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ALGORITHMIC ASPECT OF k-TUPLE DOMINATION IN GRAPHS

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Abstract. In a graph G, a vertex is said to dominate itself and all of its neighbors. For a fixed positive integer k, the k-tuple domination problem is to find a minimum sized vertex subset such that every vertex in the graph is dominated by at least k vertices in this set. The present paper studies the k-tuple domination problem in graphs from an algorithmic point of view. In particular, we give a linear-time algorithm for the 2-tuple domination problem in trees by employing a labeling method.

1. Introduction

The concept of domination in graph theory is a good model for many location problems in operations research. In a graph G, a vertex is said to *dominate* itself and all of its neighbors. A *dominating set* of $G = (V, E)$ is a subset D of V such that every vertex in V is dominated by some vertex in D. The *domination number* $\gamma(G)$ is the minimum size of a dominating set of G. Domination and its variations have been extensively studied in the literature; see [3, 7, 8].

Among the variations of domination, the k-tuple domination was introduced in [6]; also see [7, p. 189]. For a fixed positive integer k, a *k-tuple dominating set* of $G = (V, E)$ is a subset D of V such that every vertex in V is dominated by at least k vertices of D. The *k-tuple domination number* $\gamma_{\times k}(G)$ is the minimum cardinality of a k-tuple dominating set of G. The special case when $k = 1$ is the usual domination. The case when $k = 2$ was called *double domination* in [6], where exact values of the double domination numbers for some special graphs are obtained. The same paper also gives various bounds of double and k-tuple domination numbers in

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terms of other parameters. Nordhaus-Gaddum type inequality for double domination was given in [5].

The purpose of this paper is to study the k-tuple domination problem from an algorithmic point of view. In particular, we give a linear-time algorithm for the 2-tuple domination problem in trees.

We note that not every graph has a k-tuple dominating set. In fact, a graph G has a k-tuple dominating set if and only if $\delta(G)+1 \geq k$, where $\delta(G)$ is the minimum degree of a vertex in G. As any nontrivial tree has at least two leaves, we only consider 2-tuple domination for trees.

To establish our algorithm, we employ a labeling method similar to those for variations of domination in tree-type graphs; see [2, 4, 9, 10, 11, 12, 13]. Suppose $G=(V,E)$ is a graph in which every vertex v is associated with a label $M(v)=(t(v),k(v))$, where $t(v) \in \{\mathrm{B},\mathrm{R}\}$ and $k(v)$ is a nonnegative integer. The interpretation of the label is that we want to find a "dominating set" D containing all vertices u with $t(u)=\mathrm{R}$ (called *required* vertices) such that each vertex v is dominated by at least $k(v)$ vertices in D. More precisely, an *M-dominating set* of $G=(V,E)$ is a subset D of V satisfying the following conditions:

(M1) If $t(v)=\mathrm{R}$, then $v \in D$.

(M2) $|N_G[v] \cap D| \geq k(v)$ for all vertices $v \in V$, where $N_G[v]=\{v\} \cup \{u \in V : uv \in E\}$ is the *closed neighborhood* of the vertex v.

The *M-domination number* $\gamma_M(G)$ is the minimum cardinality of an M-dominating set in G. Notice that 2-tuple domination is M-domination with $M(v)=(\mathrm{B},2)$ for all vertices v in V. Also, G has an M-dominating set, i.e., $\gamma_M(G)$ is finite, if and only if $|N_G[v]| \geq k(v)$ for all vertices v in V. For instance, if G contains exactly one vertex x, then $\gamma_M(G)=0$ when $M(x)=(\mathrm{B},0)$, $\gamma_M(G)=1$ when $M(x) \in \{(\mathrm{B},1),(\mathrm{R},0),(\mathrm{R},1)\}$, and $\gamma_M(G)=\infty$ otherwise.

2. 2-Tuple Domination in Trees

To give an algorithm for the 2-tuple domination problem in trees, we in fact establish one for the M-domination problem in trees. We believe that the approach has potential for other classes of graphs. We first give the following theorem which is the base of the algorithm. Notice that it works for general graphs.

Theorem 2.1. *Suppose $G=(V,E)$ is a nontrivial graph in which every vertex v has a label $M(v)=(t(v),k(v))$. Let x be a leaf adjacent to y.*

(1) *If $k(x)>2$ or $k(y)>|N_G[y]|$, then G has no M-dominating set.*

(2) *If* $k(x) = 2$ *or* $k(y) = |N_G[y]|$, *then* $\gamma_M(G) = \gamma_{M'}(G') + 1$, *where* G' *is obtained from* G *by deleting* x *and* M' *is obtained from* M *by relabeling* y *with* $t'(y) = \mathrm{R}$ *and* $k'(y) = \max\{k(y) - 1, 0\}$.

(3) *If* $t(x) = \mathrm{R}$ *and* $k(x) < 2$ *and* $k(y) < |N_G[y]|$, *then* $\gamma_M(G) = \gamma_{M'}(G') + 1$, *where* G' *is obtained from* G *by deleting* x *and* M' *is obtained from* M *by relabeling* y *with* $k'(y) = \max\{k(y) - 1, 0\}$.

(4) *If* $M(x) = (\mathrm{B}, 1)$ *and* $k(y) < |N_G[y]|$, *then* $\gamma_M(G) = \gamma_{M'}(G')$, *where* G' *is obtained from* G *by deleting* x *and* M' *is obtained from* M *by relabeling* y *with* $t'(y) = \mathrm{R}$.

(5) *If* $M(x) = (\mathrm{B}, 0)$ *and* $k(y) < |N_G[y]|$, *then* $\gamma_M(G) = \gamma_M(G - x)$.

Proof. (1) This follows from the definition of M-domination.

(2) Suppose D' is a minimum M'-dominating set of G'. Then $y \in D'$, since $t'(y) = \mathrm{R}$. Hence, $D = D' \cup \{x\}$ is an M-dominating set of G, since $|N_G[x] \cap D| \geq 2 \geq k(x)$. Thus, $\gamma_{M'}(G') + 1 = |D'| + 1 = |D| \geq \gamma_M(G)$.

On the other hand, suppose D is a minimum M-dominating set of G. Then $x, y \in D$, since $k(x) = 2$ or $k(y) = |N_G[y]|$. Hence, $D' = D \backslash \{x\}$ is an M'-dominating set of G', since $y \in D'$ and $|N_{G'}[y] \cap D'| = |N_G[y] \cap D| - 1 \geq \max\{k(y) - 1, 0\} = k'(y)$. So, $\gamma_M(G) = |D| = |D'| + 1 \geq \gamma_{M'}(G') + 1$.

These complete the proof of $\gamma_M(G) = \gamma_{M'}(G') + 1$.

(3) Suppose D' is an M'-dominating set of G'. Then $D = D' \cup \{x\}$ is an M-dominating set of G, since $|N_G[x] \cap D| \geq 1 \geq k(x)$. Thus, $\gamma_{M'}(G') + 1 = |D'| + 1 = |D| \geq \gamma_M(G)$.

On the other hand, suppose D is a minimum M-dominating set of G. Then Then $x \in D$, since $t(x) = \mathrm{R}$. Hence, $D' = D \backslash \{x\}$ is an M'-dominating set of G', since $|N_{G'}[y] \cap D'| = |N_G[y] \cap D| - 1 \geq \max\{k(y) - 1, 0\} = k'(y)$. So, $\gamma_M(G) = |D| = |D'| + 1 \geq \gamma_{M'}(G') + 1$.

These complete the proof of $\gamma_M(G) = \gamma_{M'}(G') + 1$.

(4) Suppose D' is a minimum M'-dominating set of G'. Then $y \in D'$, since $t'(y) = \mathrm{R}$. Consequently, D' is an M-dominating set of G as $M(x) = (\mathrm{B}, 1)$. Thus, $\gamma_{M'}(G') = |D'| \geq \gamma_M(G)$.

On the other hand, suppose that D is a minimum M-dominating set of G. If $x \notin D$, then $y \in D$, since $k(x) = 1$. And so, D is an M'-dominating set of G'. Therefore, $\gamma_M(G) = |D| \geq \gamma_{M'}(G')$. We may now assume that $x \in D$. Let $D' = D \backslash \{x\}$. If $y \in D'$ and $|N_{G'}[y] \cap D'| \geq k(y)$, then D' is an M'-dominating set of G' and so $\gamma_M(G) = |D| > |D'| \geq \gamma_{M'}(G')$. So now $y \notin D'$ or $|N_{G'}[y] \cap D'| < k(y) \quad |N_G[y]| - 1 = |N_{G'}[y]|$. For the case when $y \notin D'$, let $z = y$; for the case when $y \in D'$, choose a vertex $z \in N_{G'}[y] \backslash D'$. Then, in any case, $y \in D' \cup \{z\}$ and so $D' \cup \{z\}$ is an M'-dominating set of G'. Hence, $\gamma_M(G) = |D| = |D' \cup \{z\}| \geq \gamma_{M'}(G')$.

These complete the proof of $\gamma_M(G) = \gamma_{M'}(G')$.

(5) Suppose D' is a minimum M-dominating set of $G - x$. Then D' is also an M-dominating set of G, since $t(x) = \mathrm{B}$ and $k(x) = 0$. Therefore, $\gamma_M(G - x) = |D'| \geq \gamma_M(G)$.

On the other hand, suppose that D is a minimum M-dominating set of G. If $x \notin D$, then D is also an M-dominating set of $G-x$. Thus, $\gamma_M(G) = |D| \geq \gamma_M(G-x)$. We may now assume that $x \in D$. Let $D' = D \backslash \{x\}$. If $|N_{G-x}[y] \cap D'| \geq k(y)$, then D' is an M-dominating set of $G - x$ and so $\gamma_M(G) = |D| > |D'| \geq \gamma_M(G - x)$. So now $|N_{G-x}[y] \cap D'| < k(y) \quad |N_G[y]| - 1 = |N_{G-x}[y]|$. Choose a vertex $z \in N_{G-x}[y] \backslash D'$. Then $D' \cup \{z\}$ is an M-dominating set of $G - x$. Hence, $\gamma_M(G) = |D| = |D' \cup \{z\}| \geq \gamma_M(G - x)$.

These complete the proof of $\gamma_M(G) = \gamma_M(G - x)$. ■

Based on the theorem above, we have the following linear-time algorithm for the M-domination problem in trees.

Algorithm. Find an M-dominating set of a tree.
Input. A tree $T = (V, E)$ in which each vertex v is labeled by $M(v) = (t(v), k(v))$.
Output. A minimum M-dominating set D of T.
Method.

$D \leftarrow \emptyset$;
$T' \leftarrow T$;
while (T' has at least two vertices) **do**
  choose a leaf x adjacent to y in T';
  if ($k(x) > 2$ or $k(y) > |N_{T'}[y]|$) **then**
    stop since there is no M-dominating set;
  elseif ($k(x) = 2$ or $k(y) = |N_{T'}[y]|$) **then**
    $t(y) = \mathrm{R}$ and $k(y) = \max\{k(y) - 1, 0\}$ and $D \leftarrow D \cup \{x\}$;
  elseif $\{*$ now $k(x) < 2$ and $k(y) < |N_{T'}[y]|$ $*\}$ ($t(x) = \mathrm{R}$) **then**
    $k(y) = \max\{k(y) - 1, 0\}$ and $D \leftarrow D \cup \{x\}$;
  elseif $\{*$ now $t(x) = \mathrm{B}$, $k(x) < 2$, $k(y) < |N_{T'}[y]|$ $*\}$ ($k(x) = 1$)
  then $t(y) = \mathrm{R}$;
  $T' \leftarrow T' - x$ $\{*$ delete x from T' $*\}$;
end while;
suppose the only vertex of T' is x;
if ($k(x) > 1$) **then STOP** as there is no M-dominating set;
elseif ($t(x) = \mathrm{R}$ or $k(x) = 1$) **then** $D \leftarrow D \cup \{x\}$.

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