TAIWANESE JOURNAL OF MATHEMATICS
Vol. 7, No. 3, pp. 461-475, September 2003
This paper is available online at http://www.math.nthu.edu.tw/tjm/

INTERVAL OSCILLATION CRITERIA FOR SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH DAMPING

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Abstract. We present new interval oscillation criteria for certain classes of second order nonlinear differential equations with damping that are different from most known ones in the sense that they are based on the information only on a sequence of subintervals of $[t_0, \infty)$, rather than on the whole half-line. Our results are sharper than some previous results and handle the cases which are not covered by known criteria. Finally, several examples that dwell upon the sharp conditions of our results are also included.

1. Introduction

In this paper we consider the oscillation behavior of solutions of the second order nonlinear differential equation

$$\big(r(t)y'(t)\big)' + p(t)y'(t) + q(t)f\left(y(t)\right)g\left(y'(t)\right) = 0, \tag{1.1}$$

where $t \geq t_0$, the functions r, p, q, f and g are to be specified in the following.

We recall that a function $y : [t_0, t_1) \to (-\infty, \infty)$, $t_1 > t_0$ is called a solution of Eq. (1.1) if $y(t)$ satisfies Eq. (1.1) for all $t \in [t_0, t_1)$. In the sequel it will be always assumed that solutions of Eq.(1.1) exist for any $t_0 \geq 0$. A solution $y(t)$ of Eq.(1.1) is called oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory.

Received June 19, 2001; revised March 1, 2002.
Communicated by S. B. Hsu.
2000 *Mathematics Subject Classification*: 34C10, 34C15.
Key words and phrases: Interval criteria, oscillation, second order, nonlinear differential equations, integral averaging technique.
*Supported by the NNSF of China (10171040), the NSF of Gansu Province of China (ZS011-A25-007-Z), the Foundation for University Key Teacher by Ministry of Education of China and the Teaching and Research Award Program for Outstanding Young Teachers in Higher Education Institutions of Ministry of Education of China.

When $r(t) \equiv 1$ and $p(t) \equiv 0$, Eq. (1.1) reduces to

$$y''(t) + q(t)f(y(t))g(y'(t)) = 0, \tag{1.2}$$

Eq. (1.1) has been studied by Grace and Lalli [7]. They mentioned that though stability, boundedness, and convergence to zero of all solutions of Eq. (1.2) have been investigated in the papers of Burton and Grimmer [2], Grace and Spikes [5, 6], Lalli [12], and Wong and Burton [21]. Nothing much has been known regarding the oscillatory behavior of Eq.(1.2) except for the result by Wong and Burton [21, Theorem 4] regarding oscillatory behavior of Eq. (1.2) in connection with that of the corresponding linear equation

$$y''(t) + q(t)y(t) = 0. \tag{1.3}$$

Recently, Li and Agarwal [15] and Rogovchenko [19] presented new sufficient conditions which ensure oscillatory character of Eq. (1.2). They are different from those of Grace and Lalli [7] and are applicable to other classes of equations which are not covered by the results of Grace and Lalli [7]. However, except for the results of Li and Agarwal [15], all the mentioned above oscillation results involve the interval of q and hence require the information of q on the entire half-line $[t_0, \infty)$.

From the Sturm Separation Theorem, we see that oscillation is only an interval property, i.e., if there exists a sequence of subintervals $[a_i, b_i]$ of $[t_0, \infty)$, as $a_i \to \infty$, such that for each i there exists a solution of Eq. (1.3) that has at least two zeros in $[a_i, b_i]$, then every solution of Eq. (1.3) is oscillatory.

El-Sayed [4] established an interval criterion for oscillation of a forced second-order equation, but the result is not very sharp, because a comparison with equations of constant coefficient is used in the proof. Afterwards, Wong [20] proved a general result for linear forced equation and Li and Agarwal [16] established more general results for nonlinear forced equation (1.2).

We remark that, Kong [11] and Li and Agarwal [15] employed the technique in the work of Philos [18] and obtained several interval oscillation results for second order linear equation (1.3) and nonlinear equation (1.2). However, they can not be applied to the nonlinear differential equation (1.1).

Motivated by the ideas of Kong [11] and Li and Agarwal [15], in this paper we obtain, by using a generalized Riccati technique, several new interval criteria for oscillation, that is, criteria given by the behavior of Eq. (1.1) (or of r, p, q, f and g) only on a sequence of subintervals of $[t_0, \infty)$. Our results involve the Kamenev's type condition and improve and extend the results of Kamenev [10], Li and Agarwal [15] and Philos [18]. Finally, several examples that dwell upon the sharp conditions of our results are also included. Other related oscillation results can refer to [1, 3, 6, 10, 17 and 22].

Hereinafter, we assume that

(H1) the functions $r : [t_0, \infty) \to (0, \infty)$ and $p : [t_0, \infty) \to R$ are continuous;

(H2) the function $q : [t_0, \infty) \to [0, \infty)$ is continuous and $q(t) \not\equiv 0$ on any ray $[T, \infty)$ for some $T \ge t_0$;

(H3) the function $f : R \to R$ is continuous and $yf(y) > 0$ for $y \ne 0$;

(H4) the function $g : R \to R$ is continuous and $g(y) \ge K > 0$ for $y \ne 0$.

We say that a function $H = H(t, s)$ belongs to a function class X, denoted by $H \in X$, if $H \in C(D, R_+)$, where $D = \{(t, s) : -\infty < s \le t < \infty\}$, which satisfies

$$H(t,t) = 0, \quad H(t,s) > 0, \text{ for } t > s, \tag{1.4}$$

and has partial derivatives $\partial H/\partial t$ and $\partial H/\partial s$ on D such that

$$\frac{\partial H}{\partial t} = h_1(t,s)H(t,s)^{1/2} \text{ and } \frac{\partial H}{\partial s} = -h_2(t,s)H(t,s)^{1/2}, \tag{1.5}$$

where $h_1, h_2 \in L_{loc}(D, R)$.

2. Oscillation Results for $f(x)$ with Monotonicity

In this section we always assume the following condition holds.

$$\text{(H5) there exists } f'(y) \text{ for } y \in R \text{ and } f'(y) \ge \mu > 0 \text{ for } y \ne 0. \tag{2.1}$$

First, we estabilish two interesting lemmas, which will be useful for establishing oscillation criteria for Eq. (1.1).

Lemma 2.1. *Let assumptions* (H1)-(H5) *hold. If y is a solution of Eq.* (1.1) *such that $y(t) > 0$ on $[c, b)$. For any $v \in C^1([t_0, \infty), (0, \infty))$, let*

$$u(t) = v(t)\frac{r(t)y'(t)}{f(y(t))}, \tag{2.2}$$

on $[c, b)$. Then for any $H \in X$,

$$\begin{aligned} \int_c^b H(b,s)Kv(s)q(s)ds &\le H(b,c)u(c) \\ &+\frac{1}{4\mu}\int_c^b r(s)v(s)\Big[h_2(b,s) + \left(\frac{p(s)}{r(s)} - \frac{v'(s)}{v(s)}\right)\sqrt{H(b,s)}\Big]^2 ds. \end{aligned} \tag{2.3}$$

Proof. From (1.1) and (2.2) we have for $s \in [c, b)$

$$u'(t) = -v(t)q(t)g(y'(t)) - \frac{f'(y(t))}{r(t)v(t)}u^2(t) - \frac{p(t)}{r(t)}u(t) + \frac{v'(t)}{v(t)}u(t). \tag{2.4}$$

In view of $f'(y) \geq \mu > 0$ and $g(y') \geq K > 0$, we obtain by the above equality

$$u'(t) + Kv(t)q(t) + \frac{\mu}{r(t)v(t)}u^2(t) + \frac{p(t)}{r(t)}u(t) - \frac{v'(t)}{v(t)}u(t) \quad 0. \tag{2.5}$$

Multiplying (2.5) by $H(t, s)$, integrating it with respect to s from c to t for $t \in [c, b)$, and using (1.4) and (1.5) yield

$$\begin{aligned}
&\int_c^t H(t,s)Kv(s)q(s)ds \quad -\int_c^t H(t,s)u'(s)ds \\
&\quad -\int_c^t H(t,s)\frac{\mu u^2(s)}{r(s)v(s)}ds + \int_c^t H(t,s)\left(-\frac{p(s)}{r(s)} + \frac{v'(s)}{v(s)}\right)u(s)ds \\
&= H(t,c)u(c) \\
&\quad -\int_c^t \left\{h_2(t,s)\sqrt{H(t,s)}u(s) - H(t,s)\left(\frac{p(s)}{r(s)} - \frac{v'(s)}{v(s)}\right)u(s) + H(t,s)\frac{\mu u^2(s)}{r(s)v(s)}\right\}ds \\
&= H(t,c)u(c) \\
&\quad -\int_c^t \left\{\sqrt{\frac{\mu H(t,s)}{r(s)v(s)}}u(s) + \frac{1}{2}\sqrt{\frac{r(s)v(s)}{\mu}}\left[h_2(t,s) + \left(\frac{p(s)}{r(s)} - \frac{v'(s)}{v(s)}\right)\sqrt{H(t,s)}\right]\right\}^2 ds \\
&\quad + \frac{1}{4\mu}\int_c^t r(s)v(s)\left[h_2(t,s) + \left(\frac{p(s)}{r(s)} - \frac{v'(s)}{v(s)}\right)\sqrt{H(t,s)}\right]^2 ds \\
&\quad H(t,c)u(c) + \frac{1}{4\mu}\int_c^t r(s)v(s)\left[h_2(t,s) + \left(\frac{p(s)}{r(s)} - \frac{v'(s)}{v(s)}\right)\sqrt{H(t,s)}\right]^2 ds
\end{aligned}$$

Letting $t \to b^-$ in the above, we obtain (2.3). The proof is complete.

Lemma 2.2. *Let assumptions* (H1)-(H5) *hold and suppose that y is a solution of Eq.* (1.1) *such that $y(t) > 0$ on $(a, c]$. For any $v \in C^1([t_0, \infty),\ (0, \infty))$, let $u(t)$ be defined by* (2.2) *on $(a, c]$. Then for any $H \in X$,*

$$\begin{aligned}
&\int_a^c H(s,a)Kv(s)q(s)ds \quad -H(c,a)u(c) \\
&\quad + \frac{1}{4\mu}\int_a^c r(s)v(s)\left[h_1(s,a) - \left(\frac{p(s)}{r(s)} - \frac{v'(s)}{v(s)}\right)\sqrt{H(s,a)}\right]^2 ds.
\end{aligned} \tag{2.6}$$

Proof. Similar to the proof of Lemma 2.1, we multiply (2.5) by $H(s, t)$, integrate it with respect to s from c for $t \in (a, c]$, and use (1.4) and (1.5), then we obtain that

$$
\begin{aligned}
&\int_t^c H(s,t)Kv(s)q(s)ds \quad -\int_t^c H(s,t)u'(s)ds \\
&\quad -\int_t^c H(s,t)\frac{\mu u^2(s)}{v(s)r(s)}ds + \int_t^c H(s,t)\left(-\frac{p(s)}{r(s)}+\frac{v'(s)}{v(s)}\right)u(s)ds \\
&= -H(c,t)u(c) \\
&\quad +\int_t^c \left\{ h_1(s,t)\sqrt{H(s,t)}u(s) - H(s,t)\frac{\mu u^2(s)}{r(s)v(s)} - H(s,t)\left(\frac{p(s)}{r(s)}-\frac{v'(s)}{v(s)}\right)u(s)\right\} ds \\
&= -H(c,t)u(c) - \int_t^c \frac{1}{r(s)v(s)}\left\{\left[\sqrt{\mu H(s,t)}u(s)\right]^2\right. \\
&\quad -\left(h_1(s,t)\sqrt{H(s,t)}r(s)v(s)+H(s,t)v'(s)r(s)\right)u(s) \\
&\quad \left.+\frac{1}{4\mu}r^2(s)v^2(s)\left[h_1(s,t)-\left(\frac{p(s)}{r(s)}-\frac{v'(s)}{v(s)}\right)\sqrt{H(s,t)}\right]^2\right\} ds \\
&\quad +\frac{1}{4\mu}\int_t^c r(s)v(s)\left[h_1(s,t)-\left(\frac{p(s)}{r(s)}-\frac{v'(s)}{v(s)}\right)\sqrt{H(s,t)}\right]^2 ds \\
&= -H(c,t)u(c) - \int_t^c \frac{1}{r(s)v(s)}\left\{\sqrt{\mu H(s,t)}u(s)\right. \\
&\quad \left.-\frac{1}{2\sqrt{\mu}}r(s)v(s)\left[h_1(s,t)-\left(\frac{p(s)}{r(s)}-\frac{v'(s)}{v(s)}\right)\sqrt{H(s,t)}\right]\right\}^2 ds \\
&\quad +\frac{1}{4\mu}\int_t^c r(s)v(s)\left[h_1(s,t)-\left(\frac{p(s)}{r(s)}-\frac{v'(s)}{v(s)}\right)\sqrt{H(s,t)}\right]^2 ds \\
&\quad -H(c,t)u(c)+\frac{1}{4\mu}\int_t^c r(s)v(s)\left[h_1(s,t)-\left(\frac{p(s)}{r(s)}-\frac{v'(s)}{v(s)}\right)\sqrt{H(s,t)}\right]^2 ds.
\end{aligned}
$$

Letting $t \to a^+$ in the above, we obtain (2.6). The proof is complete.

The following theorem is an immediate result from Lemmas 2.1 and 2.2.

Theorem 2.1. *Assume that* (H1)-(H5) *hold and that for some* $c \in (a,b)$ *and for some* $H \in X, v \in C^1([t_0,\infty),(0,\infty))$,

$$
\begin{aligned}
&\frac{1}{H(c,a)}\int_a^c H(s,a)Kv(s)q(s)ds + \frac{1}{H(b,c)}\int_c^b H(b,s)Kv(s)q(s)ds \\
&> \frac{1}{4\mu H(c,a)}\int_a^c r(s)v(s)\left[h_1(s,a)-\left(\frac{p(s)}{r(s)}-\frac{v'(s)}{v(s)}\right)\sqrt{H(s,a)}\right]^2 ds \qquad (2.7)\\
&\quad +\frac{1}{4\mu H(b,c)}\int_c^b r(s)v(s)\left[h_2(b,s)+\left(\frac{p(s)}{r(s)}-\frac{v'(s)}{v(s)}\right)\sqrt{H(b,s)}\right]^2 ds.
\end{aligned}
$$

Then every solution of Eq.(1.1) has at least one zero in (a, b).

Theorem 2.2. *Assume that* (H1)-(H5) *holds. If, for each $T \ge t_0$, there exist $H \in X$, $v \in C^1([t_0,\infty),(0,\infty))$ and $a, b, c \in R$ such that $T \le a < c < b$ and* (2.7) *holds, then every solution of Eq.* (1.1) *is oscillatory.*

Proof. Pick up a sequence $\{T_i\} \subset [t_0, \infty)$ such that $T_i \to \infty$ as $i \to \infty$. By the assumption, for each $i \in N$, there exist $a_i, b_i, c_i \in R$ such that $T_i \le a_i < c_i < b_i$, and (2.7) holds, where a, b, c are replaced by a_i, b_i, c_i, respectively. From Theorem 2.1, every solution $y(t)$ has at least one zero, $t_i \in (a_i, b_i)$. Noting that $t_i > a_i \ge T_i$, $i \in N$, we see that every solution has arbitrary large zeros. Thus, every solution of Eq.(1.1) is oscillatory. The proof is complete.

Theorem 2.3. *Assume that* (H1)-(H5) *hold. If*

$$\limsup_{t\to\infty} \int_l^t \Bigg[H(s,l)Kv(s)q(s) - \frac{1}{4\mu} r(s)v(s) \left(h_1(s,l) - \left(\frac{p(s)}{r(s)} - \frac{v'(s)}{v(s)} \right) \sqrt{H(s,l)} \right)^2 \Bigg] ds > 0, \tag{2.8}$$

and

$$\limsup_{t\to\infty} \int_l^t \Bigg[H(t,s)Kv(s)q(s) - \frac{1}{4\mu} r(s)v(s) \left(h_2(t,s) + \left(\frac{p(s)}{r(s)} - \frac{v'(s)}{v(s)} \right) \sqrt{H(t,s)} \right)^2 \Bigg] ds > 0, \tag{2.9}$$

for some $H \in X$, $v \in C^1([t_0,\infty),\ (0,\infty))$ and for each $l \ge t_0$, then every solution of Eq. (1.1) *is oscillatory.*

Proof. For any $T \ge t_0$, let $a = T$. In (2.8) we choose $l = a$. Then there exists $c > a$ such that

$$\int_a^c \Bigg[H(s,a)Kv(s)q(s) - \frac{1}{4\mu} r(s)v(s) \left(h_1(s,a) - \left(\frac{p(s)}{r(s)} - \frac{v'(s)}{v(s)} \right) \sqrt{H(s,a)} \right)^2 \Bigg] ds > 0. \tag{2.10}$$

In (2.9) we choose $l = c$. Then there exists $b > c$ such that

$$\int_c^b \Bigg[H(b,s)Kv(s)q(s) - \frac{1}{4\mu} r(s)v(s) \left(h_2(b,s) + \left(\frac{p(s)}{r(s)} - \frac{v'(s)}{v(s)} \right) \sqrt{H(b,s)} \right)^2 \Bigg] ds > 0. \tag{2.11}$$

Combining (2.10) and (2.11) we obtain (2.7). The conclusion thus comes from Theorem 2.2. The proof is complete.

For the case where $H := H(t-s) \in X$, we have that $h_1(t-s) = h_2(t-s)$ and denote them by $h(t-s)$. The subclass of X containing such $H(t-s)$ is denoted by X_0. Applying Theorem 2.2 to X_0, we obtain

Theorem 2.4. *Assume that* (H1)-(H5) *hold. If for each* $T \geq t_0$, *there exist* $H \in X_0$, $v \in C^1([t_0\infty),\ (0,\infty))$ *and* $a, c \in R$ *such that* $T \quad a < c$ *and*

$$
\begin{aligned}
&\int_a^c H(s-a)K\left[v(s)q(s) + v(2c-s)q(2c-s)\right] ds \\
&> \frac{1}{4\mu}\int_a^c \left[r(s)v(s) + r(2c-s)v(2c-s)\right] h^2(s-a)ds \\
&+\frac{1}{2\mu}\int_a^c \big[r(2c-s)v'(2c-s) - p(2c-s)v(2c-s) \\
&- r(s)v'(s) + p(s)v(s)\big]\, h(s-a)\sqrt{H(s-a)}ds \\
&+\frac{1}{4\mu}\int_a^c \Bigg[r(s)v(s)\left(\frac{p(s)}{r(s)} - \frac{v'(s)}{v(s)}\right)^2 \\
&+ r(2c-s)v(2c-s)\left(\frac{p(2c-s)}{r(2c-s)} - \frac{v'(2c-s)}{v(2c-s)}\right)^2\Bigg] H(s-a)ds,
\end{aligned}
\tag{2.12}
$$

then every solution of Eq. (1.1) *is oscillatory.*

Proof. Let $b = 2c - a$. Then $H(b-c) = H(c-a) = H((b-a)/2)$, and for any $w \in L[a, b]$, we have

$$\int_c^b w(s)ds = \int_a^c w(2c-s)ds.$$

Hence

$$\int_c^b H(b-s)w(s)ds = \int_a^c H(s-a)w(2c-s)ds.$$

Thus that (2.12) holds implies that (2.7) holds for $H \in X_0$, $v \in C^1([t_0,\infty),\ (0,\infty)$ and therefore every solution of Eq. (1.1) is oscillatory by Theorem 2.2. The proof is complete.

From above oscillation criteria, we can obtain different sufficient conditions for oscillation of all solutions of Eq.(1.1) by different choices of $H(t, s)$.

Let

$$H(t,s) = (t-s)^\lambda, \quad t \geq s \geq t_0,$$

where $\lambda > 1$ is a constant.

Corollary 2.1. *Assume that* (H1)-(H5) *hold. Then every solution of Eq.* (1.1) *is oscillatory provided that for each* $l \geq t_0$ *and for some* $\lambda > 1$, *there exists a function* $v \in C^1([t_0, \infty),\ (0, \infty))$ *such that the following two inequalities hold*:

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \int_l^t (s-l)^\lambda \Bigg[Kv(s)q(s) - \frac{1}{4\mu} r(s)v(s) \left(\frac{\lambda}{(s-l)} - \frac{p(s)}{r(s)} + \frac{v'(s)}{v(s)} \right)^2 \Bigg] ds > 0. \tag{2.13}$$

and

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \int_l^t (t-s)^\lambda \Bigg[Kv(s)q(s) - \frac{1}{4\mu} r(s)v(s) \left(\frac{\lambda}{(t-s)} - \frac{p(s)}{r(s)} + \frac{v'(s)}{v(s)} \right)^2 \Bigg] ds > 0. \tag{2.14}$$

Define

$$R(t) = \int_l^t \frac{1}{r(s)} ds, \quad t \geq l \geq t_0,$$

and let

$$H(t,s) = [R(t) - R(s)]^\lambda, \quad t \geq t_0,$$

where $\lambda > 1$ is constant.

By Theorem 2.3, we have the following oscillation criterion which extends Theorem 2.3 (i) of Kong [11] and Theorem 2.5 of Li and Agarwal [15].

Theorme 2.5. *Assume that* (H1)-(H5) *hold and that* $\lim_{t\to\infty} R(t) = \infty$. *Then every solution of Eq.* (1.1) *is oscillatory provided that for each* $l \geq t_0$ *and for some* $\lambda > 1$, *the following two inequalities hold*:

$$\limsup_{t\to\infty} \frac{\mu}{R^{\lambda-1}(t)} \int_l^t [R(s) - R(l)]^\lambda \left(Kq(s) - \frac{p^2(s)}{4\mu r(s)} \right) ds > \frac{\lambda^2}{4(\lambda-1)} \tag{2.15}$$

and

$$\limsup_{t\to\infty} \frac{\mu}{R^{\lambda-1}(t)} \int_l^t [R(t) - R(s)]^\lambda \left(Kq(s) - \frac{p^2(s)}{4\mu r(s)} \right) ds > \frac{\lambda^2}{4(\lambda-1)}. \tag{2.16}$$

The proof is similar to that of Theorem 2.5 of Li and Agarwal [15], we omit it here.

3. Oscillation Results for $f(x)$ Without Monotonicity

In this section we consider the oscillation of Eq. (1.1) when the function $f(y)$ is not monotone. In this case we always assume the following condition holds:

(3.1) (H5') $f(y)/y \geq \mu_0 > 0$ for $y \neq 0$, where μ_0 is a constant.

Lemma 3.1. *Let assumptions* (H1)-(H4) *and* (H5') *hold. If y is a solution of Eq.* (1.1) *such that $y(t) > 0$ on $[c, b)$. For any $v \in C^1([t_0, \infty), (0, \infty))$, let*

$$w(t) = v(t)\frac{r(t)y'(t)}{y(t)} \tag{3.2}$$

on $[c, b)$. Then for any $H \in X$,

$$\begin{aligned} \int_c^b H(b,s)K\mu_0 v(s)q(s)ds &\quad H(b,c)w(c) \\ &+\frac{1}{4}\int_c^b r(s)v(s) \ \Box\ h_2(b,s) + \left(\frac{p(s)}{r(s)} - \frac{v'(s)}{v(s)}\right)\sqrt{H(b,s)}\bigg]^2 ds. \end{aligned} \tag{3.3}$$

Proof. From (1.1) and (3.2) we have for $s \in [c, b)$

$$\begin{aligned} w'(t) =& -v(t)q(t)\frac{f(y(t))}{y(t)}g(y'(t)) - \frac{1}{r(t)v(t)}w^2(t) \\ & -\frac{p(t)}{r(t)}w(t) + \frac{v'(t)}{v(t)}w(t). \end{aligned} \tag{3.4}$$

In view of $f(y)/y \geq \mu_0 > 0$ and $g(y') \geq K > 0$, we obtain by the above equality

$$w'(t) + K\mu_0 v(t)q(t) + \frac{1}{r(t)v(t)}w^2(t) + \frac{p(t)}{r(t)}w(t) - \frac{v'(t)}{v(t)}w(t) \quad 0. \tag{3.5}$$

The rest of the proof is similar to that of Lemma 2.1. The proof is complete.

Lemma 3.2. *Let assumptions* (H1)-(H4) *and* (H5') *hold and suppose that y is a solution of Eq.* (1.1) *such that $y(t) > 0$ on $(a, c]$. For any $v \in C^1([t_0, \infty), (0, \infty))$, let $w(t)$ be difined by* (3.2) *on $(a, c]$. Then for any $H \in X$,*

$$\begin{aligned} \int_a^c H(s,a)K\mu_0 v(s)q(s)ds &\quad -H(c,a)w(c) \\ &+\frac{1}{4}\int_a^c r(t)v(s) \ \Box\ h_1(s,a) - \left(\frac{p(s)}{r(s)} - \frac{v'(s)}{v(s)}\right)\sqrt{H(s,a)}\bigg]^2 ds. \end{aligned}$$

The following theorem is an immediate result from Lemma 3.1 and Lemma 3.2.

Theorme 3.1. *Assume that* (H1)-(H4) *and* (H5') *hold and that for some* $c \in (a,b)$ *and for some* $H \in X, v \in C^1([t_0,\ \infty),(0,\infty))$,

$$\begin{aligned} &\frac{1}{H(c,a)}\int_a^c H(s,a)K\mu_0 v(s)q(s)ds + \frac{1}{H(b,c)}\int_c^b H(b,s)K\mu_0 v(s)q(s)ds \\ (3.6)\quad &> \frac{1}{4H(c,a)}\int_a^c r(s)v(s)\left[h_1(s,a) - \left(\frac{p(s)}{r(s)} - \frac{v'(s)}{v(s)}\right)\sqrt{H(s,a)}\right]^2 ds \\ &+\frac{1}{4H(b,c)}\int_c^b r(s)v(s)\left[h_2(b,s) - \frac{v'(s)}{v(s)}\sqrt{H(b,s)}\right]^2 ds. \end{aligned}$$

Then every solution of Eq. (1.1) has at least one zero in (a,b).

Theorem 3.2. *Assume that* (H1)-(H4) *and* (H5') *holds. If, for each* $T \geq t_0$, *there exist* $H \in X$, $v \in C^1([t_0,\infty),(0,\infty))$ *and* $a,b,c \in R$ *such that* $T \le a < c < b$ *and* (3.6) *holds*, *then every solution of Eq.* (1.1) *is oscillatory.*

Theorem 3.3. *Assume that* (H1)-(H4) *and* (H5') *hold. If*

$$\limsup_{t\to\infty}\int_l^t \left[H(s,l)K\mu_0 v(s)q(s) - \frac{1}{4}r(s)v(s)\left(h_1(s,l) - \left(\frac{p(s)}{r(s)} - \frac{v'(s)}{v(s)}\right)\sqrt{H(s,l)}\right)^2\right]ds > 0,$$

and

$$\limsup_{t\to\infty}\int_l^t \left[H(t,s)K\mu_0 v(s)q(s) - \frac{1}{4}r(s)v(s)\left(h_2(t,s) + \left(\frac{p(s)}{r(s)} - \frac{v'(s)}{v(s)}\right)\sqrt{H(t,s)}\right)^2\right]ds > 0,$$

for some $H \in X$, $v \in C^1([t_0,\infty),\ (0,\infty))$ *and for each* $l \geq t_0$, *then every solution of Eq.* (1.1) *is oscillatory.*

Theorem 3.4. *Assume that* (H1)-(H4) and (H5') *hold. If for each* $T \geq t_0$, *there exist* $H \in X_0$, $v \in C^1([t_0\infty), (0,\infty))$ *and* $a, c \in R$ *such that* $T \quad a < c$ *and*

$$\int_a^c H(s-a)K\mu_0\,[v(s)q(s) + v(2c-s)q(2c-s)]\,ds$$
$$> \frac{1}{4}\int_a^c [r(s)v(s) + r(2c-s)v(2c-s)]\,h^2(s-a)ds$$
$$+\frac{1}{2}\int_a^c \big[r(2c-s)v'(2c-s) - p(2c-s)v(2c-s)$$
$$- r(s)v'(s) + p(s)r(s)]\,h(s-a)\sqrt{H(s-a)}ds$$
$$+\frac{1}{4}\int_a^c \Bigg[r(s)v(s)\left(\frac{p(s)}{r(s)} - \frac{v'(s)}{v(s)}\right)^2$$
$$+r(2c-s)v(2c-s)\left(\frac{p(2c-s)}{r(2c-s)} - \frac{v'(2c-s)}{v(2c-s)}\right)^2\Bigg]H(s-a)ds,$$

then every solution of Eq. (1.1) *is oscillatory.*

Corollary 3.1. *Assume that* (H1)-(H4) *and* (H5') *hold. Then every solution of Eq.* (1.1) *is oscillatory provided that for each* $l \geq t_0$ *and for some* $\lambda > 1$, *there exists a function* $v \in C^1([t_0,\infty), (0,\infty))$ *such that the following two inequalities hold*:

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-1}}\int_l^t (s-l)^\lambda \Bigg[K\mu_0 v(s)q(s)$$
$$-\frac{1}{4}r(s)v(s)\left(\frac{\lambda}{(s-l)} + \frac{p(s)}{r(s)} - \frac{v'(s)}{v(s)}\right)^2\Bigg]ds > 0,$$

and

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-1}}\int_l^t (t-s)^\lambda \Bigg[K\mu_0 v(s)q(s)$$
$$-\frac{1}{4}r(s)v(s)\left(\frac{\lambda}{(t-s)} - \frac{p(s)}{r(s)} + \frac{v'(s)}{v(s)}\right)^2\Bigg]ds > 0.$$

Theorem 3.5. *Assume that* (H1)-(H4) *and* (H5') *hold and that* $\lim_{t\to\infty} R(t) = \infty$. *Then every solution of Eq.* (1.1) *is oscillatory provided that for each* $l \geq t_0$ *and for some* $\lambda > 1$, *the following two inequalities hold*:

$$\limsup_{t\to\infty} \frac{1}{R^{\lambda-1}(t)}\int_l^t [R(s) - R(l)]^\lambda \left(K\mu_0 q(s) - \frac{p^2(s)}{4r(s)}\right)ds > \frac{\lambda^2}{4(\lambda-1)}$$

and

$$\limsup_{t\to\infty}\frac{1}{R^{\lambda-1}(t)}\int_l^t [R(t)-R(s)]^\lambda\left(K\mu_0 q(s)-\frac{p^2(s)}{4r(s)}\right)ds>\frac{\lambda^2}{4(\lambda-1)}.$$

4. Examples

In this section we will show the applications of our oscillation criteria by two examples. We will see that the equations in the examples are oscillatory based on the results in Sections 2 and 3, though the oscillation cannot be demonstrated by the results of Huang [9], Kong [11] and Li and Agarwal [15].

Example 1. *Consider the nonlinear differential equation*

$$\begin{aligned}&\big((1+sin^2 t)y'(t)\big)'-3\sin t\cos t y'(t)\\ &+\frac{1}{(1+\cos^4 t)(1+\sin^2 t)}y(t)\left[1+y^4(t)\right]\left[1+(y'(t))^2\right]=0,\end{aligned}\tag{4.1}$$

where $t\geq 1$. Clearly,

$$f'(y)=1+5y^4\geq 1=\mu>0 \quad\text{and}\quad g(y)=1+y^2\geq 1=K \text{ for all } y\in R.$$

Let us apply Corollary 2.1 with $\lambda=2$ and $v(t)=1$. A straightforward computation yields

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t^{\lambda-1}}\int_l^t (s-l)^\lambda\left\{q(s)-\frac{1}{4\mu}r(s)\left[\frac{\lambda}{(s-l)}-\frac{p(s)}{r(s)}\right]^2\right\}ds\\ &=\limsup_{t\to\infty}\frac{1}{t}\int_l^t (s-l)^2\left[\frac{1}{(1+\cos^4 s)(1+\sin^2 s)}+\frac{3\sin s\cos s}{s-l}\right.\\ &\quad\left.-\frac{1+\sin^2 s}{(s-l)^2}-\frac{9\sin^2 s\cos^2 s}{4(1+\sin^2 s)}\right]ds\\ &=\limsup_{t\to\infty}\frac{1}{t}\int_l^t \left\{(s-l)^2\left[\frac{1}{(1+\cos^4 s)(1+\sin^2 s)}-\frac{9\sin^2 s\cos^2 s}{4(1+\sin^2 s)}\right]\right.\\ &\quad\left.+\left[3(s-l)\sin s\cos s-(1+\sin^2 s)\right]\right\}ds\\ &=\infty,\end{aligned}$$

and

$$\limsup_{t\to\infty}\frac{1}{t^{\lambda-1}}\int_l^t (t-s)^\lambda\left\{q(s)-\frac{1}{4\mu}r(s)\left[\frac{\lambda}{(s-l)}-\frac{p(s)}{r(s)}\right]^2\right\}ds$$

$$
\begin{aligned}
&= \limsup_{t\to\infty} \frac{1}{t} \int_l^t (t-s)^2 \Bigg[\frac{1}{(1+\cos^4 s)(1+\sin^2 s)} + \frac{3\sin s\cos s}{t-s} \\
&\quad - \frac{1+\sin^2 s}{(t-s)^2} - \frac{9\sin^2 s\cos^2 s}{4(1+\sin^2 s)} \Bigg] ds \\
&= \limsup_{t\to\infty} \frac{1}{t} \int_l^t \Bigg\{ (t-s)^2 \Bigg[\frac{1}{(1+\cos^4 s)(1+\sin^2 s)} - \frac{9\sin^2 s\cos^2 s}{4(1+\sin^2 s)} \Bigg] \\
&\quad + \big[3(t-s)\sin s\cos s - (1+\sin^2 s)\big] \Bigg\} ds \\
&= \infty,
\end{aligned}
$$

since

$$
\begin{aligned}
&\frac{1}{(1+\cos^4 t)(1+\sin^2 t)} - \frac{9\sin^2 t\cos^2 t}{4(1+\sin^2 t)} \\
&= \frac{4 - 9\sin^2 t\cos^2 t}{(1+\cos^4 t)(1+\sin^2 t)} = \frac{4 - 9\sin^2 t + 9\sin^4 t}{(1+\cos^4 t)(1+\sin^2 t)} > 0
\end{aligned}
$$

for any $t \in R$. Thus, assumptions (2.13) and (2.14) hold, and we conclude by Corollary 2.1 that all solutions of Eq.(4.1) are oscillatory. Observe that $y(t) = \cos t$ is such a solution.

Example 2. *Consider the nonlinear differential equation*

$$
y''(t) - \sin t y'(t) + \frac{1+\cos t}{1+\sin^2 t} y(t)\left(1+y^2(t)\right) = 0, \quad t \geq 0. \tag{4.2}
$$

Let $r(t) = 1$ and $f(y) = y(1+y^2)$. Then

$$
R(t) = t, \quad f'(y) = 1 + 3y^2 \geq 1 = \mu.
$$

Let us apply Theorem 2.5 with $\lambda = 2$. Then

$$
\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t} \int_0^t s^2 \Bigg[\frac{1+\cos s}{1+\sin^2 s} - \frac{\sin^2 s}{4} \Bigg] ds \\
&= \limsup_{t\to\infty} \frac{1}{t} \int_0^t s^2 \Bigg[\frac{4(1+\cos s) - (1+\sin^2 s)\sin^2 s}{4(1+\sin^2 s)} \Bigg] \\
&\geq \limsup_{t\to\infty} \frac{1}{t} \int_0^t \frac{s^2}{4} ds \\
&= \infty,
\end{aligned}
$$

and

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\int_0^t (t-s)^2\left(\frac{1+\cos s}{1+\sin^2 s}-\frac{\sin^2 s}{4}\right)ds\\
&=\limsup_{t\to\infty}\frac{1}{t}\int_0^t (t-s)^2\left(\frac{4(1+\cos s)-(1+\sin^2 s)\sin^2 s}{4(1+\sin^2 s)}\right)\\
&\geq \limsup_{t\to\infty}\frac{1}{t}\int_0^t \frac{(t-s)^2}{4}ds\\
&=\infty.
\end{aligned}$$

Thus, assumptions (2.15) and (2.16) hold, and we conclude by Theorem 2.5 that all solutions of Eq. (4.2) are oscillatory. Observe that $y(t) = \sin t$ is such a solution.

The important point to note here is that the recent results due to Grace and Lalli [8, Theorems 1, 2, 4, 5, 6, 7 and Corollary 1] do not apply to Eqs.(4.1) and (4.2) since $p(t)$ is oscillatory.

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