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VARIATIONAL INEQUALITIES OF ELLIPTIC AND PARABOLIC TYPE

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Abstract. This paper constitutes a short survey of existence methods for variational inequalities of parabolic type. After discussing several illustrative examples in detail, we discuss some of the common methods for proving existence of solutions to such problems: the *translation method*, *Rothe's method*, and the *penalty method*. As these methods rely on existence results for elliptic variational inequalities, we also provide a summary of basic results and techniques for static problems.

1. Introduction

The study of evolution problems where the state of the system is subject to some set of constraints has a long history and its beginnings are nearly simultaneous to the early studies of variational inequalities.

Since such problems are, by their very nature, nonlinear problems, methods complementing the semigroup theoretic approach ([4], [11], [21], [33]), used for the study of evolution equations had to be devised. These methods are mainly based on existence results for static variational inequalities and go back to theories presented in [5], [6], [29], and have been discussed in detail in various other places, e.g., [22], [39], [40].

While most of the sources already mentioned present a theory of variational inequalities usually from some fixed point of view, we shall here present a survey of several different ways to arrive at an existence theory.

We begin in Section 2 by presenting some illustrative examples of parabolic variational inequalities and establish some notation to be used throughout this paper.

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We then present a brief survey and some examples of results about static elliptic variational inequalities which will subsequently be used to derive existence results for parabolic variational inequalities. We then discuss three standard methods for proving existence of solutions to such problems: the *translation method* (Section 4), *Rothe's method* (Section 5), and the *penalty method* (Section 6). For more material on parabolic variational inequalities, see [4], [17], [22], [29], [32], and [39].

2. Examples

This section presents several examples that motivate the study of parabolic variational inequalities and indicate their range of applicability. In Section 2.1, we introduce the subject with a linear diffusion equation whose nonlinear boundary conditions represent a semipermeable boundary. We then examine two problems for the p–Laplacian, a parabolic obstacle problem (Section 2.2) and a nonlinear evolution equation (Section 2.3). These three examples guide the way to the general formulation of parabolic variational inequalities discussed in Section 3.

2.1. Diffusion with a semipermeable membrane

We begin with a model problem describing diffusion in a domain with a semi–permeable boundary ([29], [32]). Let $\quad \subset \mathbb{R}^N$ be an open bounded set with smooth boundary Γ, let the final time $T < \infty$ be given, and consider the problem of finding $u = u(x,t)$ such that

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{for} \quad (x,t) \in \quad \times (0,T), \tag{2.1}$$

$$u(x,0) = u_0(x) \quad \text{for} \quad x \in \overline{\quad}, \tag{2.2}$$

$$u \geq 0, \quad \frac{\partial u}{\partial \nu} \geq 0, \quad \text{and} \quad u\,\frac{\partial u}{\partial \nu} = 0 \quad \text{for} \quad (x,t) \in \Gamma \times (0,T), \tag{2.3}$$

where Δ is the Laplacian with respect to x. With $V = H^1(\quad)$, we look for $u \in \mathcal{V} = L^2(0,T;V)$, the Banach space of functions $v : [0,T] \to V$ with norm

$$\|v\|_{\mathcal{V}} = \left(\int_0^T \|v(s)\|_V^2 \, dt \right)^{1/2}. \tag{2.4}$$

Furthermore, we require that $f(t) \in V^*$ for a.e. $t \in (0,T)$ and that the initial datum $u_0 \in H = L^2(\quad)$.

The nonlinear boundary conditions (2.3) lead to this problem's formulation as a variational inequality. In fact, if u solves (2.1)–(2.3) and t is an arbitrarily chosen point in $(0, T)$, then $u(t)$ clearly belongs to the closed convex set $K \subset V$ defined by

(2.5) $$K = \{v \in V \mid v(x) \geq 0 \quad \text{for} \quad x \in \Gamma\}.$$

For any $v \in \mathcal{V}$ with $v(t) \in K$, multiplying both sides of (2.1) by $v(t) - u(t)$ and integrating over   produces the identity

(2.6) $$\int (u'(t) - f(t))(v(t) - u(t))\, dx = \int \Delta u(t)(v(t) - u(t))\, dx,$$

where we now write u' for the derivative $\partial u / \partial t$, since we view u as a function of time with values in V.

Using the divergence theorem and the boundary conditions (2.3), we have

(2.7) $$\begin{aligned} &\int \Delta u(t)(v(t) - u(t)) + \nabla u(t) \cdot \nabla(v(t) - u(t))\, dx \\ &\quad = \int_\Gamma (v(t) - u(t)) \frac{\partial u(t)}{\partial \nu}\, dx \geq 0, \end{aligned}$$

from which we see that

(2.8) $$\int \Delta u(t)(v(t) - u(t))\, dx \geq -\int \nabla u(t) \cdot \nabla(v(t) - u(t))\, dx.$$

Combining (2.8) with (2.6), we see that u belongs to $\mathcal{K}$ and satisfies the *parabolic variational inequality*

(2.9) $$\begin{aligned} &\int u'(t)(v(t) - u(t)) + \nabla u(t) \cdot \nabla(v(t) - u(t))\, dx \\ &\quad \geq \int f(t)(v(t) - u(t))\, dx,\ \forall\, v \in \mathcal{K},\ a.e.\ t \in (0, T), \end{aligned}$$

where $\mathcal{K}$ denotes the collection of functions $v \in \mathcal{V}$ such that $v(t) \in K$ for a.e. $t \in (0, T)$.

Although this cone $\mathcal{K}$ might appear to omit some of the boundary conditions posed in (2.3), we will see that these two problems are indeed equivalent. To this end, suppose that $u \in \mathcal{K}$ solves (2.9), and let

$$v(t) = u(t) + \varepsilon\, \zeta$$

for $t \in (0, T)$, $\varepsilon \neq 0$, and an arbitrary test function $\zeta \in C_0^\infty(\quad)$. As this function v belongs to $\mathcal{K}$, we may substitute it into (2.9) to obtain the inequality

$$\varepsilon \int (u'(t)\zeta + \nabla u(t) \cdot \nabla \zeta - f(t)\zeta)\, dx \geq 0,$$

which is actually the *equation*

$$\int (u'(t)\zeta + \nabla u(t) \cdot \nabla \zeta - f(t)\zeta)\, dx = 0,$$

since ε may be positive or negative. In the sense of distributions, u therefore satisfies the heat equation (2.1) in $\quad \times (0,T)$.

It remains to verify the boundary conditions that are not included in the definition of $\mathcal{K}$. Observe that the relations

$$\begin{aligned} &\int (u'(t)w(t) + \nabla u(t) \cdot \nabla w(t))\, dx \\ &\quad \geq \int f(t)w(t)\, dx, \ \forall w \in \mathcal{K}, \ a.e. \ t \in (0,T) \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} &\int (u'(t)(u(t)\zeta) + \nabla u(t) \cdot \nabla (u(t)\zeta))\, dx \\ &\quad = \int f(t)(u(t)\zeta)\, dx, \ \forall \zeta \in C^\infty(\overline{\ }), \ a.e. \ t \in (0,T), \end{aligned} \tag{2.11}$$

follow from (2.9) by first choosing $v = w + u$, for $w \in \mathcal{K}$, in (2.9) and then choosing $w(t) = u(t)(1 \pm \zeta)$, $\zeta \in C^\infty(\overline{\ })$, $|\zeta(x)| \cdot\ 1$ in (2.9).

Using equation (2.1), we rewrite (2.10) as

$$\begin{aligned} &\int (u'(t)w(t) + \nabla u(t) \cdot \nabla w(t))\, dx \\ &\quad \geq \int \big(u'(t) - \Delta u(t)\big)\, w(t)\, dx, \ \forall w \in \mathcal{K}, \ a.e. \ t \in (0,T), \end{aligned}$$

which is simply

$$\int (\nabla w(t) \cdot \nabla u(t) + w(t)\Delta u(t))\ dx \geq 0, \ \forall w \in \mathcal{K}, \ a.e. \ t \in (0,T). \tag{2.12}$$

We now apply the divergence theorem to (2.12) to find that

$$\int_\Gamma w(t) \frac{\partial u(t)}{\partial \nu}\, d\sigma \geq 0, \ \forall w \in \mathcal{K}, \ a.e. \ t \in [0,T],$$

i.e.,

$$\frac{\partial u}{\partial \nu} \geq 0 \quad \text{on} \quad \Gamma \times (0,T).$$

A similar argument verifies the remaining condition; we replace $f(t)$ in (2.11) with $u'(t) - \Delta u(t)$ to obtain for $a.e.\ t \in (0,T)$

$$\int (\nabla u(t) \cdot \nabla (u(t)\zeta) + u(t)\zeta\Delta u(t))\, dx = 0, \ \forall \zeta \in C^\infty(\overline{\ }), \tag{2.13}$$

to which the divergence theorem applies to deduce that

$$\int_\Gamma u(t)\zeta \frac{\partial u(t)}{\partial \nu}\, d\sigma = 0,\ \forall \zeta \in C^\infty(\overline{\ \ }),\ a.e.\ t \in (0,T).$$

This means precisely that

$$u\,\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma \times (0,T).$$

The boundary conditions (2.3) thus hold, so the original diffusion problem may either be formulated as the boundary value problem (2.1), (2.2), (2.3) or as the parabolic variational inequality (2.9).

2.2. A nonlinear obstacle problem

In many problems the diffusion coefficient will not be constant but rather will depend upon the dependent variable in some manner. A class of problems that has received much attention in recent years is obtained by replacing the Laplacian term in the integral (2.9) with a term corresponding to the p–Laplacian. To do so, we let $V = W_0^{1,p}(\ \ )$ for $p > 1$, and we use $W^{-1,q}(\ \ )$ to denote the dual of V, where p and q are conjugate exponents, $1/p + 1/q = 1$. Letting $\langle \cdot, \cdot \rangle$ denote the pairing between these spaces, we define the operator

$$A_p : W_0^{1,p}(\ \ ) \to W^{-1,q}(\ \ )$$

by

$$\langle A_p u, v\rangle = \int |\nabla u|^{p-2} \nabla u \cdot \nabla v\, dx \tag{2.14}$$

for $u, v \in W_0^{1,p}(\ \ )$. The operator A_p is defined by the p–Laplacian Δ_p,

$$\Delta_p(u) = -\nabla \cdot (|\nabla u|^{p-2} \nabla u). \tag{2.15}$$

As in the previous section, $\mathcal{V}$ denotes the space $L^2(0,T;V)$, and $\mathcal{K}$ is the set of functions $v \in \mathcal{V}$ such that $v(t) \in K$ for a.e. $t \in (0,T)$, where $K \subset V$ is a closed convex set to be specified below.

With this setup, we consider the problem of finding $\in \mathcal{K}$ with the prescribed initial value

$$u(0) = u_0 \in L^2(\ \ ) \tag{2.16}$$

and such that the inequality

$$\begin{aligned}\int u'(t)(v(t)-u(t)) + |\nabla u(t)|^{p-2}\nabla u(t)\cdot\nabla(v(t)-u(t))\,dx \\ \geq \int f(t)(v(t)-u(t))\,dx\end{aligned} \tag{2.17}$$

holds for a.e. $t \in (0,T)$ and for all $v \in \mathcal{K}$, where

$$K = \{v \in V | v \geq \psi\}, \tag{2.18}$$

for a given $\psi \in W^{1,p}(\quad)$ satisfying $\psi \cdot \quad 0$ on Γ. The closed convex set $\mathcal{K}$ represents an imposed constraint determined by the *obstacle* ψ. The existence results to follow guarantee a solution $u \in \mathcal{K}$ of the parabolic obstacle problem (2.17) for the p–Laplacian; we devote the remainder of this section to a description of the solution.

From the definition of the constraint set $\mathcal{K}$, we see that, at any time $t \in (0,T)$, $u(t)$ partitions into the two sets

$$^{+}(t) = \{x \in \quad | u(x,t) > \psi(x)\}$$

and

$$^{0}(t) = \{x \in \quad | u(x,t) = \psi(x)\}.$$

For $\varepsilon \neq 0$ and any test function $\zeta \in C_0^\infty(\ ^{+}(t))$, we follow the argument given earlier and substitute $v(t) = u(t) + \varepsilon\zeta$ into (2.17) to obtain

$$\int u'(t)\zeta + |\nabla u(t)|^{p-2}\nabla u(t)\cdot\nabla\zeta - f(t)\zeta\,dx = 0, \tag{2.19}$$

which means that the equation

$$\frac{\partial u}{\partial t} - \Delta_p u = f \tag{2.20}$$

holds in the sense of distributions, where Δ_p is the p–Laplacian defined above (2.15). The solution u of the parabolic variational inequality (2.17) therefore satisfies the partial differential equation (2.20) on

$$^{+} = \bigcup_{t\in(0,T)} \ ^{+}(t)$$

and equals the obstacle ψ on

$$^{0} = \bigcup_{t\in(0,T)} \ ^{0}(t).$$

We emphasize, however, that the boundary of ^{0}, the *free boundary* for this problem, is unknown *a priori*. In contrast to the example in Section 2.1, this problem cannot be recast as a classical boundary value problem. This example indicates the role of variational inequalities in the study of free boundary problems arising from constraints.

2.3. A nonlinear evolution equation

Using the indicator functional ϕ_K of the constraint set K defined by (2.18), we can formulate the obstacle problem of the previous section as a parabolic variational inequality over the entire space $\mathcal{V}$. Specifically, $u \in \mathcal{V}$ solves inequality (2.17) if and only if it solves the inequality

$$\begin{aligned}
&\int u'(t)(v(t)-u(t)) + |\nabla u(t)|^{p-2}\nabla u(t)\cdot\nabla(v(t)-u(t))\,dx \\
&\quad +\phi_K(v(t)) - \phi_K(u(t)) \geq \int f(t)(v(t)-u(t))\,dx, \\
&\quad \forall\, v\in\mathcal{V},\ \ a.e.\ t\in(0,T),
\end{aligned} \tag{2.21}$$

where $\phi_K : V \to \mathbb{R}\cup\{+\infty\}$ is defined by

$$\phi_K(v) = \begin{cases} 0, & \text{if} \quad v\in K, \\ +\infty, & \text{if} \quad v\notin K. \end{cases} \tag{2.22}$$

As K is convex and closed, the functional ϕ_K is convex and lower semicontinuous ([7], [15]). It is then natural to consider replacing ϕ_K in (2.21) with more general convex lower semicontinuous functionals. To explore this idea, we define the functional $\phi : V \to \mathbb{R}\cup\{+\infty\}$ by

$$\phi(v) = \alpha^{-1}\int |v|^\alpha\,dx, \tag{2.23}$$

where we choose the exponent α in accordance with the convexity requirement and with the Rellich–Kondrachov theorem ([1], [7]):

- for $p < N$, $\alpha \in (1, p^*)$, where $p^* = \frac{Np}{N-p}$ is the Sobolev conjugate of p;
- for $p \geq N$, $\alpha \in (1,\infty)$.

The functional ϕ is lower semicontinuous by Fatou's lemma and, for $\alpha > 1$, is Fréchet differentiable, with derivative $D\phi : V \to V^*$ given by

$$\langle D\phi(u), v\rangle = \int |u|^{\alpha-2}uv\,dx \quad \text{for} \quad u, v\in V, \tag{2.24}$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between V and its dual V^*.

By the existence results in the following sections, there exists a solution $u \in \mathcal{V}$ of the corresponding variational inequality

$$\begin{aligned} &\int u'(t)(v(t)-u(t)) + |\nabla u(t)|^{p-2}\nabla u(t) \cdot \nabla(v(t)-u(t))\,dx \\ &\quad +\phi(v(t)) - \phi(u(t)) \;\geq \int f(t)(v(t)-u(t))\,dx, \\ &\quad \forall v \in \mathcal{V},\ a.e.\ t \in (0,T). \end{aligned} \tag{2.25}$$

As this holds for all $v \in \mathcal{V}$, we may substitute $v(t) = u(t) + \varepsilon\,\zeta$, for $\varepsilon > 0$, into (2.25) to find that $u(t)$ satisfies

$$\begin{aligned} &\varepsilon \int u'(t)\zeta + |\nabla u(t)|^{p-2}\nabla u(t) \cdot \nabla\zeta - f(t)\zeta\,dx \\ &\quad +\phi(u(t)+\varepsilon\zeta) - \phi(u(t)) \;\geq 0,\ \forall \zeta \in C_0^\infty(\quad). \end{aligned} \tag{2.26}$$

Since ϕ is Fréchet differentiable, we have

$$\phi(u(t)+\varepsilon\zeta) - \phi(u(t)) = \langle D\phi(u(t)), \varepsilon\zeta\rangle + o(\|\varepsilon\zeta\|).$$

Substituting this into (2.26) and dividing through by ε yields

$$\begin{aligned} &\int u'(t)\zeta + |\nabla u(t)|^{p-2}\nabla u(t) \cdot \nabla\zeta - f(t)\zeta\,dx \\ &\quad +\langle D\phi(u(t)), \zeta\rangle + \frac{o(\|\varepsilon\zeta\|)}{\varepsilon} \geq 0,\ \forall\, \zeta \in C_0^\infty(\quad). \end{aligned} \tag{2.27}$$

Letting ε tend to 0 and then repeating the argument for $\varepsilon < 0$ (which reverses inequalities), we obtain

$$\begin{aligned} &\int u'(t)\zeta + |\nabla u(t)|^{p-2}\nabla u(t) \cdot \nabla\zeta - f(t)\zeta\,dx \\ &\quad +\langle D\phi(u(t)), \zeta\rangle = 0,\ \forall\, \zeta \in C_0^\infty(\quad). \end{aligned} \tag{2.28}$$

Recalling (2.24), it follows that u is a solution of the nonlinear evolution equation

$$\frac{\partial u}{\partial t} - \nabla \cdot (|\nabla u|^{p-2}\nabla u) + |u|^{\alpha-2}u = f \quad \text{in} \qquad \times (0,T), \tag{2.29}$$

with initial and boundary conditions

$$u(x,0) = u_0(x) \quad \text{and} \tag{2.30}$$

$$u(x,t) = 0 \quad \text{for} \quad x \in \Gamma. \tag{2.31}$$

In case $\alpha = 1$, the above equation (2.29) will need to be replaced by the problem

$$\frac{\partial u}{\partial t} - \nabla \cdot (|\nabla u|^{p-2}\nabla u) + \partial(|u|) \ni f \quad \text{in} \qquad \times (0, T), \tag{2.32}$$

where

$$\partial|u| = \begin{cases} 1, & \text{if } \ u > 0 \\ [-1, 1], & \text{if } \ u = 0 \\ -1, & \text{if } \ u < 0, \end{cases}$$

(see subsequent discussion for such problems).

More general *slow-fast* inequality diffusion problems with differential operators of the form

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(A(|\nabla u|^2)\nabla u\right) + \partial\phi(u),$$

with A a fast (or slow) growing function, arise naturally in many applications, as well. (See Section 3, where a static problem of this type is discussed.)

Another interesting set of applications of parabolic variational inequalities involving the p-Laplacian (or other nonlinear diffusion operators of the type just mentioned), i.e., equation (2.21), is the choice of the indicator functional ϕ_K, where the closed convex set K is given by

$$K = \{u \in W_0^{1,p}(\quad) : |\nabla u| \cdot \quad 1, \ a.e. \ x \in \quad \}.$$

Such problems, particularly for large values of p, serve as approximate models for the formation of sandpiles, see e.g., [2], [16], [36].

These examples show that, by choosing different functionals ϕ, the formulation (2.25) captures a wide variety of problems. The next section exploits this observation.

3. The General Problem

The progression of examples in Sections 2.1, 2.2, and 2.3 indicates a general formulation of parabolic variational inequalities that encompasses many different problems. Given a reflexive Banach space V and $T < \infty$, we let $\mathcal{V}$ denote the space

$$\mathcal{V} = \mathcal{L}^{\in}(\prime, \mathcal{T}; \mathcal{V}), \tag{3.1}$$

whose dual is the space

$$\mathcal{V}^* = L^2(0, T; V^*). \tag{3.2}$$

This identification of $\mathcal{V}^*$ is only possible because the underlying space V is reflexive ([8], [12]). These are standard function spaces in the treatment of evolution problems ([8], [11], [29], [39]). We require further that V be continuously embedded in some Hilbert space H, so that duality yields the pivot space structures

$$V \hookrightarrow H \hookrightarrow V^* \quad \text{and} \quad \mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*, \tag{3.3}$$

where $\mathcal{H} = \mathcal{L}^{\in}(\prime, \mathcal{T}; \mathcal{H})$. Two consequences of (3.3) will be important for us ([39]); first, the embedding

$$W := \{v \in \mathcal{V} \mid \sqsubseteq' \in \mathcal{V}^*\} \hookrightarrow \mathcal{C}\left([\prime, \mathcal{T}]; \mathcal{H}\right) \tag{3.4}$$

holds, which shows that initial data in the Hilbert space H are appropriate for the problems that we discuss. Moreover, for functions $v \in W$, we have

$$\frac{d}{dt} \| v(t) \|_V^2 = 2 \int v'(x,t)\, v(x,t)\, dx. \tag{3.5}$$

In addition to these spaces, we have an operator $A : V \to V^*$ that satisfies certain monotonicity and continuity conditions corresponding to the operators that arise in elliptic variational inequalities. To make the notation less cumbersome, we henceforth let $a(\cdot,\cdot)$ denote the form corresponding to A, i.e.,

$$a(u,v) := \langle Au, v \rangle, \quad \text{for} \quad u, v \in V,$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between V^* and V. With this notation, we recall the definitions of the relevant properties of A ([29], [39]):

Definition 3.1. An operator $A : V \to V^*$ is

- **monotone** if

$$a(u-v, u-v) \geq 0, \ \forall\, u, v \in V, \tag{3.6}$$

  and **strictly monotone** if equality forces $u = v$.

- **hemicontinuous** if the map

$$t \mapsto a(u + tv, v)$$

  is continuous for each $u, v \in V$.

- **pseudomonotone** if A is bounded and such that

$$
\begin{gathered}
u_n \rightharpoonup u \quad \text{and} \quad \limsup a(u_n, u_n - u) \leq 0 \\
\text{imply} \\
a(u, u - v) \leq \liminf a(u_n, u_n - v), \ \forall\, v \in V.
\end{gathered}
\tag{3.7}
$$

As shown in [29], pseudomonotonicity ensures that A is a continuous map from V to V^*, where V is endowed with its norm topology and V^* is given the weak topology. Although we explicitly assume pseudomonotonicity of A in the problem (3.10) stated below, it suffices to verify monotonicity and hemicontinuity, as these two properties immediately imply that A is pseudomonotone ([29], [39]). As a specific example, simple calculations show that the operator A_p, induced by the p–Laplacian and introduced in Section 2.2, is monotone and hemicontinuous, so it fits the framework outlined here.

Finally, we are given a functional $\phi : V \to \mathbb{R} \cup \{+\infty\}$ which is convex, lower semicontinuous with respect to the topology of V, and whose *effective domain* $D(\phi)$,

$$
D(\phi) := \{v \in V \mid \phi(v) < +\infty\},
\tag{3.8}
$$

is nonempty.

Note that the integrals occurring in the preceding parabolic variational inequalities gave the explicit action of V^* on V. For conciseness, as in the definition of the form $a(\cdot,\cdot)$ corresponding to A, we therefore use $\langle \cdot,\cdot \rangle$ to denote the pairing between V^* and V, so that the following is the generalization of the problems considered in Sections 2.1, 2.2, and 2.3:

Problem 3.2. *Let the spaces V, H, $\mathcal{V}$, and $\mathcal{H}$ be as described above. Suppose that the pseudomonotone operator $A : V \to V^*$ and the convex lower semicontinuous functional $\phi : V \to \mathbb{R} \cup \{+\infty\}$, with $D(\phi)$ nonempty, satisfy the coercivity condition*

$$
\lim_{\|v\| \to \infty} \frac{a(v, v - v_0) + \phi(v)}{\|v\|} = \infty,
\tag{3.9}
$$

for some $v_0 \in D(\phi)$. We seek $u \in \mathcal{V}$ such that the parabolic variational inequality

$$
\begin{gathered}
\langle u'(t) - f(t), v(t) - u(t) \rangle + a(u(t), v(t) - u(t)) \\
+\phi(v(t)) - \phi(u(t)) \geq 0, \ \forall\, v \in \mathcal{V}, \ a.e.\ t \in (0,T)
\end{gathered}
\tag{3.10}
$$

holds and such that u has the prescribed initial value

$$
u(x,0) = u_0(x) \in H.
\tag{3.11}
$$

A solution u of (3.10) necessarily belongs to the effective domain of the functional ϕ. Although we did not mention the coercivity condition (3.9) in the previous examples, such conditions arise naturally in minimization problems. They are typically used to guarantee that certain approximate solutions form a bounded set; for spaces in which bounded sets are precompact, we may then extract a convergent subsequence (in the relevant topology) and try to show that the corresponding limit solves the problem in question.

4. Elliptic Variational Inequalities

We will assume some familiarity with the theory of elliptic variational inequalities, which serves as a foundation for the results to follow. However, in order to attain a semblance of self-containment, we shall briefly review and present some of the main results on elliptic variational inequalities which we shall employ subsequently in our review of the basic existence theory of variational inequalities which are of evolution type.

4.1. Existence results

Throughout we shall assume that V is a real Banach space with its topological dual denoted by V^*, and the pairing between V^* and V, by $\langle \cdot, \cdot \rangle$. Let

$$F : V \to \mathbb{R} \cup \pm\infty = [-\infty, \infty]$$

be a functional with *effective domain*

$$D(F) = \{u \in V \mid F(u) \neq \pm\infty\}.$$

A point $u^* \in V^*$ is called a *subgradient* for the functional F at the point u provided that $u \in D(F)$ and

$$F(v) \geq F(u) + \langle u^*, v - u \rangle, \ \forall v \in V. \tag{4.1}$$

The set of all subgradients at a point $u \in D(F)$ is denoted by $\partial F(u)$ and called the *subdifferential* of F at the point u. (Concerning the properties of the subdifferential for convex functions, we refer the reader to [35] and [40].)

We shall state and prove here, one of the basic results relating minimization problems with variational inequalities. To this end we shall assume that the functional F has the following properties:

$$F, J, j : V \to (-\infty, \infty]$$

where F has the form

$$F = J + j,$$

where J and j are functionals which are lower semicontinuous with respect to a topology τ, i.e., the sets

$$\{u \mid J(u) \cdot \ a\},\ \{u \mid j(u) \cdot \ a\}$$

are closed with respect to τ for each $a \in \mathbb{R}$. Further we assume that F is coercive, i.e., that

$$F(u) \to \infty, \text{ as } \|u\| \to \infty,$$

and that bounded subsets of V are precompact with respect to the topology τ.

The topologies τ most frequently employed are the weak topology, in case V is a reflexive space, or the weak star topology, in case V is the dual of a separable space. In what is to follow, examples for both cases will be of interest.

We have the following result. We also give a brief sketch of a proof.

Theorem 4.1. *Assume the above conditions and that J is convex and $D(J) \neq \emptyset$, $D(j) = V$, with j Gâteaux differentiable, with Gâteaux derivative $j'(u)$. Then there exists $u \in D(J)$ such that*

$$F(u) = \min_{v \in V} F(v)$$

and

$$0 \in \partial J(u) + j'(u), \tag{4.2}$$

or equivalently that

$$J(v) - J(u) + \langle j'(u), v - u\rangle \geq 0,\ \forall v \in V. \tag{4.3}$$

It follows from the assumptions on F (particularly the assumption of lower semicontinuity and coercivity) that F is bounded below, say,

$$-\infty < \alpha := \inf_{v \in V} F(v).$$

We thus obtain a bounded sequence $\{u_n\}$ with

$$F(u_n) \to \alpha,$$

and therefore a subsequence $\{u_{n_j}\}$ such that, with respect to the topology τ,

$$u_{n_j} \to u,$$

and

$$F(u) = \alpha.$$

Therefore

$$F(u) = J(u) + j(u) \cdot \quad F(v) = J(v) + j(v), \ \forall v \in V.$$

Hence, for $t > 0$ and $v \in V <$ we obtain

$$0 \cdot \quad \frac{1}{t}\left(J(u + t(v-u)) - J(u)\right) + \frac{1}{t}\left(j(u + t(v-u)) - j(u)\right),$$

and, using the convexity of J and the differentiability of j, we obtain

$$0 \cdot \quad J(v) - J(u) + \langle j'(u), v - u\rangle + \frac{1}{t}o(t),$$

from which follows (4.3) and thus, by definition of the subdifferential,

$$-j'(u) \in \partial J(u),$$

i.e., we also have (4.2).

For monotone mappings we have another fundamental result ([29]), known as the Browder–Minty Theorem. It is the following:

Theorem 4.2. *Let V be a reflexive Banach space, and let $A : V \to V^*$ be a monotone hemicontinuous mapping which is bounded. Let ϕ be a convex, lower semicontinuous functional from V to $\mathbb{R} \cup \{\infty\}$ with nonempty effective domain $D(\phi)$. Finally, suppose that A and ϕ satisfy the coercivity condition*

$$\lim_{\|u\|\to\infty} \frac{\langle Au, u - u_0\rangle + \phi(u)}{\|u\|} = \infty, \tag{4.4}$$

for some $u_0 \in D(\phi)$. Then, for all $f \in V^*$, there exists a solution $u \in V$ of the variational inequality

$$\langle Au - f, v - u\rangle + \phi(v) - \phi(u) \geq 0 \quad \forall v \in V. \tag{4.5}$$

The solution is unique, whenever A is strictly monotone.

We point out important special cases of the above theorems, when in the case of Theorem 4.1 the functional j and in the case of Theorem 4.2 the functional ϕ are the indicator functionals of a convex set K in V, i.e.,

$$\phi(u) = \begin{cases} 0, & \text{for} \quad u \in K \\ \infty, & \text{for} \quad u \notin K, \end{cases} \tag{4.6}$$

with the set K closed with respect to either the topology τ (Theorem 4.1) or the topology of V (Theorem 4.2).

In these cases, the solution u of the variational inequality (4.5) clearly belongs to the set K; such sets K typically correspond to obstacles, unilateral constraints, or certain boundary conditions.

For more information on static variational inequalities, we refer to [3], [9], [14], [17], [23], [24], [27], [28], [37], and the references which they provide.

4.2. An example

Let us consider the boundary value problem

$$\text{(4.7)} \qquad \begin{cases} -\mathrm{div}(A(|\nabla u|^2)\nabla u) + F(x,u) &= 0, \quad \text{in} \\ u &= 0, \quad \text{on } \partial \end{cases},$$

where $\subset \mathbb{R}^N$ is a bounded domain with smooth boundary.

Let

$$\phi : \mathbb{R} \to \mathbb{R},\ \phi(s) = A(s^2)s.$$

Then, if $\phi(s) = |s|^{p-1}s,\ p > 1$, problem (4.7) is the stationary equation corresponding to some of the problems indicated in the previous section, and is fairly well understood and a great variety of existence results are available. These results are usually obtained using variational methods, monotone operator methods or fixed point and degree theory arguments in the Sobolev space $W_0^{1,p}(\ )$. If, on the other hand, we assume that ϕ is an odd nondecreasing function such that:

$$\phi(0) = 0,\ \phi(t) > 0,\ t > 0,$$

$$\lim_{t\to\infty} \phi(t) = \infty,$$

and

$$\phi \text{ is right continuous},$$

then a Sobolev space setting for the problem is not appropriate. The first general existence results using the theory of monotone operators in Orlicz-Sobolev spaces were obtained in [13] and in [19], [20]. Other recent work that puts the problem into this framework is contained in the papers [10] and [18].

We assume that $F : \ \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that satisfies certain growth conditions to be specified later.

A natural way of formulating the boundary value problem is a variational inequality formulation of the problem in a suitable Orlicz-Sobolev space. In order to do this we shall have need of some facts about Orlicz-Sobolev spaces which we shall give now.

Let us put $\Phi(t) = \int_0^t \phi(s)ds,\ t \in \mathbb{R}$. Then Φ is a Young (or N-) function (cf. [1], [25], [26]). Also, following these references, we denote by $\overline{\Phi}$ the conjugate Young function of Φ, i.e.,

$$\overline{\Phi}(t) = \sup\{ts - \Phi(s) : s \in \mathbb{R}\},$$

and by Φ^* the Sobolev conjugate of Φ, i.e.,

$$(\Phi^*)^{-1}(t) = \int_0^t \frac{\Phi^{-1}(s)ds}{s^{\frac{N+1}{N}}} ds, \tag{4.8}$$

provided that

$$\int_1^\infty \frac{\Phi^{-1}(s)ds}{s^{\frac{N+1}{N}}} ds = \infty. \tag{4.9}$$

Let Φ be a Young function. The Orlicz class $\tilde{L}_\Phi := \tilde{L}_\Phi(\quad)$ is the set of all (equivalence classes) of measurable functions u defined on   such that

$$\int \Phi(|u(x)|)dx < \infty.$$

The Orlicz space $L_\Phi := L_\Phi(\quad)$ is the linear hull of $\tilde{L}_\Phi$, i.e., the set of all measurable functions u on   such that

$$\int \Phi\left(\frac{|u(x)|}{k}\right) dx < \infty, \quad \text{for some} \quad k > 0.$$

Then L_Φ is a Banach space when equipped with the norm (the Luxemburg norm)

$$\|u\|_\Phi = \inf\left\{k > 0 : \int \Phi\left(\frac{|u|}{k}\right) dx \cdot \quad 1\right\},$$

or the equivalent norm (the Orlicz norm)

$$\|u\|_{(\Phi)} = \sup\left\{\left|\int uv dx\right| : v \in \tilde{L}_{\overline{\Phi}},\ \int \overline{\Phi}(|v|)dx \cdot \quad 1\right\}.$$

If Φ_1 and Φ_2 are two Young functions, one writes

$$\Phi_1 \cdot \quad \Phi_2,$$

provided there exist constants t_0 and k such that

$$\Phi_1(t) \cdot \quad \Phi_2(kt),\ t \geq t_0,$$

and one says that Φ_1 and Φ_2 are equivalent whenever

$$\Phi_1 \cdot \quad \Phi_2 \text{ and } \Phi_2 \cdot \quad \Phi_1.$$

If Φ_1 and Φ_2 are equivalent, then they determine the same Orlicz space. Also, it is the case that the imbedding

$$L_{\Phi_2} \hookrightarrow L_{\Phi_1},$$

is continuous, whenever $\Phi_1 \cdot \quad \Phi_2$.

The closure of L^∞ in L_Φ is denoted by E_Φ, which is a separable Banach space. The Orlicz-Sobolev space $W^1L_\Phi := W^1L_\Phi(\quad)$ is the set of all $u \in L_\Phi$ such that the distributional derivatives $\partial_i u = \frac{\partial u}{\partial x_i}$, $i = 1, \cdots, N$, are also in L_Φ. This is a Banach space with respect to the norm

$$\|u\|_{1,\Phi} = \|u\|_{W^1L_\Phi} = \|u\|_\Phi + \sum_{i=1}^N \|\partial_i u\|_\Phi.$$

It is known (cf. [25], [34]) that L_Φ is the dual space of $E_{\overline{\Phi}}$, i.e.,

$$L_\Phi = (E_{\overline{\Phi}})^*, \quad \text{and} \quad L_{\overline{\Phi}} = (E_\Phi)^*.$$

The space W^1E_Φ is defined similarly.

We denote by $W_0^1L_\Phi$ the closure of $C_0^\infty(\quad)$ with respect to the weak* topology.

We also mention a notion of relative growth of Young functions which will play a role in our considerations (cf. [1], [25], [26]). A Young function Φ_1 is said to grow essentially more slowly than another Young function Φ_2, abbreviated by

$$\Phi_1 \ll \Phi_2,$$

if

$$\lim_{t\to\infty} \frac{\Phi_1(t)}{\Phi_2(kt)} = 0, \ \forall k > 0.$$

Now, we formulate and extend (4.7) as a variational inequality in a suitable Orlicz-Sobolev space.

Multiplying both sides of (4.7) by $v \in C_0^\infty(\quad)$ and integrating by parts (provided these integrations may be performed), we see that the weak form of (4.7) is

$$\int A(|\nabla u|^2)\nabla u \cdot \nabla v dx + \int F(x,u) v dx = 0. \tag{4.10}$$

A natural choice of the space of test functions v is, of course, $W_0^1L_\Phi$. However, the mapping $u \mapsto L(u)$, where

$$\langle L(u), v\rangle = \int A(|\nabla u|^2)\nabla u \cdot \nabla v dx, \ v \in W_0^1L_\Phi,$$

is not necessarily defined on the whole space, we, hence, formulate (4.10) as a variational inequality.

Consider the functional

$$J : W_0^1L_\Phi \to \mathbb{R} \cup \{\infty\}, \ J(u) := \int \Phi(|\nabla u|)dx.$$

Since $\frac{\partial \Phi}{\partial \xi_i}(|\xi|) = A(|\xi|^2)\xi_i$, $i = 1, \ldots, N$, we have, at least formally,

$$\langle J'(u), v\rangle = \int \sum_{i=1}^{N} A(|\nabla u|^2)\,\partial_i\, u\partial_i v dx = \int A(|\nabla u|^2)\nabla u \cdot \nabla v dx.$$

Let us now assume that F satisfies the growth condition

$$|F(x,s)| \cdot \quad B(x) + |\Psi_0'(s)|,\ s \in \mathbb{R},\ x \in \quad , \tag{4.11}$$

where Ψ_0 is a differentiable Young function such that Ψ_0 is strictly convex,

$$\Psi_0 \ll \Phi^*, \tag{4.12}$$

and

$$B \in L_{\overline{\Psi}_0}. \tag{4.13}$$

We then may, for $u \in W^1 L_\Phi$, define $k(u) \in (W^1 L_\Phi)^*$ by

$$\langle k(u), v\rangle := \int F(x,u) v dx,\ \forall v \in W^1 L_\Phi.$$

The following lemma holds (see [27]):

Lemma 4.3. *The mapping*

$$k : W^1 L_\Phi \to (W^1 L_\Phi)^*$$

is continuous.

In many situations, it is more convenient to work in an Orlicz space which lies between L_{Φ^*} and L_{Ψ_0}. We choose a Young function Ψ such that

$$\Psi_0 \ll \Psi \ll \Phi^*. \tag{4.14}$$

We can replace, because of this ordering, Φ^* by Ψ in the proof of Lemma 4.3 and obtain:

Lemma 4.4. *If $u \in L_\Psi$, then $\psi_0(u) \in L_{\overline{\Psi}_0}$ and*

$$F(\cdot, u) \in L_{\overline{\Psi}_0} \subset L_{\overline{\Psi}}. \tag{4.15}$$

Moreover,

$$\|F(\cdot,u)\|_{\overline{\Psi}} \cdot \quad C\|F(\cdot,u)\|_{\overline{\Psi}_0} \cdot \quad \|B\|_{\overline{\Psi}_0} + \|\psi_0(|u|)\|_{\overline{\Psi}_0}.$$

We also have:

Lemma 4.5. *If $u \in L_\Psi$, then $F(\cdot, u) \in L_{\overline{\Psi}}$. The mapping*

$$k : u \mapsto k(u) = F(\cdot, u)$$

is continuous and bounded from L_Ψ to $L_{\overline{\Psi}}$.

Thus one may formulate (4.10) (at least formally) as the equation:

$$J'(u) + k(u) = 0. \tag{4.16}$$

However, J is not differentiable in general (J is not even defined on the whole space $W_0^1 L_\Phi$, since J assumes, in general, finite values only on a convex, nondense subset of $W_0^1 L_\Phi$). On the other hand, since J is convex and lower semicontinuous (as will be stated later), we replace (4.16) by the inclusion

$$0 \in \partial J(u) + k(u), \tag{4.17}$$

where ∂J is the subdifferential of J; this, in turn, is equivalent to the variational inequality

$$\begin{cases} J(v) - J(u) + \langle k(u), v - u\rangle \geq 0, \ \forall v \in W_0^1 L_\Phi \\ u \in W_0^1 L_\Phi. \end{cases} \tag{4.18}$$

The advantage of this formulation is that solutions of (4.17) are included in the effective domain of the functional J,

$$D(J) = \Big\{ u \in W_0^1 L_\Phi : J(u) = \int \Phi(|\nabla u|) dx < \infty \Big\}.$$

We therefore may consider (4.18) as the variational inequality formulation of (4.10) (and hence (4,7)).

We now proceed to discuss the existence of solutions of (4.18) and more general inequalities. We first provide some properties of the functional J, (see again [27]).

Lemma 4.6. *The functional J is convex and lower semicontinuous on $W^1 L_\Phi$. If Φ is strictly convex, then J is strictly convex on $W_0^1 L_\Phi$.*

In what follows, we consider the following variational inequality associated with the boundary value problem (4.18):

$$\begin{cases} J(v) - J(u) + \langle k(u), v - u\rangle \geq 0, \ \forall v \in K \\ u \in K, \end{cases} \tag{4.19}$$

where K is a convex subset of $W_0^1 L_\Phi$, closed with respect to the weak* topology and $0 \in K$ (in the case $K = W_0^1 L_\Phi$, (4.19) reduces to (4.18)).

We consider the problem that k is independent of u:

$$\left\{ \begin{array}{l} J(v) - J(u) + \langle k, v-u \rangle \geq 0, \ \forall v \in K \\ u \in K, \end{array} \right. \tag{4.20}$$

with $k \in L_{\overline{\Psi}}$, $\langle k, v \rangle = \int \ kvdx$, $v \in W_0^1 L_\Phi$.

We may rewrite (4.20) as

$$\left\{ \begin{array}{l} J(v) + \langle k, v \rangle \geq J(u) + \langle k, u \rangle, \ \forall v \in K \\ u \in K, \end{array} \right.$$

and see that u solves (4.20) if and only if u is a minimizer of the problem

$$\min_{v \in K} [J(v) + \langle k, v \rangle]. \tag{4.21}$$

We will indicate why (4.21) has a solution.

To accomplish this we shall, in what follows, make the additional assumption:

- $\overline{\Phi}$ satisfies a Δ_2 condition at infinity (cf. [25]), which has as a consequence that $L_{\overline{\Phi}} = E_{\overline{\Phi}}$.

It follows from the work in [19] that in the space $W_0^1 L_\Phi$ a Poincaré inequality holds and consequently that $|||\nabla u|||_\Phi$ furnishes an equivalent norm for $W_0^1 L_\Phi$. Thus for $u \in W_0^1 L_\Phi$ we shall henceforth use

$$\|u\| := |||\nabla u|||_\Phi.$$

We have the following lemma ([27]):

Lemma 4.7. *The functional J is sequentially lower semicontinuous with respect to the weak* topology of $W_0^1 L_\Phi$ and is coercive in the sense that*

$$\frac{J(u)}{\|u\|} \to \infty, \quad \text{as} \quad \|u\| \to \infty. \tag{4.22}$$

From this lemma, and Theorem 4.1 we obtain the following result:

Theorem 4.8. *For each $k \in L_{\overline{\Psi}} (\subset (W_0^1 L_\Phi)^*)$, the set U_k of solutions of (4.21) (and thus of (4.20)) is nonempty, convex, and bounded in $W_0^1 L_\Phi$, and thus precompact in L_Ψ. The solution set is a singleton, whenever Φ is strictly convex.*

In case k is dependent upon u, various assumptions may be imposed on k in order that Theorem 4.1 may be applied to deduce the existence of a solution of (4.19). We remark that conditions have been given in [10] guaranteeing that $k = j'$ for some functional j.

5. The Translation Method

The translation method, introduced by Brézis and Lions ([5], [29]), was one of the first techniques used to establish the existence of solutions to parabolic variational inequalities. This approach exploits the fact that the operator $-d/dt$ generates the semigroup of translations ([33], [38]), which leads naturally to a difference approximation scheme.

The resulting technique does not apply to problem (3.10) as stated in Section 3, due to its requirement that $u'(t)$ belong to V^* for a.e. $t \in (0, T)$. To eschew this smoothness assumption, we introduce a weak formulation of parabolic variational inequalities in Section 5.1. We use the translation method to prove the existence of weak solutions of (3.10) and then address the question of when such weak solutions actually solve (3.10).

5.1. Weak solutions of parabolic inequalities

If $u \in \mathcal{V}$ solves problem (3.10), then we clearly have

$$\begin{aligned} &\langle v'(t) - f(t), v(t) - u(t)\rangle + a(u(t), v(t) - u(t)) \\ &\quad + \phi(v(t)) - \phi(u(t)) \geq \langle v'(t) - u'(t), v(t) - u(t)\rangle, \\ &\quad \forall\, v \in \mathcal{V},\ \text{a.e. } t \in (0, T). \end{aligned} \tag{5.1}$$

It follows upon integrating inequality (5.1) from 0 to T and using (3.5),

$$\begin{aligned} &\int_0^T \langle v'(t) - u'(t), v(t) - u(t)\rangle\, dt \\ &\quad = \frac{1}{2} \| v(T) - u(T) \|_V^2 - \frac{1}{2} \| v(0) - u(0) \|_V^2. \end{aligned}$$

we obtain

$$\begin{aligned} &\int_0^T \big(\langle v'(t) - f(t), v(t) - u(t)\rangle + a(u(t), v(t) - u(t)) \big)\, dt \\ &\quad + \Phi(v) - \Phi(u) \geq 0,\ \forall\, v \in \mathcal{V} \quad \text{with} \quad v(0) = u_0, \end{aligned} \tag{5.2}$$

where $\Phi : \mathcal{V} \to \mathbb{R} \cup \{\infty\}$ is the convex lower semicontinuous functional defined by

$$\Phi(v) = \int_0^T \phi(v(t))\, dt \quad \text{for} \quad v \in \mathcal{V}. \tag{5.3}$$

The effective domain $D(\Phi)$ of Φ is defined as in (3.8).

The problem of finding $u \in \mathcal{V}$ such that (5.2) holds is the *weak* version of the parabolic variational inequality (3.10), and u is correspondingly referred to as a *weak solution*. A natural question, then, is when a weak solution actually solves the strong formulation (3.10).

Before proceeding, we establish some notation and some ancillary facts. For each $t > 0$, let $S(t)$ denote translation by t, i.e.,

$$S(t)(u(s)) = u(s-t) \quad \text{for} \quad u \in \mathcal{V},$$

from which we obtain the important family of operators

$$\left\{ \frac{I - S(t)}{t},\ t > 0 \right\}. \tag{5.4}$$

By virtue of the pivot space structure (3.3) and the fact that $\{S(t)\}$ is a semigroup of contractions, we find that the operators (5.4) are monotone:

$$\frac{1}{t}\langle (I - S(t))v, v\rangle_{\mathcal{V}} = \frac{1}{t}\langle (I - S(t))v, v_{\mathcal{H}} \geq 0, \tag{5.5}$$

where the first pairing is between $\mathcal{V}^*$ and $\mathcal{V}$ and the second is between $\mathcal{H}$ and itself. Letting $t \to 0$ in (5.5) shows that

$$\langle v', v\rangle_{\mathcal{V}} \geq 0,\ \forall\, v \in \mathcal{V} \cap D(d/dt). \tag{5.6}$$

In addition to these monotonicity results, the following compatibility of the semigroup $\{S(t)\}$ with the effective domain of Φ plays a key role in the translation method. We require $D(\Phi)$ to be invariant under $\{S(t)\}$, i.e.,

$$S(t)(D(\Phi)) \subset D(\Phi) \quad \text{for} \quad t > 0. \tag{5.7}$$

An important consequence of condition (5.7) is that, for each $v \in D(\Phi)$, the sequence $\{v_\alpha\}$ defined by

$$v_\alpha = \left(I + \alpha \frac{d}{dt} \right)^{-1} v, \quad \text{for} \quad \alpha > 0, \tag{5.8}$$

belongs to $D(\Phi) \cap D(d/dt)$ and satisfies

$$\lim_{\alpha \to 0} v_\alpha = v, \quad \langle v'_\alpha, v_\alpha - v\rangle_{\mathcal{V}} \cdot \quad 0. \tag{5.9}$$

Finally, for $t > 0$, we define the mapping $A_t : V \to V^*$ by

$$A_t = \frac{I - S(t)}{t} + A,$$

with the corresponding form $a_t(\cdot, \cdot)$,

$$a_t(u, v) = \langle A_t u, v\rangle, \quad \text{for} \quad u, v \in V.$$

5.2. Existence of weak solutions

We now prove the following result ([5]):

Theorem 5.1. *Let the spaces V, H, $\mathcal{V}$, and $\mathcal{H}$ be as described above. Suppose that the pseudomonotone operator $A : V \to V^*$ and the convex lower semicontinuous functional $\phi : V \to \mathbb{R} \cup \{+\infty\}$, with $D(\phi)$ nonempty, satisfy the coercivity condition*

$$\lim_{\|v\| \to \infty} \frac{a(v, v - v_0) + \phi(v)}{\|v\|} = \infty \tag{5.10}$$

for some $v_0 \in D(\phi)$. Further, suppose that the semigroup of translations $\{S(t)\}$ and $D(\Phi)$ satisfy the compatibility condition (5.7), where Φ is defined by (5.3). Then, for each $f \in \mathcal{V}^*$, there exists $u \in D(\Phi)$ such that

$$\begin{aligned} &\int_0^T \big(\langle v'(s) - f(s), v(s) - u(s)\rangle + a(u(s), v(s) - u(s))\big)\, ds \\ &\quad + \Phi(v) - \Phi(u) \geq 0, \ \forall v \in D(\Phi) \cap D(d/dt) \quad \textit{with} \quad v(0) = u_0. \end{aligned} \tag{5.11}$$

Proof. First, note that the operator A_t inherits the pseudomonotonicity and coercivity conditions of A, thanks to the monotonicity results verified in Section 5.1. One first verifies that the following variational inequality

$$\begin{aligned} &\int_0^T (a_t(u_t(s), v(s) - u_t(s)) - \langle f(s), v(s) - u_t(s)\rangle)\, ds \\ &\quad + \Phi(v) - \Phi(u_t) \geq 0, \ \forall v \in \mathcal{V}. \end{aligned} \tag{5.12}$$

is an *elliptic* variational inequality of the type discussed in Section 4 and hence that the Browder–Minty theorem, Theorem 4.2, may be applied to deduce the existence of a solution $u_t \in D(\Phi)$ of (5.12). Using the definition of A_t, we thus obtain $\{u_t\} \subset D(\Phi)$ such that

$$\begin{aligned} &\int_0^T \Big\langle \frac{I - S(t)}{t} u_t(s) - f(s), v(s) - u_t(s) \Big\rangle\, ds \\ &\quad + \int_0^T a(u_t(s), v(s) - u_t(s))\, ds + \Phi(v) - \Phi(u_t) \geq 0, \ \forall v \in \mathcal{V}. \end{aligned} \tag{5.13}$$

The variational inequality (5.13) and the coercivity condition (3.9) show that $\{u_t\}$ is bounded. Consequently we may assume that

$$u_t \rightharpoonup u \in \mathcal{K} \quad \text{and} \quad Au_t \rightharpoonup g \in \mathcal{V}^*.$$

The convergence of $\{Au_t\}$ follows from the boundedness of A.

We now show that the weak limit u solves (5.2). Due to the monotonicity of the operators (5.4), the inequality

$$\begin{aligned}\int_0^T \Big\langle \frac{I - S(t)}{t} v(s) - f(s), v(s) - u_t(s) \Big\rangle \, ds \\ + \int_0^T a(u_t(s), v(s) - u_t(s)) \, ds + \Phi(v) - \Phi(u_t) \geq 0, \forall v \in \mathcal{V}.\end{aligned} \tag{5.14}$$

follows from (5.13). By adding $\int_0^T a(u_t(s), v(s) - u(s)) \, ds$ to both sides of (5.14) and rearranging terms, we have

$$\begin{aligned}&\int_0^T \Big\langle \frac{I - S(t)}{t} v(s) - f(s), v(s) - u_t(s) \Big\rangle \, ds \\ &+ \int_0^T a(u_t(s), v(s) - u(s)) \, ds + \Phi(v) - \Phi(u_t) \\ &\geq \int_0^T a(u_t(s), u_t(s) - u(s)) \, ds \quad \forall v \in \mathcal{V}.\end{aligned} \tag{5.15}$$

Hence,

$$\begin{aligned}&\limsup_{t \to 0} \int_0^T a(u_t(s), u_t(s) - u(s)) \, ds \\ &\cdot \quad \int_0^T \langle v'(s) - f(s), v(s) - u(s) \rangle \, ds \\ &+\Phi(v) - \Phi(u), \ \forall \, v \in \mathcal{V}.\end{aligned} \tag{5.16}$$

For each $\alpha > 0$, define u_α as in (5.8). We may then substitute $v = u_\alpha$ into (5.16) to obtain

$$\begin{aligned}&\limsup_{t \to 0} \int_0^T a(u_t(s), u_t(s) - u(s)) \, ds \\ &\cdot \quad \int_0^T \langle u_\alpha'(s) + g(s) - f(s), u_\alpha(s) - u(s) \rangle \, ds \\ &+\Phi(u_\alpha) - \Phi(u),\end{aligned} \tag{5.17}$$

because of (5.9). Letting $\alpha \to 0$, we deduce

$$\limsup_{t \to 0} \int_0^T a(u_t(s), u_t(s) - u(s)) \, ds \cdot \ \ 0. \tag{5.18}$$

Hence, by the pseudomonotonicity of A, we obtain that

$$
\begin{aligned}
&\int_0^T a(u(s), u(s) - v(s))\, ds \\
&\quad \cdot \ \liminf_{t \to 0} \int_0^T a(u_t(s), u_t(s) - v(s))\, ds, \ \forall\, v \in \mathcal{V}.
\end{aligned} \tag{5.19}
$$

Combining (5.19) with inequality (5.16), we find that

$$
\begin{aligned}
&\int_0^T a(u(s), u(s) - v(s))\, ds \\
&\quad \cdot \ \int_0^T \langle v'(s) - f(s), v(s) - u(s) \rangle\, ds + \Phi(v) - \Phi(u), \ \forall\, v \in \mathcal{V},
\end{aligned}
$$

which is precisely the parabolic variational inequality (5.2).

Since all of the examples considered in Section 2 satisfy the requirements of Theorem 5.1, we have established the existence of weak solutions to all of these problems. Of course, one would like to know whether these weak solutions satisfy the original strong formulation (3.10).

We refer to [5] where some such results are discussed.

6. Rothe's Method

Rothe's Method ([22], [31], [41]), also known as the *method of lines* or the *method of semidiscretization*, is an extension of the backward Euler scheme for parabolic equations and is a powerful tool in both the theoretical and numerical analyses of evolution problems. To illustrate the method, we use it to solve the sample parabolic variational inequality (2.9) discussed in Section 2.1:

Problem: Find $u \in \mathcal{K}$ with $u(0) = u_0 \in H$ and such that

$$
\langle u'(t), v(t) - u(t) \rangle + a(u(t), v(t) - u(t)) \geq 0, \ \forall v \in \mathcal{K}, \ a.e.\ t \in (0, T), \tag{6.1}
$$

where

$$
K = \{ v \in V | v(x) \geq 0 \quad \text{for} \quad x \in \Gamma \}, \tag{6.2}
$$

and $\mathcal{K}$ consists of those $v \in \mathcal{V}$ such that $v(t) \in K$ for a.e. $t \in (0, T)$, and $a(\cdot, \cdot)$ is the form

$$
a(u, v) = \int \nabla u \cdot \nabla v\, dx \quad \text{for} \quad u, v \in V.
$$

As mentioned earlier, this problem models diffusion in a domain with a semipermeable boundary.

The first step of Rothe's method is to partition the time interval $[0, T]$ into n equal subintervals $[t_{i-1}, t_i]$, where $i = 1, 2, \dots, n$, $t_i = ih$, and h is the mesh width $\frac{T}{n}$. For each $i = 1, 2, \dots$, we consider the problem of finding a solution $u_i \in K$ of

$$\left\langle \frac{u_i - u_{i-1}}{h}, v - u_i \right\rangle + a(u_i, v - u_i) \geq 0, \ \forall\, v \in K. \tag{6.3}$$

with

$$u_0 = u_0(x), \ x \in \quad .$$

Rewriting inequality (6.3) in the form

$$\frac{1}{h}\langle u_i, v - u_i\rangle + a(u_i, v - u_i) \geq \frac{1}{h}\langle u_{i-1}, v(t) - u_i(t)\rangle, \ \forall\, v \in K, \tag{6.4}$$

and observing that u_{i-1} is known at each step, we see that (6.3) is an elliptic variational inequality for the bilinear form

$$u, v \longmapsto \frac{1}{h}\int uv\,dx + \int \nabla u \cdot \nabla v\,dx \tag{6.5}$$

defined on $V \times V$. For n large, the coefficient $1/h$ is large, so the form (6.5) satisfies the coercivity condition (4.4) with $v_0 = 0$. Our basic existence result (4.2) therefore applies to the elliptic variational inequality (6.3) to guarantee a solution $u_i \in K$.

We thus obtain n autonomous functions $u_i \in K$, $i = 1, \dots, n$, that may be combined to form *Rothe's function*, a proposed approximate solution of the original parabolic variational inequality:

$$u_n(x,t) = u_{i-1}(x) + \frac{t - t_{i-1}}{h}(u_i(x) - u_{i-1}(x)), \quad t \in [t_{i-1}, t_i]. \tag{6.6}$$

Observe that Rothe's function $u_n(x,t)$ is linear in time over each subinterval $[t_{i-1}, t_i]$; the time variable plays the role of a homotopy parameter connecting u_{i-1} at time t_{i-1} to u_i at time t_i.

To show that $u_n(x,t)$ converges to a solution $u(x,t)$ of (6.1) as $n \to \infty$, we establish some necessary estimates and then apply the Arzelà–Ascoli Theorem. Thus, for $j \geq 2$, we take $v = u_j$ in the inequality (6.3) for $i = j - 1$ and $v = u_{j-1}$ in the inequality for $i = j$ to produce

$$\left\langle \frac{u_{j-1} - u_{j-2}}{h}, u_j - u_{j-1} \right\rangle + a(u_{j-1}, u_j - u_{j-1}) \geq 0, \tag{6.7}$$

$$\left\langle \frac{u_j - u_{j-1}}{h}, u_{j-1} - u_j \right\rangle + a(u_j, u_{j-1} - u_j) \geq 0. \tag{6.8}$$

Adding inequalities (6.7) and (6.8) yields

$$\frac{1}{h}\|u_j - u_{j-1}\|^2 + \|\nabla(u_j - u_{j-1})\|^2 \cdot \ \frac{1}{h}\langle u_{j-1} - u_{j-2}, u_j - u_{j-1}\rangle, \tag{6.9}$$

from which we obtain

$$\|u_j - u_{j-1}\|^2 + 2h\|\nabla(u_j - u_{j-1})\|^2 \cdot \ \|u_{j-1} - u_{j-2}\|^2, \quad j \geq 2 \tag{6.10}$$

by applying the Cauchy–Schwarz inequality and using the elementary inequality $2ab \cdot \ a^2 + b^2$. The norm $\|\cdot\|$ being used here is the norm of $H = L^2(\ )$.

For the case $j = 1$, we choose $v = u_0$ in (6.3) to get

$$\frac{1}{h}\|u_1 - u_0\|^2 + \|\nabla(u_1 - u_0)\|^2 \cdot \ |\langle \nabla u_0, \nabla(u_1 - u_0)\rangle|.$$

If the initial datum u_0 belongs to $H^2(\ ) \cap H_0^1(\ )$, integration by parts reveals that

$$|\langle \nabla u_0, \nabla(u_1 - u_0)\rangle| \cdot \ |\langle \Delta u_0, u_1 - u_0\rangle| \cdot \ \|\Delta u_0\|\, \|u_1 - u_0\|,$$

so that we have the basic bound

$$\left\|\frac{u_1 - u_0}{h}\right\| \cdot \ \|\Delta u_0\|. \tag{6.11}$$

Combining this estimate with inequality (6.10) shows that

$$\left\|\frac{u_i - u_{i-1}}{h}\right\| \cdot \ C, \quad i = 1, 2, \ldots, n, \tag{6.12}$$

for some constant C that is independent of n. Upon choosing $v = u_i$ in (6.3), a similar uniform bound involving the norm of V follows:

$$\|u_i\|_V \cdot \ C \quad i = 1, 2, \ldots, n. \tag{6.13}$$

These bounds provide a uniform estimate on the derivative u_n', since

$$u_n' = \frac{u_i - u_{i-1}}{h}.$$

Thus, (6.12) says that

$$\|u_n'\| \cdot \ C \quad \text{for} \quad t \in [0, T], \tag{6.14}$$

which immediately gives the equicontinuity result

$$|u_n(t) - u_n(\tau)| \cdot \ C\,|t - \tau|, \quad \text{for} \quad t, \tau \in [0, T]$$

for the family $\{u_n\}$, $n \in \mathbb{N}$. The Arzelà–Ascoli Theorem and the compact embedding of $H_0^1(\quad)$ into $L^2(\quad)$ then guarantee that u_n converges to some function u in the space $C([0,T], L^2(\quad))$. In fact, u is Lipschitz continuous and therefore differentiable almost everywhere in $[0,T]$.

It remains to show that the limit u solves the parabolic variational inequality (6.1). To accomplish this, we next define $\overline{u}_n(t)$ to be the step function

$$\overline{u}_n(t) = u_i \quad \text{for} \quad t \in [t_{i-1}, t_i]. \tag{6.15}$$

It follows from (6.13) that $\{\overline{u}_n\}$ has a subsequence that converges weakly in $H_0^1(\quad)$, which we relabel as $\{\overline{u}_n\}$. Moreover, (6.12) yields the bound

$$|\overline{u}_n(t) - u_n(t)| \cdot \quad \frac{C}{n},$$

from which it follows that the weak limit of this sequence is u. By exploiting the bound (6.14) in a similar fashion, we see that u_n' converges weakly to u' in $L^2(0,T;L^2(\quad))$.

In terms of u_n and $\overline{u}_n$, the elliptic variational inequality (6.3) is

$$\langle u_n'(t), v(t) - \overline{u}_n(t)\rangle + a(\overline{u}_n(t), v(t) - \overline{u}_n(t)) \geq 0, \ \forall\, v \in \mathcal{K}, \tag{6.16}$$

which holds almost everywhere in $[0,T]$. For arbitrary points τ_1 and τ_2 in $[0,T]$, integrating (6.16) from τ_1 to τ_2 gives

$$\int_{\tau_1}^{\tau_2} \langle u_n'(t), v(t) - \overline{u}_n(t)\rangle + a(\overline{u}_n(t), v(t) - \overline{u}_n(t))\, dt \geq 0, \ \forall v \in \mathcal{K}. \tag{6.17}$$

Taking $\liminf$ as $n \to \infty$ in this inequality, we have

$$\int_{\tau_1}^{\tau_2} \langle u'(t), v(t) - u(t)\rangle + a(u(t), v(t) - u(t))\, dt \geq 0, \ \forall v \in \mathcal{K}, \tag{6.18}$$

since $\overline{u}_n \to u$ and $u_n' \to u$ in $L^2(0,T;L^2(\quad))$ and the bilinear form $a(\cdot,\cdot)$ is weakly lower semicontinuous. As (6.18) holds for any τ_1 and τ_2, the parabolic variational inequality (6.1) follows, proving that u is the desired solution.

Rothe's method for parabolic variational inequalities (3.10) may thus be summarized as the following algorithm:

- For a given integer n, divide the time interval $[0,T]$ into equal intervals of width $h = \frac{T}{n}$.
- For each $i = 1, \ldots, n$, obtain a solution $u_i \in K$ of the *elliptic* variational inequality

$$\left\langle \frac{u_i - u_{i-1}}{h} - f, v - u_i \right\rangle + a(u_i, v - u_i) \geq 0 \quad \forall v \in K, \tag{6.19}$$

where $u_{i-1} \in K$ is known.

- Construct Rothe's function $u_n(x,t)$ (6.6) and prove that $\{u_n\}$ converges to a solution u of (3.10) as $n \to \infty$.

The first two steps of this procedure are simple, as long as there are existence results for elliptic variational inequalities involving the particular operator A and the function spaces in question. The details of the third step, however, will depend heavily on the particular problem under consideration. A general result that follows from an application of Rothe's method is the following ([22]):

Theorem 6.1. *Let the spaces V, H, $\mathcal{V}$, and $\mathcal{H}$ be as described before. Suppose that the pseudomonotone operator $A : V \to V^*$ and the convex lower semicontinuous functional $\phi : V \to \mathbb{R} \cup \{+\infty\}$, with $D(\phi)$ nonempty, satisfy the coercivity condition*

$$\lim_{\|v\| \to \infty} \frac{a(v, v - v_0) + \phi(v)}{\|v\|} = \infty \tag{6.20}$$

for some $v_0 \in D(\phi)$, and suppose that there exists $z_0 \in H$ satisfying

$$\langle z_0, v \rangle + a(u_0, v) + \phi(v) - \phi(u_0) \geq \langle f(0), v - u_0 \rangle \ \forall v \in V \tag{6.21}$$

for the initial datum $u_0 \in H$. Finally, suppose that $f : [0,T] \times H \to H$ is Lipschitz. Then there exists a unique $u \in L^\infty(0,T;V) \cap C([0,T];H)$ with $u' \in L^\infty(0,T;H)$ such that

$$\begin{aligned} &\langle u'(t) - f(t, u(t), v(t) - u(t) \rangle + a(u(t), v(t) - u(t)) \\ &\quad + \phi(v(t)) - \phi(u(t)) \geq 0, \ \forall v \in \mathcal{V}, \ a.e. \ t \in (0,T) \end{aligned} \tag{6.22}$$

and $u(0) = u_0 \in H$.

We remark that Kačur, [22], actually proves this theorem for the more general case of a maximal monotone operator A. Operators of this type arise in many evolution problems and are much more general than the pseudomonotone operators considered here. For instance, maximal monotone operators, such as the fundamental example provided by the subdifferential of a convex function, are generally multivalued, whereas we have only considered single–valued operators from V to V^*. We refer to [6] for a thorough treatment of such operators and their fundamental role in evolution problems on Hilbert spaces.

Although this technique and the translation method of the previous section both employ a difference approximation of u', we emphasize that the two approaches are quite different. Rothe's method produces strong solutions $u \in C([0,T];H)$, whereas the translation method only provides weak solutions, whose regularity must then be investigated. In addition, the constructive nature of Rothe's method renders it effective in numerical analysis and computation. For more on this aspect of the method, as well as applications to a wide variety of evolution problems, we refer the reader to [22], [41].

7. The Penalty Method

Penalization is another common approach to variational inequalities, see, e.g., [5], [17], [23], [29], [32] for some detailed discussions. The underlying idea is to replace the inequality under consideration with a sequence of equations involving a *penalty operator* P whose kernel is the closed, convex set $D(\phi)$, where ϕ is the convex lower semicontinuous functional in the inequality. In the common case in which ϕ is the indicator functional of a constraint set K, for example, the approximating equations thus penalize admissible functions for violating the constraint that K represents. One then shows that the sequence of solutions obtained converges to a solution of the original variational inequality.

For simplicity of exposition, we restrict attention to the obstacle problem for the p–Laplacian that was discussed in Section 2.2. After showing how to use the penalty method to solve this particular problem, we describe how to apply it to the more general situation of problem (3.2).

We are thus interested in finding $u \in \mathcal{K}$ with $u(0) = u_0 \in L^2(\quad)$ and such that the parabolic variational inequality

$$\langle u' - f, v - u\rangle + a(u, v - u) \geq 0, \ \forall v \in \mathcal{K}, \tag{7.1}$$

holds, where $V = W_0^{1,p}(\quad)$, $\mathcal{V} = L^2(0,T;V)$, and $\langle \cdot,\cdot \rangle$ denotes the pairing between $\mathcal{V}^*$ and $\mathcal{V}$, given explicitly by

$$\langle u, v\rangle = \int_0^T \int (u(t), v(t))\, dx\, dt, \ \ u \in \mathcal{V}^*, \ v \in \mathcal{V}, \tag{7.2}$$

where the integrand $\int (u(t), v(t)) dx$ is the action of $u(t) \in V^*$ on $v(t) \in V$. We emphasize that this pairing is the same as that used in Section 5; this is necessary because the penalty method for the parabolic variational inequality (7.1) requires the solution of parabolic equations. In the preceding sections, we used associated elliptic problems to treat the parabolic variational inequalities of interest.

In addition, $a(\cdot,\cdot)$ is the form

$$a(u,v) = \int_0^T \int |\nabla u(t)|^{p-2} \nabla u(t) \cdot \nabla v(t)\, dx\, dt, \quad \text{for} \quad u, v \in \mathcal{V}, \tag{7.3}$$

and

$$A_p : \mathcal{V} \to \mathcal{V}^*$$

is given by

$$\langle A_p u, v \rangle := a(u, v),$$

K is the constraint set

$$K = \{v \in V | v \geq \psi\}, \tag{7.4}$$

for a given $\psi \in W^{1,p}(\ \ )$ satisfying $\psi \cdot\ \ 0$ on Γ, and $\mathcal{K}$ is the set of all $v \in \mathcal{V}$ with $v(t) \in K$ for a.e. $t \in (0, T)$.

For this particular problem, we define the appropriate *penalty operator* $P : \mathcal{V} \to \mathcal{V}^*$ by

$$(Pv)(t) := -(\psi - v(t))^+, \ v \in \mathcal{V}, \ t \in [0, T], \tag{7.5}$$

where $(\psi - v(t))^+$ denotes the positive part of $\psi - v(t) \in V$, i.e.,

$$(\psi - v(t))^+ := \max\{\psi - v(t), 0\}.$$

This truncation operation leaves $V = W_0^{1,p}(\ \ )$ invariant, so it follows from the pivot space structure (3.3) that P maps $\mathcal{V}$ into $\mathcal{V}^*$.

Note that the kernel of P is precisely the constraint set K. In addition, we have the following:

$$\begin{aligned}
&\langle Pu - Pv, u - v \rangle \\
&= \int_0^T \int (-(\psi - u(t))^+ + (\psi - v(t))^+)(u(t) - v(t))\, dx\, dt \\
&= \int_0^T \int_{\{u,v<\psi\}} (u(t) - v(t))^2\, dx\, dt \\
&= \int_0^T \int_{\{u<\psi\ \ v\}} -(\psi - u(t))(u(t) - v(t))\, dx\, dt \\
&+ \int_0^T \int_{\{v<\psi\ \ u\}} (\psi - v(t))(u(t) - v(t))\, dx\, dt \\
&+ \int_0^T \int_{\{u,v>\psi\}} 0 \cdot (u(t) - v(t))\, dx, dt,
\end{aligned}$$

which is nonnegative because the first three integrands are clearly nonnegative and the last integral vanishes. Thus, P is monotone. Since the sum of a pseudomonotone operator and a monotone operator is pseudomonotone ([29], [39]), it follows that $A_p + □P$ is pseudomonotone for any positive scalar □.

Let us choose $v_0 \in K$, then, since $Pv_0 = 0$, and P is monotone, we obtain the lower bound

$$
\begin{aligned}
a(v, v - v_0) + \square\langle Pv, v - v_0\rangle &= a(v, v - v_0) + \square\langle Pv - Pv_0, v - v_0\rangle \\
&\geq a(v, v - v_0) \\
&\geq \|v\|^p - \|v\|^{\frac{p}{q}}\|v_0\|, \quad \text{for} \quad v \in V.
\end{aligned} \tag{7.6}
$$

Therefore, $A_p + \square P$ also satisfies the desired coercivity condition. Finally, a simple calculation verifies that P is hemicontinuous.

For $\varepsilon > 0$, we now consider the associated *penalized problem*

$$
u' - \Delta_p u + \frac{1}{\varepsilon} Pu = f \quad \text{in} \qquad \times (0, T), \tag{7.7}
$$

which is understood in the sense of distributions. Since $A_p + \frac{1}{\varepsilon}P$ is pseudomonotone and coercive, equation (7.7) has a solution u_ε in $\mathcal{V} \cap D(d/dt)$ ([29]), in the sense that

$$
\langle u'_\varepsilon + A_p u_\varepsilon - f, v\rangle = \frac{1}{\varepsilon}\langle -Pu_\varepsilon, v\rangle, \ \forall v \in \mathcal{V},
$$

where, as above, $\langle \cdot, \cdot \rangle$ denotes the pairing between $\mathcal{V}^*$ and $\mathcal{V}$. One can show that a subsequence of the resulting set $\{u_\varepsilon\}$ of solutions of (7.7) converges as $\varepsilon \to 0$ to a solution of the parabolic variational inequality (7.1).

This is accomplished by noting that the boundedness of the sequence $\{u_\varepsilon\}$ follows from the coercivity condition (3.9) and, as shown in ([29]), we can conclude from the identity

$$
\langle Pu_\varepsilon, v\rangle = \varepsilon\langle u'_\varepsilon + A_p u_\varepsilon, v\rangle, \ \forall v \in \mathcal{V},
$$

that

$$
\|Pu_\varepsilon(t)\|_{\mathcal{V}^*} \cdot \ C\varepsilon, \tag{7.8}
$$

for some constant $C > 0$. Consequently, we may extract a subsequence $\{u_{\varepsilon_n}\}$ such that $\varepsilon_n \to 0$, $u_{\varepsilon_n} \to u$ in $\mathcal{V}$ for some $u \in \mathcal{V}$, and $Pu_{\varepsilon_n} \to 0$ in $\mathcal{V}^*$, as $n \to \infty$.

Since P is monotone, we have

$$
\langle Pv - Pu_{\varepsilon_n}, v - u_{\varepsilon_n}\rangle \geq 0, \ \forall\, v \in \mathcal{V},
$$

from which we obtain

$$
\langle Pv, v - u\rangle \geq 0, \ \forall\, v \in \mathcal{V} \tag{7.9}
$$

after passing to the limit as $n \to \infty$. For any $w \in \mathcal{V}$ and $s > 0$, substituting $v = u + sw$ into (7.9), yields

$$
\langle P(u + sw), sw\rangle \geq 0.
$$

Dividing by s and using the hemicontinuity of P we see that

$$\langle Pu, w\rangle \geq 0,\ \forall w \in \mathcal{V},$$

from which it follows that $Pu = 0$. The weak limit u therefore belongs to $\mathcal{K}$.

Using equation (7.7) and the monotonicity of P, we have

$$\langle u'_{\varepsilon_n} + A_p u_{\varepsilon_n} - f, u - u_{\varepsilon_n}\rangle = \frac{1}{\varepsilon_n}\langle Pu - Pu_{\varepsilon_n}\rangle u - u_{\varepsilon_n}\rangle \geq 0,$$

which yields

$$\limsup_{n\to\infty}\langle A_p u_{\varepsilon_n}, u_{\varepsilon_n} - u\rangle \cdot \ \limsup_{n\to\infty}\langle f - u'_{\varepsilon_n}, u_{\varepsilon_n} - u\rangle = 0.$$

We may thus use the pseudomonotonicity of A_p to obtain

$$\langle A_p u, u - v\rangle \cdot \ \liminf_{n\to\infty}\langle A_p u_{\varepsilon_n}, u_{\varepsilon_n} - v\rangle \cdot \ \langle f - u', u - v\rangle,\ \forall v \in \mathcal{V}, \tag{7.10}$$

which is exactly inequality (7.1). The function u obtained from this penalization process therefore solves the parabolic obstacle problem for the p–Laplacian and $u \in \mathcal{V}$, $u' \in \mathcal{V}^*$, hence, $u \in C([0, T]; H)$. Thus, the solution u has the same regularity property as that obtained by Rothe's method.

Due to the nature of the constraint set K in this particular example, it was easy to identify the appropriate penalty operator P (7.5). For the general parabolic variational inequality (3.10), the penalty operator P may be defined by

$$P = J(I - P_{D(\phi)}), \tag{7.11}$$

where J is the duality map and $P_{D(\phi)}$ denotes projection onto the closed convex set $D(\phi)$. This definition assures that P is monotone and hemicontinuous, properties that were essential in the argument above.

An aspect of the penalty method that we have not pursued is its effectiveness in treating problems of regularity. Regularity results for the penalized equations may be exploited to deduce more detailed information about the solution u of the variational inequality of interest. We refer to ([17]) and the references therein for specific results in this direction.

Finally, the penalty method may also be applied to variational inequalities of hyperbolic type. Mignot and Puel illustrate this approach in [30], and more general discussions of this class of variational inequalities may be found in [5], [22], [29].

References

1. R. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

2. G. Aronsson, L. C. Evans, and Y. Wu, Fast/slow diffusion and growing sandpiles, *J. Differential Equations*, **131** (1996), 304-335.
3. C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities: Applications to Free Boundary Problems*, Wiley, New York, 1984.
4. V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Editura Academiei Republicii Socialiste România, Bucharest, 1976.

   Translated from the Romanian.
5. H. Brézis, Problèmes unilatéraux, *J. Math. Pures Appl.* (9), **51** (1972), 1-168.
6. H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Publishing Co., Amsterdam, 1973.
7. H. Brézis, *Analyse fonctionnelle: théorie et applications*, Masson, Paris, 1983.
8. T. Cazenave and A. Haraux, *An Introduction to Semilinear Evolution Equations*, Oxford University Press, New York, 1998.
9. M. Chipot, Variational Inequalities and Flow in Porous Media, *vol. 52 of Applied Math. Sciences, Springer*, New York, 1984.
10. P. Clément, M. Garcia-Huidobro, R. Manásevich, and K. Schmitt, Mountain pass type solutions for quasilinear elliptic equations, *Calc. Variations and PDE*, **11** (2000), 33-62.
11. E. DiBenedetto, *Degenerate Parabolic Equations*, Springer, Berlin, 1993.
12. N. Dinculeanu, *Vector Measures*, Pergamon Press, Oxford, 1967.
13. T. Donaldson, Nonlinear elliptic boundary value problems in Orlicz-Sobolev spaces, *J. Differential Equations*, **10** (1971), 507-528.
14. G. Duvaut and J. L. Lions, *Les inéquations en mécanique et en physique*, Dunod, Paris, 1972.
15. I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, North Holland, American Elsevier, New York, 1976.
16. L. C. Evans, M. Feldman, and F. Gariepy, Fast/slow diffusion and collapsing sandpiles, *J. Differential Equations*, **137** (1997), 166-209.
17. A. Friedman, *Variational Principles and Free Boundary Value Problems*, Wiley-Interscience, New York, 1983.
18. M. García-Huidobro, V. Le, R. Manásevich, and K. Schmitt, On principal eigenvalues for quasilinear elliptic differential operators: an Orlicz-Sobolev space setting, *Nonl. Differential Equations and Applications*, **6** (1999), 207-225.
19. J. Gossez, Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) growing coefficients, *Trans. Amer. Math. Soc.* **190** (1974), 163-205.
20. ———, A strongly nonlinear elliptic problem in Orlicz-Sobolev spaces, *Proc. Symp. Pure Mathematics, Amer. Math. Soc.* **45** (1986), 455-462.

21. E. Hille and R. Phillips, Functional Analysis and Semigroups, *AMS Colloquium Publ.* vol. 31. 3rd ed., Providence, 1968.

22. J. Kačur, *Method of Rothe in Evolution Equations*, Teubner Verlagsgesellschaft, Leipzig, 1985.

23. N. Kikuchi and J. Oden, *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1988.

24. D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities*, *Acad. Press*, New York, 1980.

25. M. A. Krasnosel'skii and Y. B. Rutickii, *Convex Functions and Orlicz Spaces*, Noordhoff, Groningen, 1961.

26. A. Kufner, O. John, and S. Fučík, *Function Spaces*, Noordhoff, Leyden, 1977.

27. V. Le and K. Schmitt, Quasilinear elliptic equations and inequalities with rapidly growing coefficients, *J. London Math. Soc.* **62** (2000), 852-872.

28. V. K. Le and K. Schmitt, *Global Bifurcation in Variational Inequalities: Applications to Obstacle and Unilateral Problems*, Springer-Verlag, New York, 1997.

29. J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969.

30. F. Mignot and J.-P. Puel, Inéquations variationelles et quasivariotionelles hyperboliques du premier ordre, *J. Math. Pures Appl.* **55** (1976), 353-378.

31. H. Nagase, On an application of Rothe's method to nonlinear parabolic variational inequalities, *Funkcial. Ekvac.* **32** (1989), 273-299.

32. D. Pascali and S. Sburlan, *Nonlinear Mappings of Monotone Type*, Sijthoff and Noordhoff, Bucharest, 1978.

33. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York, 1983.

34. M. Rao and Z. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, New York, 1985.

35. T. R. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1970.

36. J. Rodrigues and L. Santos, A parabolic quasi-variational inequality arising in a superconductivity model, *Ann. Scuola Norm. Pisa Cl. Sci.* **29** (2000), 153-169.

37. J. F. Rodrigues, *Obstacle Problems in Mathematical Physics*, North-Holland, Amsterdam, 1987.

38. R. E. Showalter, *Hilbert Space Methods for Partial Differential Equations*, Pitman, London, 1977.

39. ———, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, American Math. Soc., Providence, 1997.

40. E. Zeidler, *Nonlinear Functional Analysis and its Applications, Vol.3: Variational Methods and Optimization*, Springer, Berlin, 1985.

41. A. Ženíšek, *Nonlinear Elliptic and Evolution Problems and their Finite Element Approximations*, Academic Press Inc., London, 1990.

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