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ON THE EXISTENCE OF STRONG SOLUTIONS TO SOME SEMILINEAR ELLIPTIC PROBLEMS

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Abstract. We study the following semilinear elliptic problem:

$$\begin{cases} \displaystyle\sum_{i,j=1}^{N} a_{ij}(x,u)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x,u)\frac{\partial u}{\partial x_i} + c(x,u)u = f(x) \quad \text{in } B, \\ u = 0 \qquad \text{on } \partial B, \end{cases}$$

where B is a ball in $\mathbb{R}^N$, $N \geq 3$, $a_{ij} = a_{ij}(x,r) \in C^{0,1}(\bar{B} \times \mathbb{R})$, a_{ij}, $\partial a_{ij}/\partial x_i$, $\partial a_{ij}/\partial r$, b_i, $c \in L^\infty(B \times \mathbb{R})$, with $i,j = 1,2,\cdots,N$ and $c \cdot\ 0$, and $f \in L^p(B)$. For each p, $p \geq N$, there exists a strong solution $u \in W^{2,p}(B) \cap W_0^{1,p}(B)$ provided the oscillations of a_{ij} with respect to r are sufficiently small. Moreover, for $N/2 < p < N$, if $\|f\|_{L^p}$ is small enough, then the existence result remains hold.

1. Introduction

Let   be an open set in $\mathbb{R}^N$, $N \geq 3$. $W^{m,p}(\ ) = \{u \in L^p(\ )|$ weak derivatives $D^\alpha u \in L^p(\ )$ for all $|\alpha| \cdot\ m\}$, $W_0^{m,p}(\ )$ is the closure of $C_0^\infty(\ )$ in $W^{m,p}(\ )$ and $W_{\text{loc}}^{m,p}(\ )$ is the space consisting of functions belonging to $W^{m,p}(\ ')$ for all $\ ' \subset\ $. $H^m(\ ) = W^{m,2}(\ )$, $H_0^m(\ ) = W_0^{m,2}(\ )$. $B_R(y)$ is the open ball in $\mathbb{R}^N$ of radius R centered at y. $B_R^+(y) = B_R(y) \cap \mathbb{R}_+^N = \{x = (x_1, \cdots, x_N) \in B_R(y) | x_N > 0\}$.

We investigate the following semilinear elliptic problem in a $C^{1,1}$ domain $\ \subset \mathbb{R}^N$, $N \geq 3$:

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$$(1.1)\quad \begin{cases} Lu = \sum_{i,j=1}^{N} a_{ij}(x,u)\dfrac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x,u)\dfrac{\partial u}{\partial x_i} + c(x,u)u = f(x) \quad \text{in} \quad , \\ u = 0 \qquad \text{on } \partial \quad , \end{cases}$$

where $f \in L^p(\ )$.

Define the mapping F in $W^{2,p}(\ ) \cap W_0^{1,p}(\ )$ by letting $u = F(v)$ be the unique solution in $W^{2,p}(\ ) \cap W_0^{1,p}(\ )$ to the *linear* elliptic problem:

$$(1.2)\quad \begin{cases} L_v u = \sum_{i,j=1}^{N} a_{ij}(x,v)\dfrac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x,v)\dfrac{\partial u}{\partial x_i} + c(x,v)u = f(x) \quad \text{in} \quad , \\ u = 0 \qquad \text{on } \partial \quad . \end{cases}$$

The unique solvability of problem (1.2) is guaranteed by the linear existence result [4, Theorem 9.15] under appropriate coefficients conditions. We notice here that F is well-defined for $p > N/2$ and is continuous in the topology of $H^1(\ )$ [3]. One then intends to find a fixed point of F. Observe that the well-known regularity theorem of Agmon-Douglis-Nirenberg [1] asserts that

$$(1.3) \qquad \|u\|_{W^{2,p}(\ )} \cdot \ C(\|u\|_{L^p(\ )} + \|L_v u\|_{L^p(\ )}),$$

where C is a constant depending on the moduli of continuity of the coefficients $a_{ij}(x, v(x))$ on $\bar{\ }$, etc. If $a_{ij}(x,v) = a_{ij}(x)$, then the constant C in (1.3) is independent of v; furthermore, there exists a constant C independent of v such that

$$(1.4) \qquad \|u\|_{W^{2,p}(\ )} \cdot \ C\|L_v u\|_{L^p(\ )}.$$

Applying the Schauder fixed point theorem, one can readily obtain a solution to problem (1.1). However, for the case that a_{ij} depends on both x and v, the constant C in (1.3) varies with v.

Our main idea is to make the constant in (1.3) be independent of v. When is a ball B in $\mathbb{R}^N$, a global $W^{2,p}$ estimate for $u \in W^{2,p}(B) \cap W_0^{1,p}(B)$ is established in Section 2 under stronger coefficients conditions on a_{ij} with $a_{ij} = a_{ij}(x,r) \in C^{0,1}(\bar{B} \times \mathbb{R})$ and sufficiently small oscillations with respect to r. In Section 3, the global $W^{2,p}$ estimate together with the maximum principle [2] for the solution of problem (1.2),

$$\sup |u| \cdot \ C\|f\|_{L^N(\ )},$$

leads directly to the existence of solutions to problem (1.1) in B provided $p \geq N$. Moreover, for $p < N$, if $\|f\|_{L^p}$ is small enough, then the existence result can be also asserted. Besides, existence of solutions in some other specific domains is also considered in this paper.

2. $W^{2,p}$ Estimates

Recall that an operator L in (1.1) is said to be elliptic in Ω if there exists $\lambda > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x,r)\xi_i\xi_j \ge \lambda|\xi|^2 \quad \text{for } (r,\xi)\in\mathbb{R}\times\mathbb{R}^N \text{ and a.e. } x\in\Omega. \tag{2.1}$$

For a fixed point $x\in\mathbb{R}^N$, we denote osc $a_{ij}(x,r)$ the oscillation of a_{ij} with respect to r in $\mathbb{R}$, that is, osc $a_{ij}(x,r) = \sup\{a_{ij}(x,r_1) - a_{ij}(x,r_2) | r_1,\ r_2\in\mathbb{R}\}$, and let

$$\text{osc } a(x,r) = \max_{1\le i,j\le N} \text{osc } a_{ij}(x,r).$$

For $v\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$, let $L_v u$ be given by (1.2). We start this section by observing an interior $W^{2,p}$ estimate in an open set $\Omega\subset\mathbb{R}^N$ for $u\in W^{2,p}_{\text{loc}}(\Omega)\cap L^p(\Omega)$, with $L_v u\in L^p(\Omega)$, which will then be applied to derive a global $W^{2,p}$ estimate for $u\in W^{2,p}(B)\cap W_0^{1,p}(B)$, with $L_v u\in L^p(B)$, in a ball $B\subset\mathbb{R}^N$ in Proposition 2.2.

Notice that the interior $W^{2,p}$ estimate for the linear case formulated in Theorem 9.11 [4, p. 235] is derived by a uniform perturbation of the coefficients $a_{ij}(x)$ in the neighborhoods of finite points in Ω. In the present case that $a_{ij} = a_{ij}(x,u)$, an interior $W^{2,p}$ estimate can be established along the same line provided the oscillations of a_{ij} with respect to r are sufficiently small. Therefore, we have the following lemma in which K is a constant depending only on N, p, and satisfying

$$\|D^2 w\|_{L^p(\Omega)} \le K\|\Delta w\|_{L^p(\Omega)}, \tag{2.2}$$

where $w\in W_0^{2,p}(\Omega)$ [4].

Lemma 2.1. *Let Ω be an open set in $\mathbb{R}^N$ and the coefficients of L satisfy, for a positive constant Λ,*

$$a_{ij}\in C^{0,1}(\Omega\times\mathbb{R}),\ b_i,\ c\in L^\infty(\Omega\times\mathbb{R}),\ |a_{ij}|,\ |b_i|,\ |c|\le\Lambda, \tag{2.3}$$

where $i,j = 1,\cdots,N$. Suppose that

$$\text{osc } a(x,r) \le \frac{\lambda}{4K} \qquad \forall x\in\Omega, \tag{2.4}$$

where K is given by (2.2). *Then if $u\in W^{2,p}_{\text{loc}}(\Omega)\cap L^p(\Omega)$ and $L_v u\in L^p(\Omega)$, with $1<p<\infty$, we have for any domain $\Omega'\subset\subset\Omega$ the estimate*

$$\|u\|_{W^{2,p}(\Omega')} \le C(\|u\|_{L^p(\Omega)} + \|L_v u\|_{L^p(\Omega)}), \tag{2.5}$$

where C is a constant $($independent of $v)$ depending on N, p, λ, Λ, $\;'$, with respect to x on $\;'$. ∎

To simplify the boundary estimate, we refrain   to be a ball in $\mathbb{R}^N$. Thus, we can further derive a local boundary estimate which together with Lemma 2.1 enables us to establish the following global estimate.

Proposition 2.2. *Let B be a ball in $\mathbb{R}^N$ and the operator L satisfy* (2.3) *with $a_{ij}(x,r) \in C^{0,1}(\bar{B} \times \mathbb{R})$. Suppose that*

$$\text{osc } a(x,r) \cdot \quad \frac{\lambda}{4K} \qquad \forall x \in B, \tag{2.6}$$

$$\text{osc } a(x,r) < \frac{\lambda}{8N^2K} \qquad \forall x \in \partial B, \tag{2.7}$$

where K is given by (2.2)*. Then if $u \in W^{2,p}(B) \cap W_0^{1,p}(B)$ and $L_v u \in L^p(B)$, with $1 < p < \infty$, we have the estimate*

$$\|u\|_{W^{2,p}(B)} \cdot \quad C(\|u\|_{L^p(B)} + \|L_v u\|_{L^p(B)}), \tag{2.8}$$

where C is a constant $($independent of $v)$ depending on N, p, λ, Λ, ∂B, B and the moduli of continuity of the coefficients $a_{ij}(x,r)$ with respect to x on $\bar{B}$.

Proof. For simplicity, let B be the unit ball $B_1(0)$ with its boundary $\mathcal{S}$:

$$\mathcal{S} = \partial B = \left\{ x = (x_1, \cdots, x_N) \in \mathbb{R}^N \Big| \sum_{i=1}^{N} x_i^2 = 1 \right\}.$$

Now we claim that $\mathcal{S} \in C^{1,1}$. For any $x^0 = (x_1^0, \cdots, x_N^0) \in \mathcal{S}$, there exists an integer k, $1 \cdot \quad k \cdot \quad N$, such that $x_0 \in \mathcal{S}_k^+$ or $x_0 \in \mathcal{S}_k^-$, where

$$\mathcal{S}_k^+ = \left\{ x \in \mathcal{S} | \textstyle\sum_{i \neq k} x_i^2 \cdot \quad \frac{N-1}{N}, x_k > 0 \right\},$$
$$\mathcal{S}_k^- = \left\{ x \in \mathcal{S} | \textstyle\sum_{i \neq k} x_i^2 \cdot \quad \frac{N-1}{N}, x_k < 0 \right\};$$

for otherwise we would have $\sum_{i=1}^N x_i^2 > 1$, a contradiction. Without loss of generality, we can assume $x_0 \in \mathcal{S}_N^+$. Write

$$\begin{aligned} x_0 \;\; = \;\; & (\cos\theta_1 \sin\theta_2 \cdots \sin\theta_{N-1},\; \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{N-1}, \\ & \cos\theta_2 \sin\theta_3 \cdots \sin\theta_{N-1},\; \cos\theta_3 \sin\theta_4 \cdots \sin\theta_{N-1}, \\ & \cos\theta_4 \sin\theta_5 \cdots \sin\theta_{N-1}, \cdots,\; \cos\theta_{N-2} \sin\theta_{N-1},\; \cos\theta_{N-1}) \end{aligned}$$

for some θ_i, $0 \cdot \ \theta_{N-1} \cdot \ \tan^{-1}\sqrt{N-1}$, $0 \cdot \ \theta_i < 2\pi$, $i = 1, \cdots, N-2$, where θ_{N-1} is the angle from the positive x_N-axis to x_0. Rotate the coordinate axes, the rotated axes being denoted as the x'_1-, $\cdots$, x'_N-axis, by the mapping $\mathbb{R}_{x_0}$ defined by $x' = x\mathbf{O}_N$, where

$$\mathbf{O}_3 = \begin{bmatrix} \cos\theta_1\cos\theta_2 & -\sin\theta_1 & \cos\theta_1\sin\theta_2 \\ \sin\theta_1\cos\theta_2 & \cos\theta_1 & \sin\theta_1\sin\theta_2 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix},$$

$$\mathbf{O}_k = \left[\begin{matrix} \mathbf{O}_{k-1} & \mathbf{0} \\ \mathbf{0} & 1 \end{matrix}\right] \begin{bmatrix} \mathbf{I}_{k-2} & 0 & 0 \\ 0\cdots0 & \cos\theta_{k-1} & \sin\theta_{k-1} \\ 0\cdots0 & -\sin\theta_{k-1} & \cos\theta_{k-1} \end{bmatrix}, \quad k = 4, \cdots, N,$$

here $\mathbf{I}_{k-2}$ being the $(k-2)\times(k-2)$ identity matrix, such that x_0 is converted into the point $(0, \cdots, 0, 1)$. Define a mapping $\boldsymbol{\psi} = \boldsymbol{\psi}_{x_0} = \boldsymbol{\psi}_{(0,\cdots,0,1)} \circ \mathbb{R}_{x_0}$ in a neighborhood $\mathcal{N} = \mathcal{N}_{x_0} = \mathbb{R}_{x_0}^{-1}(\mathcal{N}_{(0,\cdots,0,1)}) \subset \mathbb{R}^N$, where

$$\boldsymbol{\psi}_{(0,\cdots,0,1)} = \frac{1}{r_0}\left(x'_1, \cdots, x'_{N-1}, \sqrt{1 - \sum_{i\neq N} x'^2_i} - x'_N\right), \quad 0 < r_0 \cdot \ \sqrt{\frac{N-1}{N}},$$

and

$$\mathcal{N}_{(0,\cdots,0,1)} = \left\{ x' \in \mathbb{R}^N \Big| \sum_{i\neq N} x'^2_i < r_0^2,\ \sqrt{1 - \sum_{i\neq N} x'^2_i} - \sqrt{r_0^2 - \sum_{i\neq N} x'^2_i} < x_N < \sqrt{1 - \sum_{i\neq N} x'^2_i} + \sqrt{r_0^2 - \sum_{i\neq N} x'^2_i} \right\}.$$

Then $\boldsymbol{\psi}$ is a diffeomorphism from $\mathcal{N}$ onto the unit ball $B_1(0)$ in $\mathbb{R}^N$ such that $\boldsymbol{\psi}(\mathcal{N} \cap B) \subset \mathbb{R}^N_+$, $\boldsymbol{\psi}(\mathcal{N} \cap \partial B) \subset \partial\mathbb{R}^N_+$, $\boldsymbol{\psi} \in C^{1,1}(\mathcal{N})$, $\boldsymbol{\psi}^{-1} \in C^{1,1}(B_1(0))$. Under the mapping $y = \boldsymbol{\psi}(x) = (\psi_1(x), \cdots, \psi_N(x))$, let $\tilde{u}(y) = u(x)$, $\tilde{v}(y) = v(x)$ and $\tilde{L}_{\tilde{v}}\tilde{u}(y) = L_v u(x)$, where

$$\tilde{L}_{\tilde{v}}\tilde{u} = \sum_{i,j=1}^{N} \tilde{a}_{ij}(y, \tilde{v}(y)) \frac{\partial^2 \tilde{u}}{\partial y_i \partial y_j} + \sum_{i=1}^{N} \tilde{b}_i(y, \tilde{v}(y)) \frac{\partial \tilde{u}}{\partial y_i} + \tilde{c}(y, \tilde{v}(y))\tilde{u}(y) \quad \text{in } B_1^+(0)$$

and

$$\begin{aligned} \tilde{a}_{ij}(y, \tilde{v}(y)) &= \sum_{r,s} \frac{\partial \psi_i}{\partial x_r} \frac{\partial \psi_j}{\partial x_s} a_{rs}(x, v(x)), \\ \tilde{b}_i(y, \tilde{v}(y)) &= \sum_{r,s} \frac{\partial^2 \psi_i}{\partial x_r \partial x_s} a_{rs}(x, v(x)) + \sum_r \frac{\partial \psi_i}{\partial x_r} b_r(x, v(x)), \\ \tilde{c}(y, \tilde{v}(y)) &= c(x, v(x)), \end{aligned}$$

so that $\tilde{L}$ satisfies conditions similar to (2.1) and (2.3) with constants $\tilde{\lambda}$, $\tilde{\Lambda}$ depending on λ, Λ and $\boldsymbol{\psi}$. Furthermore, $\tilde{u} \in W^{2,p}(B_1^+(0))$, $\tilde{u} = 0$ on $B_1(0) \cap \partial\mathbb{R}_+^N$ in the sense of $W^{1,p}(B_1^+(0))$.

Notice that $D\boldsymbol{\psi} = D\boldsymbol{\psi}_{(0,\cdots,0,1)} D\mathbb{R}_{x_0}$ and $\tilde{a} = (D\boldsymbol{\psi})a(D\boldsymbol{\psi})^T$, where

$$\begin{aligned} D\boldsymbol{\psi} &= \left[\frac{\partial \psi_i}{\partial x_j}\right], \ D\boldsymbol{\psi}_{(0,\cdots,0,1)} = \left[\frac{\partial \psi_i}{\partial x'_j}\right], \\ D\mathbb{R}_{x_0} &= \left[\frac{\partial x'_i}{\partial x_j}\right], \ \tilde{a} = [\tilde{a}_{ij}], \ i,j = 1, \cdots, N. \end{aligned}$$

We can obtain from a further computation of $\tilde{a}$ that

$$\text{(2.9)} \qquad \operatorname{osc} \tilde{a}(0,r) < \frac{N^2}{r_0^2} \cdot \operatorname{osc} a(x_0, r).$$

Now we will choose $\tilde{\lambda} > 0$ properly. For all $\xi = (\xi_1, \cdots, \xi_N) \in \mathbb{R}^N$,

$$\begin{aligned} \sum_{i,j=1}^N \tilde{a}_{ij}\xi_i\xi_j &= \xi\tilde{a}\xi^T = (\xi(D\boldsymbol{\psi}))a(\xi(D\boldsymbol{\psi}))^T \geq \lambda|\xi(D\boldsymbol{\psi})|^2 \\ &= \frac{\lambda}{r_0^2}\left(\sum_{i\neq N}\xi_i^2 + \Big(1 + \sum_{i\neq N} X_i^2\Big)\xi_N^2 - 2\sum_{i\neq N}\xi_i\xi_N X_i\right) \\ &\geq \frac{\lambda}{r_0^2}\left((1-\epsilon)\sum_{i\neq N}\xi_i^2 + (1 + (1 - \frac{1}{\epsilon})\sum_{i\neq N} X_i^2)\xi_N^2\right) \end{aligned}$$

for any $\epsilon > 0$, where $X_i = x'_i/\sqrt{1 - \sum_{i\neq N} x'^2_i}$, $i = 1, \cdots, N-1$. Choose $0 < \epsilon < 1$ such that $1 + (1-(1/\epsilon))\sum_{i\neq N} X_i^2 > 1-\epsilon$, i.e., $\sum_{i\neq N} X_i^2 < \epsilon^2/(1-\epsilon)$ and so $\tilde{\lambda} = \lambda(1-\epsilon)/r_0^2$. Since $\sum_{i\neq N} X_i^2 < r_0^2/(1-r_0^2)$ in $\mathcal{N}_{(0,\cdots,0,1)}$, we can take $\epsilon^2/(1-\epsilon) = r_0^2/(1-r_0^2)$ to obtain

$$\text{(2.10)} \qquad \tilde{\lambda} = \lambda \cdot \frac{2 - r_0^2 - \sqrt{4r_0^2 - 3r_0^4}}{2r_0^2(1-r_0^2)}.$$

In view of the proof of Theorem 9.13 [4, p. 239], the oscillations of $\tilde{a}_{ij}(0,r)$ with respect to $r \in \mathbb{R}$, corresponding to condition (2.4), must be less than $\tilde{\lambda}/8K$, that is,

$$\text{(2.11)} \qquad \operatorname{osc} \tilde{a}(0,r) \cdot \ \frac{\tilde{\lambda}}{8K}.$$

In view of (2.9) and (2.10), inequality (2.11) holds provided

$$\text{osc } a(x_0, r) \cdot \quad \frac{\lambda}{16N^2K} \cdot \frac{2 - r_0^2 - \sqrt{4r_0^2 - 3r_0^4}}{1 - r_0^2}. \tag{2.12}$$

Since the right-hand side of (2.12) increases to $\lambda/8N^2K$ as $r_0 \to 0$, there exists r_0 small enough such that, under hypothesis (2.7), inequality (2.12) holds. Thus, using the same deduction as in the proof of Lemma 2.1, we obtain, on returning to our original coordinates, a local boundary estimate in a neighborhood, say $\tilde{\mathcal{N}}$. For an arbitrary ball B in $\mathbb{R}^N$, by means of a linear transformation from B onto the unit ball and following the arguments as stated above we can also arrive at such an estimate. Finally, by covering ∂B with a finite number of such neighborhoods $\tilde{\mathcal{N}}$ and using also the interior estimate (2.5), the desired estimate (2.8) follows immediately. ■

Corollary 2.3. *Under the hypotheses of Proposition* 2.2 *with* B *replaced by the ellipsoid*

$$\mathcal{E} = \left\{ x = (x_1, \cdots, x_N) \in \mathbb{R}^N | \sum_{i=1}^{N} \left(\frac{x_i - c_i}{r_i} \right)^2 < 1 \right\},$$

and with (2.7) *replaced by*

$$\text{osc } a(x, r) < \frac{\min r_i}{\max r_i} \cdot \frac{\lambda}{8N^2K} \qquad \forall x \in \partial\mathcal{E}, \tag{2.13}$$

the same conclusion (2.8) *remains valid.*

Proof. Let $T : \mathbb{R}^N \to \mathbb{R}^N$ be given by

$$T(x) = \left(\frac{x_1 - c_1}{r_1}, \cdots, \frac{x_N - c_N}{r_N} \right).$$

Then T is a diffeomorphism from $\mathcal{E}$ onto the unit ball $B_1(0)$ in $\mathbb{R}^N$. For any $x^0 = (x_1^0, \cdots, x_N^0) \in \partial\mathcal{E}$, there exists an integer k, $1 \cdot \; k \cdot \; N$, such that $x_0 \in \Gamma_k^+$ or $x_0 \in \Gamma_k^-$, where $\Gamma_k^+ = T^{-1}(\mathcal{S}_k^+)$, $\Gamma_k^- = T^{-1}(\mathcal{S}_k^-)$. Thus, there is a neighborhood $\mathcal{U} = \mathcal{U}_{x_0} = T^{-1}(\mathcal{N}_{T(x_0)})$ and a diffeomorphism $\boldsymbol{\phi} = \boldsymbol{\phi}_{x_0} = \boldsymbol{\psi}_{T(x_0)} \circ T$ from $\mathcal{U}$ onto the unit ball $B_1(0)$ in $\mathbb{R}^N$ such that $\boldsymbol{\phi}(\mathcal{U} \cap \mathcal{E}) \subset \mathbb{R}_+^N$, $\boldsymbol{\phi}(\mathcal{U} \cap \partial\mathcal{E}) \subset \partial\mathbb{R}_+^N$, $\boldsymbol{\phi} \in C^{1,1}(\mathcal{U})$, $\boldsymbol{\phi}^{-1} \in C^{1,1}(B_1(0))$. The desired estimate (2.8) can be similarly derived by following the proof in Proposition 2.2. ■

Remark 2.4. Proposition 2.2 remains valid with B replaced by an ovaloid in $\mathbb{R}^N$. (An ovaloid in $\mathbb{R}^N$ is a rectangle in $\mathbb{R}^N$ with rounded comers.) ■

3. Existence of Strong Solutions

The results of the preceding section will now be applied to establish the existence of solutions of the following semilinear elliptic problem:

$$(3.1)\quad \begin{cases} Lu = \sum_{i,j=1}^{N} a_{ij}(x,u)\dfrac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x,u)\dfrac{\partial u}{\partial x_i} + c(x,u)u = f(x) \quad \text{in } B, \\ u = 0 \qquad \text{on } \partial B, \end{cases}$$

where $f \in L^p(B)$.

For the moment, we suppose $a_{ij} \in C^{0,1}(\bar{B} \times \mathbb{R})$, a_{ij}, $\partial a_{ij}/\partial x_i$, $\partial a_{ij}/\partial r$, b_i, c are bounded Carathédory functions, with $c \cdot\ 0$, and $f \in L^p(B)$, with $p > N/2$. Consider the mapping F which assigns to $v \in W^{2,p}(B) \cap W_0^{1,p}(B)$ the solution $u \in W^{2,p}(B) \cap W_0^{1,p}(B)$ to the equation

$$(3.2)\quad L_v u = \sum_{i,j=1}^{N} a_{ij}(x,v)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x,v)\frac{\partial u}{\partial x_i} + c(x,v)u = f(x) \quad \text{in } B.$$

(F is well-defined provided $p > N/2$.)

Since $W^{2,p}(B) \cap W_0^{1,p}(B)$ is continuously imbedded in $H^1(B)$, by the ellipticity of L, the mapping $F : W^{2,p}(B) \cap W_0^{1,p}(B) \longrightarrow W^{2,p}(B) \cap W_0^{1,p}(B)$ is continuous in the topology of $H^1(B)$ [3]. Together with estimate (2.8) and the maximum principle for equation (3.2):

$$(3.3)\qquad \sup_B |u| \cdot\ M\|f\|_{L^N(B)},$$

where M is a constant depending on N, diam B, λ and Λ [2], (the maximum principle is only valid for $p \geq N$), we have the following existence result.

Theorem 3.1. *Let B be a ball in $\mathbb{R}^N$ and suppose $a_{ij} \in C^{0,1}(\bar{B} \times \mathbb{R})$, a_{ij}, $\partial a_{ij}/\partial x_i$, $\partial a_{ij}/\partial r$, b_i, $c \in L^\infty(B \times \mathbb{R})$, with $i, j = 1, \cdots, N$ and $c \cdot\ 0$. Then, for $p \geq N$, there exists a solution $u \in W^{2,p}(B) \cap W_0^{1,p}(B)$ to problem* (3.1) *under hypotheses* (2.6) *and* (2.7).

Proof. Consider the solution $u = F(v)$ for $v \in W^{2,p}(B) \cap W_0^{1,p}(B)$. Since $f \in L^p(B)$ with $p \geq N$, it follows from (2.8) and (3.3) that there exists a constant $k > 0$ such that

$$\|u\|_{W^{2,p}} \cdot\ k \qquad \text{for all } u = F(v),\ v \in W^{2,p}(B) \cap W_0^{1,p}(B).$$

Let

$$\mathcal{K} = \{v \in W^{2,p}(B) \cap W_0^{1,p}(B) |\ \|v\|_{W^{2,p}} \cdot\ k\}.$$

Then F is a continuous mapping from $\mathcal{K}$ into $\mathcal{K}$ in the topology of $H^1(B)$. Moreover, since $W^{2,p}(B)$ is a reflexive space and $W^{1,p}(B)$ is continuously imbedded in $H^1(B)$, $\mathcal{K}$ is weakly compact in $H^1(B)$ and hence it is closed in $H^1(B)$. Also, since $W^{2,p}(B) \hookrightarrow W^{1,p}(B)$ is a compact imbedding, $\mathcal{K}$ is a compact set in $H^1(B)$. We conclude from the Schauder fixed point theorem that there exists a solution to problem (3.1) in $\mathcal{K}$. ■

In the sequel, we shall show that if $\|f\|_{L^p}$ is sufficiently small, then the existence result of problem (3.1) still holds.

Lemma 3.2. *Let* $a_{ij} \in C^{0,1}(\bar{B} \times \mathbb{R})$, a_{ij}, $\partial a_{ij}/\partial x_i$, $\partial a_{ij}/\partial r$, b_i, $c \in L^\infty(B \times \mathbb{R})$, *with* $i, j = 1, \cdots, N$ *and* $c \cdot\ 0$. *Then*, *under hypotheses* (2.6) *and* (2.7), *there exists a constant* C *independent of* u *and* v *such that*, *for all* $v \in \mathcal{K} = \{v \in W^{2,p}(B) \cap W_0^{1,p}(B) \mid \|v\|_{W^{2,p}} \cdot\ k\}$,

$$\|u\|_{W^{2,p}} \cdot\ C\|L_v u\|_{L^p} \tag{3.4}$$

for all $u \in W^{2,p}(B) \cap W_0^{1,p}(B)$.

Proof. We argue by contradiction. If (3.4) is not true, then for all $m > 0$ there exist sequences $(w_m) \subset W^{2,p}(B) \cap W_0^{1,p}(B)$ and $(v_m) \subset \mathcal{K}$ satisfying

$$\|w_m\|_{W^{2,p}} \geq m\|L_{v_m} w_m\|_{L^p}.$$

We will claim that there exists a sequence $(u_m) \subset W^{2,p}(B) \cap W_0^{1,p}(B)$ satisfying

$$\|u_m\|_{L^p} = 1; \ \|L_{v_m} u_m\|_{L^p} \to 0. \tag{3.5}$$

Let $z_m = w_m/\|w_m\|_{W^{2,p}}$. Then $\|z_m\|_{W^{2,p}} = 1$ and

$$\|L_{v_m} z_m\|_{L^p} = \frac{\|L_{v_m} w_m\|_{L^p}}{\|w_m\|_{W^{2,p}}} \cdot\ \frac{1}{m}\frac{\|w_m\|_{W^{2,p}}}{\|w_m\|_{W^{2,p}}} = \frac{1}{m}.$$

Thus

$$\|L_{v_m} z_m\|_{L^p} \to 0 \qquad \text{as } m \to \infty.$$

From Proposition 2.2, there exists $M > 0$ independent of (v_m) such that

$$\|z_m\|_{W^{2,p}} \cdot\ M(\|z_m\|_{L^p} + \|L_{v_m} z_m\|_{L^p}).$$

Hence, for any $\epsilon > 0$, we have

$$\|z_m\|_{W^{2,p}} \cdot\ M\epsilon + M\|z_m\|_{L^p} \qquad \text{as } m \to \infty.$$

It follows that

$$\|z_m\|_{L^p} \geq \frac{1}{M}\|z_m\|_{W^{2,p}} - \epsilon = \frac{1}{M} - \epsilon \qquad \text{as } m \to \infty.$$

Since ϵ is arbitrary, we have

$$\|z_m\|_{L^p} \geq \frac{1}{M} \qquad \text{as } m \to \infty.$$

Let $u_m = z_m/\|z_m\|_{L^p}$. Then

$$\|u_m\|_{L^p} = 1; \;\; \|L_{v_m} u_m\|_{L^p} \to 0.$$

Thus we get a sequence $(u_m) \subset W^{2,p}(B) \cap W_0^{1,p}(B)$ satisfying (3.5) and

(3.6) $$\|u_m\|_{W^{2,p}} \cdot \;\; M(\|u_m\|_{L^p} + \|L_{v_m} u_m\|_{L^p}).$$

Combining (3.5) with (3.6), we know that (u_m) is bounded in $W^{2,p}(B)$ and thus there exists a subsequence, denoted again by (u_m), converging weakly to a function $u \in W^{2,p}(B) \cap W_0^{1,p}(B)$. Moreover, since $W^{2,p}(B) \hookrightarrow W^{1,p}(B)$ is a compact imbedding, (u_m) converges to u in $L^p(B)$ satisfying $\|u\|_{L^p} = 1$. Similarly, since (v_m) is bounded in $W^{2,p}(B)$, we can extract a subsequence, denoted also by (v_m), such that $v_m \to v$ a.e. and $v_m \to v$ in $W^{1,p}(B)$ for some $v \in W^{2,p}(B) \cap W_0^{1,p}(B)$. Also, since a_{ij}, $\partial a_{ij}/\partial x_i$, $\partial a_{ij}/\partial r$, b_i and c are bounded Carathédory functions, by Lebesgue's dominated convergence theorem, we have

$$\int_B a_{ij}(v_m)\frac{\partial u_m}{\partial x_j}\frac{\partial \phi}{\partial x_i} + \int_B \left(\frac{\partial a_{ji}}{\partial x_j}(v_m) + \frac{\partial a_{ji}}{\partial r}(v_m)\frac{\partial v_m}{\partial x_j} - b_i(v_m)\right)\frac{\partial u_m}{\partial x_i}\phi$$
$$+ \int_B (-c(v_m))u_m\phi \to \int_B a_{ij}(v)\frac{\partial u}{\partial x_j}\frac{\partial \phi}{\partial x_i} + \int_B \left(\frac{\partial a_{ji}}{\partial x_j}(v) + \frac{\partial a_{ji}}{\partial r}(v)\frac{\partial v}{\partial x_j}\right.$$
$$\left. - b_i(v)\right)\frac{\partial u}{\partial x_i}\phi + \int_B (-c(v))u\phi$$

for all $\phi \in C_0^\infty(B)$. Hence $L_v u = 0$ and $u = 0$ by the uniqueness assertion, which contradicts the condition $\|u\|_{L^p} = 1$. ■

Theorem 3.3. *Let B be a ball in $\mathbb{R}^N$ and suppose $a_{ij} \in C^{0,1}(\bar{B} \times \mathbb{R})$, a_{ij}, $\partial a_{ij}/\partial x_i$, $\partial a_{ij}/\partial r$, b_i, $c \in L^\infty(B \times \mathbb{R})$, with $i, j = 1, \cdots, N$ and $c \cdot \;\; 0$. Then, for $p > N/2$, there exists a positive constant C_0 such that if*

$$\|f\|_{L^p(B)} \cdot \;\; C_0,$$

there exists a solution $u \in W^{2,p}(B) \cap W_0^{1,p}(B)$ to problem (3.1) *under hypotheses* (2.6) *and* (2.7).

Proof. Consider the set

$$\mathcal{K} = \left\{ v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \mid \|v\|_{W^{2,p}} \le k \right\}.$$

It follows from Lemma 3.2 that there exists a constant $C > 0$ independent of $v \in \mathcal{K}$ such that

$$\|u\|_{W^{2,p}} \le C\|f\|_{L^p} \qquad \text{for all } u = F(v),\ v \in \mathcal{K}.$$

Choose a constant $C_0 > 0$ such that $CC_0 \le k$. Hence if $\|f\|_{L^p} \le C_0$, we have $\|u\|_{W^{2,p}} \le k$. It follows readily from the Schauder fixed point theorem that there exists a solution of problem (3.1) in $\mathcal{K}$. ■

Remark 3.4. For $p \ge N$, since $W^{2,p}(\Omega)$ is imbedded in $C^1(\bar{\Omega})$ for a bounded $C^{1,1}$ domain Ω, the constant C in estimate (1.3) can be chosen to be independent of v with v restricted to some bounded set in $W^{2,p}(\Omega)$. Then, together with the maximum principle, Theorem 3.3 remains valid with B replaced by Ω provided $p \ge N$ without any restrictions on the oscillations of a_{ij} with respect to r.

Remark 3.5. Theorems 3.1 and 3.2 remain valid with B replaced by the ellipsoid $\mathcal{E}$ in Corollary 2.3 and with (2.7) replaced by (2.13).

Remark 3.6. Theorems 3.1 and 3.2 remain valid with B replaced by an ovaloid in $\mathbb{R}^N$.

Remark 3.7. For any bounded domain Ω with a sufficiently smooth boundary, although the diffeormorphism $\boldsymbol{\psi}$ in Proposition 2.2 is not explicitly observed, it seems that the existence of strong solutions $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ to problem (3.1) in Ω remains valid provided the oscillations of a_{ij} with respect to r are sufficiently small.

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