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LINEAR FUNCTIONAL EQUATIONS IN A HILBERT MODULE

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Abstract. We prove the generalized Hyers-Ulam-Rassias stability of the invertible mapping in a Banach module over a unital Banach algebra in the spirit of Gavruta, and prove the generalized Hyers-Ulam-Rassias stability of linear functional equations in a Hilbert module over a unital C^*-algebra in the spirit of Gavruta.

INTRODUCTION

In 1940, S.M. Ulam [9] raised the following question: Under what conditions does there exist an additive mapping near an approximately additive mapping?

Let E_1 and E_2 be Banach spaces. Consider $f : E_1 \to E_2$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x + y) - f(x) - f(y)\| \cdot \ \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in E_1$. Th.M. Rassias [7] showed that there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \cdot \ \frac{2\epsilon}{2 - 2^p}||x||^p$$

for all $x \in E_1$.

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In 1994, Gavruta showed in [3] that the following: Let G be an abelian group and X a Banach space. Denote by $\varphi : G \times G \to [0, \infty)$ a function such that

$$\widetilde{\varphi}(x, y) = \sum_{j=1}^{\infty} 2^{-j} \varphi(2^{j-1}x, 2^{j-1}y) < \infty$$

for all $x, y \in G$. If $f : G \to X$ is a mapping satisfying

$$\|f(x + y) - f(x) - f(y)\| \cdot \ \varphi(x, y)$$

for all $x, y \in G$, then there exists a unique additive mapping $T : G \to X$ such that

$$\|f(x) - T(x)\| \cdot \ \tilde{\varphi}(x, x)$$

for all $x \in G$.

In this paper, let A be a unital Banach algebra with norm $|\cdot|$, $A_1 = \{a \in A \mid \ |a| = 1\}$, and ${}_A\mathcal{H}$ a left Banach A-module with norm $\|\cdot\|$. Throughout this paper, assume that $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are mappings such that $F(tx)$ and $G(tx)$ are continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_A\mathcal{H}$.

We are going to prove the generalized Hyers-Ulam-Rassias stability of the invertible mapping in a Banach module over a unital Banach algebra in the spirit of Gavruta.

Lemma 1. *Let $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be a mapping for which there exists a function $\varphi : {}_A\mathcal{H} \times {}_A\mathcal{H} \to [0, \infty)$ such that*

(i) $$\tilde{\varphi}(x, y) := \textstyle\sum_{k=0}^{\infty} 2^{-k} \varphi(2^k x, 2^k y) < \infty,$$

$$\|F(ax + ay) - aF(x) - aF(y)\| \cdot \ \varphi(x, y)$$

for all $a \in A_1$ and all $x, y \in {}_A\mathcal{H}$. Then there exists a unique A-linear mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ such that

(ii) $$\|F(x) - T(x)\| \cdot \ \frac{1}{2}\tilde{\varphi}(x, x)$$

for all $x \in {}_A\mathcal{H}$.

Proof. Put $a = 1 \in A_1$. By the Gavruta result [3], there exists a unique additive mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfying (ii). The mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ was given by $T(x) = \lim_{n\to\infty} \frac{F(2^n x)}{2^n}$ for all $x \in {}_A\mathcal{H}$. By the same reasoning as the proof of [7, Theorem], the additive mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is $\mathbb{R}$-linear.

By the assumption, for each $a \in A_1$,

$$\|F(2^n ax) - 2aF(2^{n-1}x)\| \cdot \ \varphi(2^{n-1}x, 2^{n-1}x)$$

for all $x \in {}_A\mathcal{H}$. Using the fact that for each $a \in A$ and each $z \in {}_A\mathcal{H}$ $\|az\| \cdot K|a| \cdot \|z\|$ for some $K > 0$, one can show that

$$\|aF(2^n x) - 2aF(2^{n-1}x)\| \cdot \ K|a| \cdot \|F(2^n x) - 2F(2^{n-1}x)\| \cdot \ K\varphi(2^{n-1}x, 2^{n-1}x)$$

for all $a \in A_1$ and all $x \in {}_A\mathcal{H}$. So

$$\begin{aligned} \|F(2^n ax) - aF(2^n x)\| \cdot & \ \|F(2^n ax) - 2aF(2^{n-1}x)\| + \|2aF(2^{n-1}x) - aF(2^n x)\| \\ \cdot & \ \varphi(2^{n-1}x, 2^{n-1}x) + K\varphi(2^{n-1}x, 2^{n-1}x) \end{aligned}$$

for all $a \in A_1$ and all $x \in {}_A\mathcal{H}$. Thus $2^{-n}\|F(2^n ax) - aF(2^n x)\| \to 0$ as $n \to \infty$ for all $a \in A_1$ and all $x \in {}_A\mathcal{H}$. Hence

$$T(ax) = \lim_{n\to\infty} \frac{F(2^n ax)}{2^n} = \lim_{n\to\infty} \frac{aF(2^n x)}{2^n} = aT(x)$$

for each $a \in A_1$. So

$$T(ax) = |a|T(\frac{a}{|a|}x) = |a|\frac{a}{|a|}T(x) = aT(x)$$

for all $a \in A (a \neq 0)$ and all $x \in {}_A\mathcal{H}$. Hence

$$T(ax + by) = T(ax) + T(by) = aT(x) + bT(y)$$

for all $a, b \in A$ and all $x, y \in {}_A\mathcal{H}$. So the unique $\mathbb{R}$-linear mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is an A-linear mapping, as desired. ■

Theorem 2. *Let $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be mappings for which there exists a function $\varphi : {}_A\mathcal{H} \times {}_A\mathcal{H} \to [0, \infty)$ satisfying (i) such that*

$$\begin{aligned} \|F(ax + ay) - aF(x) - aF(y)\| & \ \cdot \ \varphi(x, y), \\ G(ax + ay) - aG(x) - aG(y)\| & \ \cdot \ \varphi(x, y)\| \end{aligned}$$

for all $a \in A_1$ and all $x, y \in {}_A\mathcal{H}$. Assume that $F(2^n x) = 2^n F(x)$ and $G(2^n x) = 2^n G(x)$ for all positive integers n and all $x \in {}_A\mathcal{H}$. Then the mappings $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are A-linear mappings. Furthermore, if the mappings $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfy the inequalities

$$\begin{aligned} \|F \circ G(x) - x\| & \ \cdot \ \varphi(x, x), \\ \|G \circ F(x) - x\| & \ \cdot \ \varphi(x, x) \end{aligned}$$

for all $x \in {}_A\mathcal{H}$, then the mapping G is the inverse of the mapping F.

Proof. By the same method as the proof of Lemma 1, one can show that there exists a unique A-linear mapping $L : {}_A\mathcal{H} \to {}_A\mathcal{H}$ such that

$$\|G(x) - L(x)\| \cdot \ \frac{1}{2}\widetilde{\varphi}(x, x)$$

for all $x \in {}_A\mathcal{H}$.

By the assumption,

$$\begin{aligned} T(x) &= \lim_{n\to\infty} \frac{F(2^n x)}{2^n} = F(x), \\ L(x) &= \lim_{n\to\infty} \frac{G(2^n x)}{2^n} = G(x) \end{aligned}$$

for all $x \in {}_A\mathcal{H}$, where the mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is given in the proof of Lemma 1. Hence the A-linear mappings T and L are the mappings F and G, respectively. So the mappings $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are A-linear mappings.

Now by the assumption,

$$\begin{aligned} \|F \circ G(2^n x) - 2^n x\| &\ \cdot \ \varphi(2^n x, 2^n x), \\ \|G \circ F(2^n x) - 2^n x\| &\ \cdot \ \varphi(2^n x, 2^n x) \end{aligned}$$

for all positive integers n and all $x \in {}_A\mathcal{H}$. Thus

$$\begin{aligned} 2^{-n}\|F \circ G(2^n x) - 2^n x\| &\to 0, \\ 2^{-n}\|G \circ F(2^n x) - 2^n x\| &\to 0 \end{aligned}$$

as $n \to \infty$ for all $x \in {}_A\mathcal{H}$. Hence

$$\begin{aligned} F \circ G(x) &= \lim_{n\to\infty} \frac{F \circ G(2^n x)}{2^n} = x, \\ G \circ F(x) &= \lim_{n\to\infty} \frac{G \circ F(2^n x)}{2^n} = x \end{aligned}$$

for all $x \in {}_A\mathcal{H}$. So the mapping G is the inverse of the mapping F. ■

From now on, let A be a unital C^*-algebra with norm $|\cdot|$, A_1^+ the set of positive elements in A_1, and ${}_A\mathcal{H}$ a left Hilbert A-module with norm $\|\cdot\|$.

Now we are going to prove the generalized Hyers-Ulam-Rassias stability of linear functional equations in a Hilbert module over a unital C^*-algebra in the spirit of Gavruta.

Lemma 3. *Let $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be a mapping for which there exists a function $\varphi : {}_A\mathcal{H} \times {}_A\mathcal{H} \to [0, \infty)$ satisfying (i) such that*

$$\|F(ax + ay) - aF(x) - aF(y)\| \cdot \ \varphi(x, y)$$

for all $a \in A_1^+ \cup \{i\}$ and all $x, y \in {}_A\mathcal{H}$. Then there exists a unique A-linear operator $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfying (ii).

Proof. By the same reasoning as the proof of Lemma 1, there exists a unique $\mathbb{R}$-linear mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfying (ii).

By the same method as the proof of Lemma 1, one can obtain that

$$T(ax) = \lim_{n\to\infty} \frac{F(2^n ax)}{2^n} = \lim_{n\to\infty} \frac{aF(2^n x)}{2^n} = aT(x)$$

for each $a \in A_1^+ \cup \{i\}$. So

$$\begin{aligned} T(ax) = |a|T(\frac{a}{|a|}x) = &\ |a|\frac{a}{|a|}T(x) = aT(x), \quad \forall a \in A^+ (a \neq 0),\ \forall x \in {}_A\mathcal{H}, \\ T(ix) = &\ iT(x), \quad \forall x \in {}_A\mathcal{H}. \end{aligned}$$

For any element $a \in A$, $a = \dfrac{a+a^*}{2} + i\dfrac{a-a^*}{2i}$, and $\frac{a+a^*}{2}$ and $\frac{a-a^*}{2i}$ are self-adjoint elements, furthermore, $a = (\frac{a+a^*}{2})^+ - (\frac{a+a^*}{2})^- + i(\frac{a-a^*}{2i})^+ - i(\frac{a-a^*}{2i})^-$, where $(\frac{a+a^*}{2})^+$, $(\frac{a+a^*}{2})^-$, $(\frac{a-a^*}{2i})^+$, and $(\frac{a-a^*}{2i})^-$ are positive elements (see [2, Lemma 38.8]). So

$$\begin{aligned} T(ax) &= T((\frac{a+a^*}{2})^+ x - (\frac{a+a^*}{2})^- x + i(\frac{a-a^*}{2i})^+ x - i(\frac{a-a^*}{2i})^- x) \\ &= (\frac{a+a^*}{2})^+ T(x) + (\frac{a+a^*}{2})^- T(-x) + (\frac{a-a^*}{2i})^+ T(ix) + (\frac{a-a^*}{2i})^- T(-ix) \\ &= (\frac{a+a^*}{2})^+ T(x) - (\frac{a+a^*}{2})^- T(x) + i(\frac{a-a^*}{2i})^+ T(x) - i(\frac{a-a^*}{2i})^- T(x) \\ &= ((\frac{a+a^*}{2})^+ - (\frac{a+a^*}{2})^- + i(\frac{a-a^*}{2i})^+ - i(\frac{a-a^*}{2i})^-)T(x) = aT(x) \end{aligned}$$

for all $a \in A$ and all $x \in {}_A\mathcal{H}$. Hence

$$T(ax + by) = T(ax) + T(by) = aT(x) + bT(y)$$

for all $a, b \in A$ and all $x, y \in {}_A\mathcal{H}$. So the unique $\mathbb{R}$-linear mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is an A-linear operator, as desired. ∎

Theorem 4. *Let $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be a mapping for which there exists a function $\varphi : {}_A\mathcal{H} \times {}_A\mathcal{H} \to [0, \infty)$ satisfying (i) such that*

$$\|F(ax + ay) - aF(x) - aF(y)\| \cdot \ \varphi(x, y)$$

for all $a \in A_1^+ \cup \{i\}$ and all $x, y \in {}_A\mathcal{H}$. Assume that $F(2^n x) = 2^n F(x)$ for all positive integers n and all $x \in {}_A\mathcal{H}$. Then the mapping $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is an

A-linear operator. Furthermore, (1) if the mapping $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfies the inequality

$$\|F(x) - F^*(x)\| \cdot \ \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, then the mapping $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a self-adjoint operator, if the mapping $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfies the inequality

$$\|F \circ F^*(x) - F^* \circ F(x)\| \cdot \ \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, then the mapping $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a normal operator, if the mapping $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfies the inequalities

$$\|F \circ F^*(x) - x\| \ \cdot \ \varphi(x, x),$$
$$\|F^* \circ F(x) - x\| \ \cdot \ \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, then the mapping $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a unitary operator, and if the mapping $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfies the inequalities

$$\|F \circ F(x) - F(x)\| \ \cdot \ \varphi(x, x),$$
$$\|F^*(x) - F(x)\| \ \cdot \ \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, then the mapping $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a projection.

Proof. By the assumption,

$$T(x) = \lim_{n\to\infty} \frac{F(2^n x)}{2^n} = F(x)$$

for all $x \in {}_A\mathcal{H}$, where the operator $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is given in the proof of Lemma 3. So the A-linear operator $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is the mapping $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$.

(1) By the assumption,

$$\|F(2^n x) - F^*(2^n x)\| \cdot \ \varphi(2^n x, 2^n x)$$

for all positive integers n and all $x \in {}_A\mathcal{H}$. Thus $2^{-n}\|F(2^n x) - F^*(2^n x)\| \to 0$ as $n \to \infty$ for all $x \in {}_A\mathcal{H}$. Hence

$$F(x) = \lim_{n\to\infty} \frac{F(2^n x)}{2^n} = \lim_{n\to\infty} \frac{F^*(2^n x)}{2^n} = F^*(x)$$

for all $x \in {}_A\mathcal{H}$. So the A-linear mapping F is a self-adjoint operator.

(2) By the assumption,

$$\|F \circ F^*(2^n x) - F^* \circ F(2^n x)\| \cdot \ \varphi(2^n x, 2^n x)$$

for all positive integers n and all $x \in {}_A\mathcal{H}$. Thus $2^{-n}\|F \circ F^*(2^n x) - F^* \circ F(2^n x)\| \to 0$ as $n \to \infty$ for all $x \in {}_A\mathcal{H}$. Hence

$$F \circ F^*(x) = \lim_{n\to\infty} \frac{F \circ F^*(2^n x)}{2^n} = \lim_{n\to\infty} \frac{F^* \circ F(2^n x)}{2^n} = F^* \circ F(x)$$

for all $x \in {}_A\mathcal{H}$. So the A-linear mapping F is a normal operator.

(3) By the assumption,

$$\begin{aligned} \|F \circ F^*(2^n x) - 2^n x\| &\cdot \varphi(2^n x, 2^n x), \\ \|F^* \circ F(2^n x) - 2^n x\| &\cdot \varphi(2^n x, 2^n x) \end{aligned}$$

for all positive integers n and all $x \in {}_A\mathcal{H}$. Thus

$$\begin{aligned} 2^{-n}\|F \circ F^*(2^n x) - 2^n x\| &\to 0, \\ 2^{-n}\|F^* \circ F(2^n x) - 2^n x\| &\to 0 \end{aligned}$$

as $n \to \infty$ for all $x \in {}_A\mathcal{H}$. Hence

$$\begin{aligned} F \circ F^*(x) &= \lim_{n\to\infty} \frac{F \circ F^*(2^n x)}{2^n} = x, \\ F^* \circ F(x) &= \lim_{n\to\infty} \frac{F^* \circ F(2^n x)}{2^n} = x \end{aligned}$$

for all $x \in {}_A\mathcal{H}$. So the A-linear mapping F is a unitary operator.

(4) By the assumption,

$$\begin{aligned} \|F \circ F(2^n x) - F(2^n x)\| &\cdot \varphi(2^n x, 2^n x), \\ \|F^*(2^n x) - F(2^n x)\| &\cdot \varphi(2^n x, 2^n x) \end{aligned}$$

for all positive integers n and all $x \in {}_A\mathcal{H}$. Thus

$$\begin{aligned} 2^{-n}\|F \circ F(2^n x) - F(2^n x)\| &\to 0, \\ 2^{-n}\|F^*(2^n x) - F(2^n x)\| &\to 0 \end{aligned}$$

as $n \to \infty$ for all $x \in_A \mathcal{H}$. Hence

$$\begin{aligned} F \circ F(x) &= \lim_{n\to\infty} \frac{F \circ F(2^n x)}{2^n} = \lim_{n\to\infty} \frac{F(2^n x)}{2^n} = F(x), \\ F^*(x) &= \lim_{n\to\infty} \frac{F^*(2^n x)}{2^n} = \lim_{n\to\infty} \frac{F(2^n x)}{2^n} = F(x) \end{aligned}$$

for all $x \in {}_A\mathcal{H}$. So the A-linear mapping F is a projection. ∎

Remark. When the inequalities

$$\|F(ax+ay)-aF(x)-aF(y)\| \cdot \varphi(x,y)$$

in the statements of the above results are replaced by the inequalities

$$\|aF(x+y)-F(ax)-F(ay)\| \cdot \varphi(x,y)$$

or the inequalities

$$\begin{aligned} \|F(x+y)-F(x)-F(y)\| &\cdot \varphi(x,y), \\ \|F(ax)-aF(x)\| &\cdot \varphi(x,x) \end{aligned}$$

the results do also hold. The proofs are similar to the proofs of the results.

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