbert's confluent hypergeometric function and (2.2) slides to

$$\begin{aligned}\mathbf{\Psi}_2[a; c_1, c_2; z_1, z_2] &= \sum_{j_1=0}^{\infty} \frac{(a)_{j_1} z_1^{j_1}}{(c_1)_{j_1} j_1!}\, {}_1F_1(a + j_1; c_2; z_2)\\ &= \sum_{j_2=0}^{\infty} \frac{(a)_{j_2} z_2^{j_2}}{(c_2)_{j_2} j_2!}\, {}_1F_1(a + j_2; c_1; z_1).\end{aligned} \tag{2.4}$$

The generalized Kampé de Fériet's function in m variables $z_1, \ldots, z_m$ is defined as follows (Srivastava and Panda [18, p. 1127]):

$$\begin{aligned}&F^{\mu:\nu_1;\cdots;\nu_m}_{\rho:\sigma_1;\cdots;\sigma_m}\left[\begin{matrix} a_1, \ldots, a_\mu : b_{11}, \ldots, b_{1\nu_1}; \cdots; b_{m1}, \ldots, b_{m\nu_m};\\ c_1, \ldots, c_\rho : d_{11}, \ldots, d_{1\sigma_1}; \cdots; d_{m1}, \ldots, d_{m\sigma_m};\end{matrix}\ z_1, \ldots, z_m\right]\\ &= \sum_{j_1,\cdots,j_m=0}^{\infty} \frac{\prod_{i=1}^{\mu}(a_i)_{j_1+\cdots+j_m} \prod_{i=1}^{\nu_1}(b_{1i})_{j_1} \cdots \prod_{i=1}^{\nu_m}(b_{mi})_{j_m} z_1^{j_1} \cdots z_m^{j_m}}{\prod_{i=1}^{\rho}(c_i)_{j_1+\cdots+j_m} \prod_{i=1}^{\sigma_1}(d_{1i})_{j_1} \cdots \prod_{i=1}^{\sigma_m}(d_{mi})_{j_m}\, j_1! \cdots j_m!}\end{aligned} \tag{2.5}$$

where the series is convergent if $\mu + \nu_k < \rho + \sigma_k + 1, k = 1, \ldots, m$ or $\mu + \nu_k = \rho + \sigma_k + 1, k = 1, \ldots, m$ with either $\mu > \rho$ and $|z_1|^{1/(\mu-\rho)} + \cdots + |z_m|^{1/(\mu-\rho)} < 1$ or $\mu \cdot \rho$ and $\max\{|z_1|, \ldots, |z_m|\} < 1$. Further generalization of the multivariable generalized Kampé de Fériet's function, which is referred to in the literature as generalized Lauricella function, is due to Srivastava and Daoust [15, p. 454]. It may be recorded here that under certain conditions the generalized Kampé de Fériet's function reduces to Lauricella functions F_A, F_B, F_C, F_D and generalized hypergeometric function of one variable. For $\rho = \nu_1 = \cdots = \nu_m = 0$ and $\mu = \sigma_1 = \cdots = \sigma_m = 1$, the generalized Kampé de Fériet's function reduces to the confluent hypergeometric function of several variables. Substituting $\mu = \rho = \nu_1 = \cdots = \nu_m = \sigma_1 = \cdots = \sigma_m = 1$ in (2.5), the generalized Kampé de Fériet's function in m variables $z_1, \ldots, z_m$ simplifies to

$$\begin{aligned}&F^{1:1;\cdots;1}_{1:1;\cdots;1}\left[\begin{matrix} a : b_1; \cdots; b_m;\\ c : d_1; \cdots; d_m;\end{matrix}\ z_1, \ldots, z_m\right]\\ &= \sum_{j_1,\cdots,j_m=0}^{\infty} \frac{(a)_{j_1+\cdots+j_m} (b_1)_{j_1} \cdots (b_m)_{j_m} z_1^{j_1} \cdots z_m^{j_m}}{(c)_{j_1+\cdots+j_m} (d_1)_{j_1} \cdots (d_m)_{j_m}\, j_1! \cdots j_m!}.\end{aligned} \tag{2.6}$$

If $b_1 = d_1, \ldots, b_{m-1} = d_{m-1}$, then (2.6) reduces to

$$F^{1:1;\cdots;1}_{1:1;\cdots;1}\left[\begin{array}{l} a: b_1;\cdots;b_{m-1};b_m; \\ c: b_1;\cdots;b_{m-1};d_m; \end{array} z_1,\dots,z_m\right]$$

$$= F^{1:0;1}_{1:0;1}\left[\begin{array}{l} a: -;b_m; \\ c: -;d_m; \end{array} \textstyle\sum_{i=1}^{m-1} z_i, z_m\right] \tag{2.7}$$

$$= \sum_{j=0}^{\infty} \frac{(a)_j (b_m)_j z_m^j}{(c)_j (d_m)_j j!}\, {}_1F_1\Big(a+j; c+j; \sum_{i=1}^{m-1} z_i\Big).$$

Next we will give a result which has been used in Section 4 to derive moment expression.

Lemma 2.1. *For* $\mathrm{Re}(\alpha_i) > 0$, $i = 1,\dots,m$ *and* $\mathrm{Re}(\beta) > 0$,

(2.8)

$$\underset{\substack{0<x_i<1 \\ 0<\sum_{i=1}^m x_i<1}}{\int\cdots\int} \prod_{i=1}^m x_i^{\alpha_i-1}\Big(1-\sum_{i=1}^m x_i\Big)^{\beta-1}$$

$$\times \mathbf{\Psi}_2^{(m+1)}\left[a; c_1,\dots,c_{m+1}; \delta_1 x_1,\dots,\delta_m x_m, \delta_{m+1}\Big(1-\sum_{i=1}^m x_i\Big)\right] dx_1\cdots dx_m$$

$$= \frac{\prod_{i=1}^m \Gamma(\alpha_i)\Gamma(\beta)}{\Gamma(\sum_{i=1}^m \alpha_i+\beta)} F^{1:1;\cdots;1}_{1:1;\cdots;1}\left[\begin{array}{rl} a: & \alpha_1;\cdots;\alpha_m;\beta; \\ \sum_{i=1}^m \alpha_i+\beta: & c_1;\cdots;c_m;c_{m+1}; \end{array} \delta_1,\dots,\delta_{m+1}\right]$$

where $\mathbf{\Psi}_2^{(m+1)}$ *and* $F^{1:1;\cdots;1}_{1:1;\cdots;1}$ *are the confluent hypergeometric and Kampé de Fériet's functions of several variables respectively.*

Proof. Expanding $\mathbf{\Psi}_2^{(m+1)}$ using (2.2), integrating $x_1\dots,x_m$ with the help of Dirichlet integral and using (2.6) we get the desired result. ■

The matrix-variate distributions such as Wishart, non-central Wishart, beta and Dirichlet involve multivariate gamma function. Since we will be defining and using these distributions to derive our distributional results, it will not be out of context to define multivariate gamma function. The multivariate gamma function, denoted by $\Gamma_p(a)$, is defined as

$$\Gamma_p(a) = \int_{A>0} \mathrm{etr}(-A)\det(A)^{a-\frac{p+1}{2}}\, dA, \tag{2.9}$$

where $\mathrm{Re}(a) > \frac{p-1}{2}$, and the integral is over the space of $p\times p$ symmetric positive definite matrices. By evaluating the integral in (2.9), the multivariate gamma function can be expressed as product of ordinary gamma functions

$$\Gamma_p(a) = \pi^{\frac{p(p-1)}{4}} \prod_{i=1}^{p} \Gamma\Big(a - \frac{i-1}{2}\Big), \mathrm{Re}(a) > \frac{p-1}{2}. \tag{2.10}$$

Finally, we define Wishart and non-central Wishart distributions and state some of their properties. These definitions and results have been taken from Gupta and Nagar [7, Chapter 3].

Definition 2.1. A $p \times p$ symmetric positive definite random matrix S is said to have Wishart distribution with parameters p, $n(\geq p)$ and $\Sigma\,(p \times p) > 0$, denoted by $S \sim W_p(n, \Sigma)$, if its p.d.f. is given by

$$\Big\{2^{\frac{np}{2}}\Gamma_p\Big(\frac{n}{2}\Big)\det(\Sigma)^{\frac{n}{2}}\Big\}^{-1} \mathrm{etr}\Big(-\frac{\Sigma^{-1}S}{2}\Big)\det(S)^{\frac{n-p-1}{2}},\ S > 0. \tag{2.11}$$

Definition 2.2. A $p \times p$ symmetric positive definite random matrix S is said to have a non-central Wishart distribution with parameters p, $n(\geq p)$, $\Sigma\,(p \times p) > 0$ and Θ, denoted by $S \sim W_p(n, \Sigma, \Theta)$, if its p.d.f. is given by

$$\begin{aligned} &\Big\{2^{\frac{np}{2}}\Gamma_p\Big(\frac{n}{2}\Big)\det(\Sigma)^{\frac{n}{2}}\Big\}^{-1} \mathrm{etr}\Big(-\frac{\Theta}{2}\Big)\,\mathrm{etr}\Big(-\frac{\Sigma^{-1}S}{2}\Big) \\ &\times \det(S)^{\frac{n-p-1}{2}}\,{}_0F_1\Big(\frac{n}{2}; \frac{\Theta\Sigma^{-1}S}{4}\Big),\ S > 0 \end{aligned} \tag{2.12}$$

where ${}_0F_1$ is the Bessel function of matrix argument.

For $\Theta = 0$, the non-central Wishart distribution reduces to Wishart distribution. Further, when $\Sigma = I_p$ and $\Theta = \mathrm{diag}(\theta^2, 0, \ldots, 0)$, the p.d.f. of $S = (s_{ij})$ simplifies to

$$\Big\{2^{\frac{np}{2}}\Gamma_p\Big(\frac{n}{2}\Big)\Big\}^{-1} \exp\Big(-\frac{\theta^2 + \mathrm{tr}\,S}{2}\Big)\det(S)^{\frac{n-p-1}{2}}\,{}_0F_1\Big(\frac{n}{2}; \frac{\theta^2 s_{11}}{4}\Big), \tag{2.13}$$

where $S > 0$, $n \geq p$ and ${}_0F_1$ is the Bessel function of scalar argument.

Theorem 2.1. *Let $S \sim W_p(n, \Sigma, \Theta)$. Partition S, Σ and Θ as*

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}, \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, \Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix}$$

where S_{11}, Σ_{11} and Θ_{11} are $q \times q$ matrices. Then, $S_{11} \sim W_q(n, \Sigma_{11}, \Theta_{11})$.

Theorem 2.2. *Let $S_1, \ldots, S_r$ be independent random matrices, $S_i \sim W_p(n_i, \Sigma, \Theta_i)$, $i = 1, \ldots, r$. Then, $\sum_{i=1}^{r} S_i \sim W_p(\sum_{i=1}^{r} n_i, \Sigma, \sum_{i=1}^{r} \Theta_i)$.*

3. Non-Central Matrix-Variate Dirichlet Distribution

Let $X_1, \ldots, X_{r+1}$ be independent symmetric positive definite random matrices of order p. Define the transformation $\sum_{i=1}^{r+1} X_i = CC'$ and $U_i = C^{-1} X_i C'^{-1}$, $i = 1, \ldots, r$ where the matrix C is lower triangular with positive diagonal elements. If $X_i \sim W_p(n_i, \Sigma)$, $i = 1, \ldots, r+1$, then the joint distribution of $U_1, \ldots, U_r$ is matrix-variate Dirichlet (Olkin and Rubin [14]). Further, if $X_{r+1} \sim W_p(n_{r+1}, \Sigma, \Theta)$ and $X_i \sim W_p(n_i, \Sigma)$, $i = 1, \ldots, r$, then the random matrices $U_1, \ldots, U_r$ follow a non-central matrix-variate Dirichlet distribution (Asoo [2], Troskie [19], De Waal [3] and Gupta and Nagar [6]).

In this section we will derive the joint probability density function of $U_1, \ldots, U_r$ when each X_i has a non-central Wishart distribution of rank one.

Theorem 3.1. *Let $X_1, \ldots, X_{r+1}$ be independent symmetric positive definite random matrices, $X_i \sim W_p(n_i, \Sigma, \Theta_i)$ where $\Theta_i = \operatorname{diag}(\theta_i^2, 0, \ldots, 0)$, $i = 1, \ldots, r+1$. Define $\sum_{i=1}^{r+1} X_i = CC'$ and $X_i = CU_iC'$, $i = 1, \ldots, r$ where the matrix C is lower triangular with positive diagonal elements. Then, the joint p.d.f. of $U_1, \ldots, U_r$ is given by*

$$\frac{\Gamma_p(\frac{1}{2}\sum_{i=1}^{r+1} n_i)}{\prod_{i=1}^{r+1}\Gamma_p(\frac{1}{2}n_i)} \exp\Big(-\frac{\sum_{i=1}^{r+1}\theta_i^2}{2}\Big)\prod_{i=1}^{r}\det(U_i)^{\frac{n_i-p-1}{2}}\det\Big(I_p - \sum_{i=1}^{r} U_i\Big)^{\frac{n_{r+1}-p-1}{2}}$$

$$\times \mathbf{\Psi}_2^{(r+1)}\Big[\frac{\sum_{i=1}^{r+1} n_i}{2}; \frac{n_1}{2}, \ldots, \frac{n_{r+1}}{2}; \frac{\theta_1^2 u_{111}}{2}, \ldots, \frac{\theta_r^2 u_{11r}}{2}, \frac{\theta_{r+1}^2(1-\sum_{i=1}^{r} u_{11i})}{2}\Big],$$

$$0 < U_i < I_p, i = 1, \ldots, r, \sum_{i=1}^{r} U_i < I_p,$$

where $U_i = (u_{\alpha\beta i}), i = 1, \ldots, r$ and $\mathbf{\Psi}_2^{(r+1)}$ is the confluent hypergeometric function in $r+1$ variables.

Proof. The random matrix U_i is invariant under the transformation $X_i \to \Sigma^{-\frac{1}{2}} X_i (\Sigma^{-\frac{1}{2}})'$, where $\Sigma^{-\frac{1}{2}}$ is a lower triangular matrix such that $\Sigma^{\frac{1}{2}}(\Sigma^{\frac{1}{2}})' = \Sigma$. Hence, we can assume with out loss of generality that $\Sigma = I_p$, that is, $X_i \sim W_p(n_i, I_p, \Theta_i)$. Using independence and (2.13) the joint p.d.f. of $X_1, \ldots, X_{r+1}$ is given by

$$\prod_{i=1}^{r+1}\Big\{2^{\frac{n_i p}{2}}\Gamma_p\Big(\frac{n_i}{2}\Big)\Big\}^{-1}\exp\Big(-\frac{\sum_{i=1}^{r+1}\theta_i^2}{2}\Big)\operatorname{etr}\Big(-\frac{\sum_{i=1}^{r+1} X_i}{2}\Big)$$

$$\times\prod_{i=1}^{r+1}\det(X_i)^{\frac{n_i-p-1}{2}}\prod_{i=1}^{r+1}{}_0F_1\Big(\frac{n_i}{2}; \frac{\theta_i^2 x_{11i}}{4}\Big),\ X_i > 0,\ n_i \geq p, i = 1, \ldots, r.$$

Making the transformation $\sum_{i=1}^{r+1} X_i = CC'$, and $X_i = CU_iC'$, $i = 1, \dots, r$ where $C = (c_{ij})$ is a lower triangular matrix, $c_{ii} > 0$, with the Jacobian $J(X_1, \dots, X_{r+1} \to U_1, \dots, U_r, C) = 2^p \prod_{i=1}^p c_{ii}^{(p+1)(r+1)-i}$ and integrating with respect to $c_{ij}, 1 \cdot\ j \cdot\ i \cdot\ p$, we get the joint density of $U_1, \dots, U_r$ as

$$
\begin{aligned}
&2^p \prod_{i=1}^{r+1} \Big\{ 2^{\frac{n_i p}{2}} \Gamma_p\Big(\frac{n_i}{2}\Big)\Big\}^{-1} \exp\Big(-\frac{\sum_{i=1}^{r+1}\theta_i^2}{2}\Big) \\
&\times \prod_{i=1}^r \det(U_i)^{\frac{n_i-p-1}{2}} \det\Big(I_p - \sum_{i=1}^r U_i\Big)^{\frac{n_{r+1}-p-1}{2}} I_1 I_2 \prod_{i=2}^p I_{3i},
\end{aligned} \tag{3.1}
$$

where $0 < U_i < I_p, i = 1, \dots, r, \sum_{i=1}^r U_i < I_p$. Further, using results on integration and (2.3), it is easy to see that

$$
\begin{aligned}
I_1 &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\Big(-\frac{\sum_{i>j}^p c_{ij}^2}{2}\Big) \prod_{i>j}^p dc_{ij} = (\sqrt{2\pi})^{\frac{p(p-1)}{2}}, \\
I_2 &= \int_0^{\infty} \exp\Big(-\frac{c_{11}^2}{2}\Big) c_{11}^{\sum_{i=1}^{r+1} n_i - 1} \prod_{i=1}^r {}_0F_1\Big(\frac{n_i}{2}; \frac{\theta_i^2 c_{11}^2 u_{11i}}{4}\Big) \\
&\quad \times {}_0F_1\Big(\frac{n_{r+1}}{2}; \frac{\theta_{r+1}^2 c_{11}^2 (1 - \sum_{i=1}^r u_{11i})}{4}\Big) dc_{11} = 2^{\frac{\sum_{i=1}^{r+1} n_i}{2} - 1} \Gamma\Big(\frac{\sum_{i=1}^{r+1} n_i}{2}\Big) \\
&\quad \times \mathbf{\Psi}_2^{(r+1)}\Big[\frac{\sum_{i=1}^{r+1} n_i}{2}; \frac{n_1}{2}, \dots, \frac{n_{r+1}}{2}; \frac{\theta_1^2 u_{111}}{2}, \dots, \frac{\theta_r^2 u_{11r}}{2}, \frac{\theta_{r+1}^2 (1 - \sum_{i=1}^r u_{11i})}{2}\Big], \\
I_{3i} &= \int_0^{\infty} \exp\Big(-\frac{c_{ii}^2}{2}\Big) c_{ii}^{\sum_{i=1}^{r+1} n_i - i} dc_{ii} = 2^{\frac{\sum_{i=1}^{r+1} n_i - i - 1}{2}} \Gamma\Big(\frac{\sum_{i=1}^{r+1} n_i - i + 1}{2}\Big).
\end{aligned}
$$

Finally, substituting I_1, I_2 and I_{3i} in (3.1) and simplify the resulting expression, we obtain the desired result. ■

If $(U_1, \dots, U_r)$ has the p.d.f. given in Theorem 3.1, then we will write

$$
(U_1, \dots, U_r) \sim D_p^I\Big(\frac{n_1}{2}, \dots, \frac{n_r}{2}; \frac{n_{r+1}}{2}; \theta_1^2, \dots, \theta_r^2; \theta_{r+1}^2\Big).
$$

Corollary 3.1.1. *Let $X_1, \dots, X_{r+1}$ be independent random matrices, $X_i \sim W_p(n_i, \Sigma), i = 1, \dots, r$ and $X_{r+1} \sim W_p(n_{r+1}, \Sigma, \Theta)$ where $\Theta = \operatorname{diag}(\theta^2, 0, \dots, 0)$. Define $\sum_{i=1}^{r+1} X_i = CC'$ and $X_i = CU_iC'$, $i = 1, \dots, r$ where the matrix C is lower triangular with positive diagonal elements. Then, the joint p.d.f. of $U_1, \dots, U_r$ is given by*

$$
\begin{aligned}
&\frac{\Gamma_p(\frac{1}{2}\sum_{i=1}^{r+1} n_i)}{\prod_{i=1}^{r+1} \Gamma_p(\frac{1}{2} n_i)} \exp\Big(-\frac{\theta^2}{2}\Big) \prod_{i=1}^r \det(U_i)^{\frac{n_i-p-1}{2}} \det\Big(I_p - \sum_{i=1}^r U_i\Big)^{\frac{n_{r+1}-p-1}{2}} \\
&\times {}_1F_1\Big(\frac{\sum_{i=1}^{r+1} n_i}{2}; \frac{n_{r+1}}{2}; \frac{\theta^2 (1 - \sum_{i=1}^r u_{11i})}{2}\Big),
\end{aligned}
$$

where $0 < U_i < I_p$, $i = 1, \ldots, r$, $\sum_{i=1}^{r} U_i < I_p$ *and* ${}_1F_1$ *is the confluent hypergeometric function.*

The above density function was first derived by Troskie [19]. The joint distribution of $(U_1, \ldots, U_r)$ in this case is called "linear non-central matrix-variate Dirichlet Distribution."

Corollary 3.1.2. *Let* $X_1, X_2, \ldots, X_{r+1}$ *be independent symmetric positive definite random matrices,* $X_i \sim W_p(n_i, \Sigma)$, $i = 1, \ldots, r+1$. *Define* $\sum_{i=1}^{r+1} X_i = CC'$ *and* $X_i = CU_iC'$, $i = 1, \ldots, r$ *where the matrix* $C = (c_{ij})$ *is lower triangular with* $c_{ii} > 0$. *Then, the random matrices* $U_1, \ldots, U_r$ *follow a matrix variate Dirichlet type I distribution with joint p.d.f.*

$$\frac{\Gamma_p(\frac{1}{2}\sum_{i=1}^{r+1} n_i)}{\prod_{i=1}^{r+1}\Gamma_p(\frac{1}{2}n_i)} \prod_{i=1}^{r} \det(U_i)^{\frac{n_i-p-1}{2}} \det\Big(I_p - \sum_{i=1}^{r} U_i\Big)^{\frac{n_{r+1}-p-1}{2}},$$

where $0 < U_i < I_p, i = 1, \ldots, r, \sum_{i=1}^{r} U_i < I_p$.

Corollary 3.1.3. *Let* X_1 *and* X_2 *be independent random matrices,* $X_i \sim W_p(n_i, \Sigma, \Theta_i)$, $\Theta_i = \operatorname{diag}(\theta_i^2, 0, \ldots, 0)$, $i = 1, 2$. *Define* $X_1 + X_2 = CC'$ *and* $X_1 = CUC'$, *where the matrix* $C = (c_{ij})$ *is lower triangular with* $c_{ii} > 0$. *Then, the p.d.f. of* $U = (u_{\alpha\beta})$ *is given by*

$$\frac{\Gamma_p[\frac{1}{2}(n_1+n_2)]}{\Gamma_p(\frac{1}{2}n_1)\Gamma_p(\frac{1}{2}n_2)} \exp\Big(-\frac{\theta_1^2+\theta_2^2}{2}\Big) \det(U)^{\frac{n_1-p-1}{2}} \det(I_p - U)^{\frac{n_2-p-1}{2}}$$
$$\times \mathbf{\Psi}_2\Big[\frac{n_1+n_2}{2}; \frac{n_1}{2}, \frac{n_2}{2}; \frac{\theta_1^2 u_{11}}{2}, \frac{\theta_2^2(1-u_{11})}{2}\Big], 0 < U < I_p,$$

where $\mathbf{\Psi}_2$ *is the Humbert's confluent hypergeometric function.*

The above distribution is designated by $U \sim B_p^I(\frac{1}{2}n_1, \frac{1}{2}n_2; \theta_1^2; \theta_2^2)$. The distribution of U, in this case, is called "doubly non-central matrix-variate beta distribution", e.g., see Gill and Siotani [4], Amey and Gupta [1] and Kabe [9, 10]. The above density, using (2.4), can also be written as

$$\frac{\Gamma_p[\frac{1}{2}(n_1+n_2)]}{\Gamma_p(\frac{1}{2}n_1)\Gamma_p(\frac{1}{2}n_2)} \exp\Big(-\frac{\theta_1^2+\theta_2^2}{2}\Big) \det(U)^{\frac{n_1-p-1}{2}} \det(I_p - U)^{\frac{n_2-p-1}{2}}$$
$$\times \sum_{j=0}^{\infty} \frac{(\frac{1}{2}(n_1+n_2))_j\,(\frac{1}{2}\theta_1^2 u_{11})^j}{(\frac{1}{2}n_1)_j\, j!} {}_1F_1\Big(\frac{n_1+n_2}{2}+j; \frac{n_2}{2}; \frac{\theta_2^2(1-u_{11})}{2}\Big).$$

Corollary 3.1.4. *Let the random matrices* X_1 *and* X_2 *be independent,* $X_1 \sim W_p(n_1, \Sigma)$ *and* $X_2 \sim W_p(n_2, \Sigma, \Theta)$, *where* $\Theta = \operatorname{diag}(\theta^2, 0, \ldots, 0)$. *Define* $X_1 +$

$X_2 = CC'$ and $X_1 = CUC'$, where the matrix C is lower triangular with positive diagonal elements. Then, the p.d.f. of $U = (u_{\alpha\beta})$ is given by

$$\frac{\Gamma_p[\frac{1}{2}(n_1+n_2)]}{\Gamma_p(\frac{1}{2}n_1)\Gamma_p(\frac{1}{2}n_2)} \exp\Big(-\frac{\theta^2}{2}\Big) \det(U)^{\frac{n_1-p-1}{2}} \det(I_p - U)^{\frac{n_2-p-1}{2}}$$

$$\times {}_1F_1\Big(\frac{n_1+n_2}{2}; \frac{n_2}{2}; \frac{\theta^2(1-u_{11})}{2}\Big),\ 0 < U < I_p.$$

The above distribution, designated by $U \sim B_p^I(\frac{1}{2}n_1, \frac{1}{2}n_2; \theta^2)$, is called the "linear non-central matrix-variate Beta Distribution." The density function of U given above was first derived by Kshirsagar [12], also see Khatri and Pillai [11].

Corollary 3.1.5. *Let the $p \times p$ independent random matrices X_1 and X_2 have Wishart distribution, $X_i \sim W_p(n_i, \Sigma)$, $i = 1, 2$. Define $X_1 + X_2 = CC'$ and $X_1 = CUC'$, where the matrix C is lower triangular with positive diagonal elements. Then, the random matrix U has a matrix-variate beta type I distribution, $U \sim B_p^I(\frac{1}{2}n_1, \frac{1}{2}n_2)$, with the p.d.f.*

$$\frac{\Gamma_p[\frac{1}{2}(n_1+n_2)]}{\Gamma_p(\frac{1}{2}n_1)\Gamma_p(\frac{1}{2}n_2)} \det(U)^{\frac{n_1-p-1}{2}} \det(I_p - U)^{\frac{n_2-p-1}{2}}, 0 < U < I_p.$$

4. Properties

In this section we will study certain properties of the non-central matrix-variate Dirichlet distribution derived in the last section.

Theorem 4.1. *Let $(U_1, \dots, U_r) \sim D_p^I(\frac{1}{2}n_1, \dots, \frac{1}{2}n_r; \frac{1}{2}n_{r+1}; \theta_1^2, \dots, \theta_r^2; \theta_{r+1}^2)$. Partition the matrix U_i as*

$$U_i = \begin{pmatrix} U_{11i} & U_{12i} \\ U_{21i} & U_{22i} \end{pmatrix},\ U_{11i}\,(q \times q),\ i = 1, \dots, r.$$

Then, $(U_{111}, \dots, U_{11r}) \sim D_q^I(\frac{1}{2}n_1, \dots, \frac{1}{2}n_r; \frac{1}{2}n_{r+1}; \theta_1^2, \dots, \theta_r^2; \theta_{r+1}^2)$.

Proof. We will use *synthetic representation* of the random matrices $U_1, \dots, U_r$ to prove this theorem. According to the Theorem 3.1, the random matrices $U_1, \dots, U_r$ can be represented as $X_i = CU_iC'$, $i = 1, \dots, r$ and $\sum_{i=1}^{r+1} X_i = CC'$ where the matrix C is lower triangular with positive diagonal elements and $X_1, \dots, X_{r+1}$ are independent random matrices, $X_i \sim W_p(n_i, \Sigma, \Theta_i)$, $\Theta_i = \operatorname{diag}(\theta_i^2, 0, \dots, 0)$, $i = 1, \dots, r+1$.

Now, partition the matrices X_i, Σ, Θ_i and C as

$$X_i = \begin{pmatrix} X_{11i} & X_{12i} \\ X_{21i} & X_{22i} \end{pmatrix}, \ \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

$$\Theta_i = \begin{pmatrix} \Theta_{11i} & \Theta_{12i} \\ \Theta_{21i} & \Theta_{22i} \end{pmatrix}, \ C = \begin{pmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{pmatrix}$$

where $X_{11i}, \Sigma_{11}, \Theta_{11i}$ and C_{11} are matrices of order $q \times q$. Using these partitions we have $X_{11i} = C_{11} U_{11i} C'_{11}$, $i = 1, \ldots, r$, and $\sum_{i=1}^{r+1} X_{11i} = C_{11} C'_{11}$. From Theorem 2.1 it is clear that $X_{11i} \sim W_q(n, \Sigma_{11}, \Theta_{11i})$ where $\Theta_{11i} = \operatorname{diag}(\theta_i^2, 0, \ldots, 0)$, $i = 1, \ldots, r+1$. Now, application of Theorem 3.1 yields the density of $(U_{111}, \ldots, U_{11r})$. ■

Corollary 4.1.1. *The joint p.d.f. of* $u_{111}, \ldots, u_{11r}$ *is given by*

$$\frac{\Gamma(\frac{1}{2}\sum_{i=1}^{r+1} n_i)}{\prod_{i=1}^{r+1}\Gamma(\frac{1}{2}n_i)} \exp\Big(-\frac{\sum_{i=1}^{r+1}\theta_i^2}{2}\Big) \prod_{i=1}^{r} u_{11i}^{\frac{n_i-2}{2}} \Big(1 - \sum_{i=1}^{r} u_{11i}\Big)^{\frac{n_{r+1}-2}{2}}$$
$$\times \Psi_2^{(r+1)}\Big[\frac{\sum_{i=1}^{r+1} n_i}{2}; \frac{n_1}{2}, \ldots, \frac{n_{r+1}}{2}; \frac{\theta_1^2 u_{111}}{2}, \ldots, \frac{\theta_r^2 u_{11r}}{2}, \frac{\theta_{r+1}^2 (1 - \sum_{i=1}^{r} u_{11i})}{2}\Big],$$

where $0 < u_{11i} < 1, i = 1, \ldots, r, \sum_{i=1}^{r} u_{11i} < 1$ *and* u_{11i} *is the first element on the principal diagonal of* U_i, $i = 1, \ldots, r$.

Theorem 4.2. *If* $(U_1, \ldots, U_r) \sim D_p^I(\frac{1}{2}n_1, \ldots, \frac{1}{2}n_r; \frac{1}{2}n_{r+1}; \theta_1^2, \ldots, \theta_r^2; \theta_{r+1}^2)$, *then for* $1 \cdot s \cdot r$, $(U_1, \ldots, U_s) \sim D_p^I(\frac{1}{2}n_1, \ldots, \frac{1}{2}n_s; \frac{1}{2}\sum_{i=s+1}^{r+1} n_i; \theta_1^2, \ldots, \theta_s^2; \sum_{i=s+1}^{r+1}\theta_i^2)$.

Proof. Using the synthetic representation, $(U_1, \ldots, U_r)$ can be represented in terms of independent non-central Wishart matrices $X_1, \ldots, X_{r+1}$. Define $\sum_{i=s+1}^{r+1} X_i = X$. Then, $X_1, \ldots, X_s$ and X are independent, $X_i \sim W_p(n_i, \Sigma, \Theta_i)$, $\Theta_i = \operatorname{diag}(\theta_i^2, 0, \ldots, 0)$, $i = 1, \ldots, s$. Also, from Theorem 2.2, $X \sim W_p(\sum_{i=s+1}^{r+1} n_i, \Sigma, \Theta)$ where the non-centrality matrix $\Theta = \operatorname{diag}(\sum_{i=s+1}^{r+1}\theta_i^2, 0, \ldots, 0)$. Further, $\sum_{i=1}^{s} X_i + X = CC'$ and $X_i = CU_iC'$, $i = 1, \ldots, s$. Now, using Theorem 3.1, we obtain the joint density of $U_1, \ldots, U_s$. ■

Corollary 4.2.1. *If* $(U_1, \ldots, U_r) \sim D_p^I(\frac{1}{2}n_1, \ldots, \frac{1}{2}n_r; \frac{1}{2}n_{r+1}; \theta_1^2, \ldots, \theta_r^2; \theta_{r+1}^2)$, *then for* $1 \cdot s \cdot r$, $U_s \sim B_p^I(\frac{1}{2}n_s, \frac{1}{2}\sum_{i=1(\neq s)}^{r+1} n_i; \theta_s^2; \sum_{i=1(\neq s)}^{r+1}\theta_i^2)$.

The conditional p.d.f. of $(U_{s+1}, \ldots, U_r)$ given $(U_1, \ldots, U_s)$, $1 \cdot s \cdot r$, is given by

$$\frac{\text{p.d.f. of } (U_1, \ldots, U_r)}{\text{p.d.f. of } (U_1, \ldots, U_s)}$$

which can be obtained explicitly by substituting density functions of $(U_1, \dots, U_r)$ and $(U_1, \dots, U_s)$, where $(U_1, \dots, U_k) \sim D_p^I(\frac{1}{2}n_1, \dots, \frac{1}{2}n_k; \frac{1}{2}\sum_{i=k+1}^{r+1} n_i;\ \theta_1^2, \dots, \theta_k^2; \sum_{i=k+1}^{r+1}\theta_i^2)$, $1 \cdot\ \ k \cdot\ \ r$.

In the next theorem, we will derive the joint density of partial sums of matrices which are jointly distributed as non-central matrix-variate Dirichlet.

Theorem 4.3. *Let* $(U_1, \dots, U_r) \sim D_p^I(\frac{1}{2}n_1, \dots, \frac{1}{2}n_r; \frac{1}{2}n_{r+1}; \theta_1^2, \dots, \theta_r^2; \theta_{r+1}^2)$ *and, for* $i = 1, \dots, \ell$, *define*

$$\sum_{j=r_{i-1}^*+1}^{r_i^*} U_j,\ \theta_{(i)}^2 = \sum_{j=r_{i-1}^*+1}^{r_i^*} \theta_j^2,\ n_{(i)} = \sum_{j=r_{i-1}^*+1}^{r_i^*} n_j,\ r_0^* = 0,\ r_i^* = \sum_{j=1}^{i} r_j,$$

Then, $(U_{(1)}, \dots, U_{(\ell)}) \sim D_p^I(\frac{1}{2}n_{(1)}, \dots, \frac{1}{2}n_{(\ell)}; \frac{1}{2}n_{r+1}; \theta_{(1)}^2, \dots, \theta_{(\ell)}^2; \theta_{r+1}^2)$.

Proof. In this case too we will use the synthetic representation of the random matrices $U_1, \dots, U_r$. Define $X_{(i)} = \sum_{j=r_{i-1}^*+1}^{r_i^*} X_j$, $i = 1, \dots, \ell$. Then, $X_{(1)}, \dots, X_{(\ell)}$ and X_{r+1} are independently distributed, $X_{r+1} \sim W_p(n_{r+1}, \Sigma, \Theta_{r+1})$, and from Theorem 2.2, $X_{(i)} \sim W_p(n_{(i)}, \Sigma, \Theta_{(i)})$, $\Theta_{(i)} = \operatorname{diag}(\theta_{(i)}^2, 0, \dots, 0)$, $i = 1, \dots, \ell$. Further, $\sum_{i=1}^{\ell} X_{(i)} + X_{r+1} = CC'$ and $X_{(i)} = CU_{(i)}C'$, $i = 1, \dots, \ell$. Now, using Theorem 3.1, we get $(U_{(1)}, \dots, U_{(\ell)}) \sim D_p^I(\frac{1}{2}n_{(1)}, \dots, \frac{1}{2}n_{(\ell)}; \frac{1}{2}n_{r+1}; \theta_{(1)}^2, \dots, \theta_{(\ell)}^2; \theta_{r+1}^2)$ ■

When $\ell = 1$, $\sum_{i=1}^{r} U_i \sim B_p^I(\frac{1}{2}\sum_{i=1}^{r} n_i, \frac{1}{2}n_{r+1}; \sum_{i=1}^{r}\theta_i^2; \theta_{r+1}^2)$.

Theorem 4.4. *If* $(U_1, \dots, U_r) \sim D_p^I(\frac{1}{2}n_1, \dots, \frac{1}{2}n_r; \frac{1}{2}m; \theta_1^2, \dots, \theta_r^2; \theta_{r+1}^2)$, *then*

$$\begin{aligned} &E\Big[\prod_{i=1}^{r} \det(U_i)^{\frac{n_i}{2}}\Big]^h \\ &= \frac{\Gamma_p[\frac{1}{2}(n+m)]\prod_{i=1}^{r}\Gamma_p[\frac{1}{2}n_i(1+h)]}{\prod_{i=1}^{r}\Gamma_p(\frac{1}{2}n_i)\Gamma_p[\frac{1}{2}n(1+h)+\frac{1}{2}m]} \exp\Big(-\frac{\sum_{i=1}^{r+1}\theta_i^2}{2}\Big) \\ &\quad \times F_{1:1;\cdots;1}^{1:1;\cdots;1}\left[\begin{matrix} \frac{1}{2}(n+m) : \frac{1}{2}n_1(1+h); \cdots; \frac{1}{2}n_r(1+h); \frac{1}{2}m; \\ \frac{1}{2}n(1+h)+\frac{1}{2}m : \frac{1}{2}n_1; \cdots; \frac{1}{2}n_r; \frac{1}{2}m; \end{matrix} \quad \frac{\theta_1^2}{2}, \dots, \frac{\theta_{r+1}^2}{2}\right], \end{aligned}$$

and

$$E\Big[\det\Big(I_p - \sum_{i=1}^{r} U_i\Big)^h\Big]$$

$$= \frac{\Gamma_p[\frac{1}{2}(n+m)]\Gamma_p(\frac{1}{2}m+h)}{\Gamma_p(\frac{1}{2}m)\Gamma_p[\frac{1}{2}(n+m)+h]} \exp\Big(-\frac{\sum_{i=1}^{r+1}\theta_i^2}{2}\Big)\sum_{s=0}^{\infty}\frac{(\frac{1}{2}(n+m))_s(\frac{1}{2}m+h)_s(\theta_{r+1}^2)^s}{(\frac{1}{2}(n+m)+h)_s(\frac{1}{2}m)_s s!}$$
$$\times {}_1F_1\Big(\frac{n+m}{2}+s;\frac{n+m}{2}+h+s;\frac{\theta_1^2+\cdots+\theta_r^2}{2}\Big)$$

where $n=\sum_{i=1}^{r} n_i$*,* $F_{1:1;\cdots;1}^{1:1;\cdots;1}$ *and* ${}_1F_1$ *are the generalized Kampé de Fériet's and confluent hypergeometric functions respectively.*

Proof. (i) From the p.d.f of $U_1,\ldots,U_r$ given in Theorem 3.1, we have

$$E\Big[\prod_{i=1}^{r}\det(U_i)^{\frac{n_i}{2}}\Big]^h = \frac{\Gamma_p[\frac{1}{2}(n+m)]\prod_{i=1}^{r}\Gamma_p[\frac{1}{2}n_i(1+h)]}{\prod_{i=1}^{r}\Gamma_p(\frac{1}{2}n_i)\Gamma_p[\frac{1}{2}n(1+h)+\frac{1}{2}m]}\exp\Big(-\frac{\sum_{i=1}^{r+1}\theta_i^2}{2}\Big)$$
$$\times \underset{\substack{0<U_i<I_p\\ 0<\sum_{i=1}^{r}U_i<I_p}}{\int\cdots\int} \Big[\frac{\Gamma_p[\frac{1}{2}n(1+h)+\frac{1}{2}m]}{\prod_{i=1}^{r}\Gamma_p[\frac{1}{2}n_i(1+h)]\Gamma_p(\frac{1}{2}m)}.$$
$$\times\prod_{i=1}^{r}\det(U_i)^{\frac{n_i(1+h)-p-1}{2}}\det\Big(I_p-\sum_{i=1}^{r}U_i\Big)^{\frac{m-p-1}{2}}\Big]$$
$$\times\mathbf{\Psi}_2^{(r+1)}\Big[\frac{n+m}{2};\frac{n_1}{2},\ldots,\frac{n_r}{2},\frac{m}{2};\frac{\theta_1^2u_{111}}{2},\ldots,\frac{\theta_r^2u_{11r}}{2},$$
$$\frac{\theta_{r+1}^2(1-\sum_{i=1}^{r}u_{11i})}{2}\Big]\prod_{i=1}^{r}dU_i.$$

The first factor in above integral (terms in brackets) is matrix-variate Dirichlet density with parameters $\frac{1}{2}n_1(1+h),\ldots,\frac{1}{2}n_r(1+h);\frac{1}{2}m$. The second factor involves only $u_{111},\ldots,u_{11r}$. Thus, integrating terms in the bracket over all the elements of U_j, $j=1,\ldots,r$ except the first element of each U_j, $j=1,\ldots,r$, we obtain

$$E\Big[\prod_{i=1}^{r}\det(U_i)^{\frac{n_i}{2}}\Big]^h = \frac{\Gamma_p[\frac{1}{2}(n+m)]\prod_{i=1}^{r}\Gamma_p[\frac{1}{2}n_i(1+h)]}{\prod_{i=1}^{r}\Gamma_p(\frac{1}{2}n_i)\Gamma_p[\frac{1}{2}n(1+h)+\frac{1}{2}m]}\exp\Big(-\frac{\sum_{i=1}^{r+1}\theta_i^2}{2}\Big)$$
$$\times\frac{\Gamma[\frac{1}{2}n(1+h)+\frac{1}{2}m]}{\prod_{i=1}^{r}\Gamma[\frac{1}{2}n_i(1+h)]\Gamma(\frac{1}{2}m)}\underset{\substack{0<u_{11i}<1\\ 0<\sum_{i=1}^{r}u_{11i}<1}}{\int\cdots\int}\prod_{i=1}^{r}u_{11i}^{\frac{n_i(1+h)}{2}-1}\Big(1-\sum_{i=1}^{r}u_{11i}\Big)^{\frac{m}{2}-1}$$
$$\times\mathbf{\Psi}_2^{(r+1)}\Big[\frac{n+m}{2};\frac{n_1}{2},\ldots,\frac{n_r}{2},\frac{m}{2};\frac{\theta_1^2u_{111}}{2},\ldots,\frac{\theta_r^2u_{11r}}{2},$$
$$\frac{\theta_{r+1}^2(1-\sum_{i=1}^{r}u_{11i})}{2}\Big]\prod_{i=1}^{r}du_{11i}.$$

Now, evaluation of the above integral using (2.8) yields the desired result.
(ii) Following similar steps, we have

$$E\Big[\det\Big(I_p-\sum_{i=1}^{r}U_i\Big)^h\Big]=\frac{\Gamma_p[\frac{1}{2}(n+m)]\Gamma_p(\frac{1}{2}m+h)}{\Gamma_p(\frac{1}{2}m)\Gamma_p[\frac{1}{2}(n+m)+h]}\exp\Big(-\frac{\sum_{i=1}^{r+1}\theta_i^2}{2}\Big)$$
$$\times F^{1:1;\cdots;1}_{1:1;\cdots;1}\left[\begin{matrix}\frac{1}{2}(n+m):\frac{1}{2}n_1;\cdots;\frac{1}{2}n_r;\frac{1}{2}m+h;\\ \frac{1}{2}(n+m)+h:\frac{1}{2}n_1;\cdots;\frac{1}{2}n_r;\frac{1}{2}m;\end{matrix}\ \frac{\theta_1^2}{2},\dots,\frac{\theta_{r+1}^2}{2}\right].$$

Simplifying the generalized Kampé de Fériet's function using (2.7) we get the desired result. ∎

Alternatively, the above moment expression can be obtained by noting that $\sum_{i=1}^{n}U_i$ has a doubly non-central matrix-variate Beta distribution.

Javier and Gupta [8] derived certain asymptotic expansion of the matrix variate Dirichlet type I distribution. Gupta, Cardeño and Nagar [5] derived similar results for the matrix-variate Kummer-Dirichlet distributions. Here, we give asymptotic expansion for the non-central matrix-variate Dirichlet distribution derived in Section 3.

Theorem 4.5. *Let* $(U_1,\dots,U_r)\sim D_p^I(\frac{1}{2}n_1,\dots,\frac{1}{2}n_r;\frac{1}{2}n_{r+1};\theta_1^2,\dots,\theta_r^2;\theta_{r+1}^2)$ *and* $W=(W_1,\dots,W_r)$ *where* $W_i=\frac{1}{2}n_{r+1}U_i$, $i=1,\dots,r$. *Then*, W *is asymptotically distributed as a product of independent non-central Wishart densities*; *more specifically*

$$\lim_{n_{r+1}\to\infty}f(W)=\prod_{i=1}^{r}\frac{\exp(-\frac{1}{2}\theta_i^2)\operatorname{etr}(-W_i)\det(W_i)^{\frac{n_i-p-1}{2}}\,{}_0F_1(\frac{1}{2}n_i;\frac{1}{2}\theta_i^2w_{11i})}{\Gamma_p(\frac{1}{2}n_i)},$$

where $f(W)$ *denotes the density of the matrix* W.

Proof. Transforming $W_i=\frac{1}{2}n_{r+1}U_i$, $i=1,\dots,r$, with Jacobian $J(U_1,\dots,U_r\to W_1,\dots,W_r)=(\frac{1}{2}n_{r+1})^{-\frac{1}{2}rp(p+1)}$ in the joint density of $(U_1,\dots,U_r)$ given in Theorem 3.1, the density $f(W)$ of $W=(W_1,\dots,W_r)$ is obtained as

$$\frac{\Gamma_p(\frac{1}{2}\sum_{i=1}^{r+1}n_i)}{\Gamma_p(\frac{1}{2}n_{r+1})}\exp\Big(-\frac{\sum_{i=1}^{r+1}\theta_i^2}{2}\Big)\Big(\frac{n_{r+1}}{2}\Big)^{-\frac{1}{2}p\sum_{i=1}^{r}n_i}$$
$$\times\Big\{\prod_{i=1}^{r}\frac{\det(W_i)^{\frac{n_i-p-1}{2}}}{\Gamma_p(\frac{1}{2}n_i)}\Big\}\det\Big(I_p-\frac{2}{n_{r+1}}\sum_{i=1}^{r}W_i\Big)^{\frac{n_{r+1}-p-1}{2}}$$
$$\times\Psi_2^{(r+1)}\Big[\frac{\sum_{i=1}^{r+1}n_i}{2};\frac{n_1}{2},\dots,\frac{n_{r+1}}{2};\frac{\theta_1^2w_{111}}{n_{r+1}},\dots,\frac{\theta_r^2w_{11r}}{n_{r+1}},\frac{\theta_{r+1}^2}{2}\Big(1-\frac{2}{n_{r+1}}\sum_{i=1}^{r}w_{11i}\Big)\Big]$$

where $W_i = (w_{\alpha\beta i}), i = 1, \ldots, r$. The result follows since

$$\lim_{n_{r+1}\to\infty} \frac{\Gamma_p(\frac{1}{2}\sum_{i=1}^{r+1} n_i)}{\Gamma_p(\frac{1}{2}n_{r+1})} \Big(\frac{n_{r+1}}{2}\Big)^{-\frac{p\sum_{i=1}^{r} n_i}{2}} = 1,$$

$$\lim_{n_{r+1}\to\infty} \det\Big(I_p - \frac{2}{n_{r+1}}\sum_{i=1}^{r} W_i\Big)^{\frac{n_{r+1}-p-1}{2}} = \operatorname{etr}\Big(-\sum_{i=1}^{r} W_i\Big)$$

and

$$\lim_{n_{r+1}\to\infty} \mathbf{\Psi}_2^{(r+1)}\Big[\frac{\sum_{i=1}^{r+1} n_i}{2}; \frac{n_1}{2}, \ldots, \frac{n_{r+1}}{2}; \frac{\theta_1^2 w_{111}}{n_{r+1}}, \ldots, \frac{\theta_r^2 w_{11r}}{n_{r+1}}, \frac{\theta_{r+1}^2(1 - \frac{2}{n_{r+1}}\sum_{i=1}^{r} w_{11i})}{2}\Big]$$

$$= \exp\Big(\frac{\theta_{r+1}^2}{2}\Big)\prod_{i=1}^{r} {}_0F_1\Big(\frac{n_i}{2}; \frac{\theta_i^2 w_{11i}}{2}\Big). \qquad \blacksquare$$

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