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ON THE JENSEN'S INEQUALITY FOR CONVEX FUNCTIONS ON THE CO-ORDINATES IN A RECTANGLE FROM THE PLANE

M. Klaričić Bakula and J. Pečarić

Abstract. Several inequalities of Jensen's type for functions convex on the co-ordinates are given. Obtained results generalize the coresponding results of S. S. Dragomir given in [2].

1. Introduction

Let $I = [a, b]\,,\ a < b$, be an interval in $\mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ a convex function. The following double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known in the literature as Hadamard's inequality for convex functions (see for example [4, p. 10] or [4, p. 137]).

In paper [2] Dragomir considered an inequality of Hadamard's type for convex functions on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$. A function $f : [a, b]\times[c, d] \to \mathbb{R},\ [a, b]\times[c, d] \subset \mathbb{R}^2$ with $a < b$ and $c < d$, is called *convex on the co-ordinates* if the partial mappings $f_y : [a, b] \to \mathbb{R}$ defined as $f_y(u) := f(u, y)\,,$ and $f_x : [c, d] \to \mathbb{R}$ defined as $f_x(v) := f(x, v)\,,$ are convex where defined for all $y \in [c, d]$ and $x \in [a, b]$.

In [2] Dragomir has proved the following theorem:

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Theorem A. *Suppose that* $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$, *where* $a<b$ *and* $c<d$, *is convex on the co-ordinates on* $[a,b]\times[c,d]$. *Then one has the inequalities:*

$$\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\
&\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy\\
&\le \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right.\\
&\quad\left.\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right]\\
&\le \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}
\end{aligned}\tag{1.2}$$

The above inequalities are sharp.

The goal of this paper is to give a generalization of the above result and few other results from [2]. In Section 2 we give a generalization of Theorem A which involves weight functions and also nonlinear transformations of the base intervals. Using the obtained results, in Section 3 we establish some other interesting Jensen-type inequalities for convex functions on the co-ordinates. In Section 4 we introduce two functions which are closely connected with the integral Jensen's inequality. And in the end, in Section 5, we give some results related to those given in Section 2, but now for functions whose second partial derivatives are convex on the co-ordinates.

2. Jensen's Inequality

Let $(\Omega,\mathcal{A},\mu)$ be a measure space, let $g:\Omega\to I$, $I\subset\mathbb{R}$, be a function from $L^\infty(\mu)$ and let $p:\Omega\to\mathbb{R}$ be a nonnegative function from $L^1(\mu)$ such that $\int_\Omega pd\mu\neq 0$. Then for any convex function $\varphi:I\to\mathbb{R}$ inequality

$$\varphi\left(\frac{1}{\int_\Omega pd\mu}\int_\Omega pgd\mu\right)\le\frac{1}{\int_\Omega pd\mu}\int_\Omega p\varphi(g)\,d\mu\tag{2.1}$$

holds. This inequality is a variant of the well known integral Jensen's inequality (see [4, p. 45] or [3, p. 10]). Of course, if set I is bounded, then function g need only to be measurable. If μ is the standard Lebesgue measure and $\Omega = [a,b]$, then from (2.1) we obtain

$$\varphi\left(\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\,g(x)\,dx\right) \le \frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)\,\varphi(g(x))\,dx. \tag{2.2}$$

Discrete version of Jensen's inequality (see [3, p. 6] or [4, p. 43]) can be obtained from (2.1) if we choose $\Omega = \{1,2,...,n\},\ g(i) = x_i,\ p(i) = 1$ and $\mu(\{i\}) = p_i$ for $i = 1,2,...,n$. In that case (2.1) becomes

$$\varphi\left(\frac{1}{\sum_{k=1}^{n} p_k}\sum_{k=1}^{n} p_k x_k\right) \le \frac{1}{\sum_{k=1}^{n} p_k}\sum_{k=1}^{n} p_k\varphi(x_k). \tag{2.3}$$

If $I = [c,d]$, where $-\infty < c < d < +\infty$, and function φ is continuous, then the converse of the integral Jensen's inequality states

$$\frac{1}{\int_\Omega p\,d\mu}\int_\Omega p\varphi(g)\,d\mu \le \frac{d-\overline{g}}{d-c}\varphi(c) + \frac{\overline{g}-c}{d-c}\varphi(d), \tag{2.4}$$

where $\overline{g} = \frac{1}{\int_\Omega pd\mu}\int_\Omega pgd\mu$ (see [4, p. 98] or [3, p. 9]).

Before we use (2.1) to obtain a generalization of Theorem A, we introduce some notation. Throughout the rest of the paper we assume that:

(i) $(\Omega_1, \mathcal{A}, \mu)$ and $(\Omega_2, \mathcal{B}, \nu)$ are measure spaces;

(ii) $p : \Omega_1 \to \mathbb{R}$, $p \in L^1(\mu)$, and $w : \Omega_2 \to \mathbb{R}$, $w \in L^1(\nu)$, are nonnegative functions such that $\int_{\Omega_1} pd\mu \neq 0$ and $\int_{\Omega_2} wd\nu \neq 0$;

(iii) $g : \Omega_1 \to I,\ g \in L^\infty(\mu)$, and $h : \Omega_2 \to J,\ h \in L^\infty(\nu)$, $I, J \subset \mathbb{R}$;

(iv) $\varphi : I \times J \to \mathbb{R}$ is convex on the coordinates on $I \times J$.

Theorem 1. *Let φ, g, h, p and w be as the above. Then we have the following inequalities:*

$$\begin{aligned}\varphi\left(\overline{g},\overline{h}\right) &\le \frac{1}{2}\left\{\frac{1}{P}\int_{\Omega_1} p\varphi\left(g,\overline{h}\right)d\mu + \frac{1}{W}\int_{\Omega_2} w\varphi\left(\overline{g},h\right)d\nu\right\} \\ &\le \frac{1}{PW}\int_{\Omega_1}\int_{\Omega_2} pw\varphi(g,h)\,d\mu d\nu,\end{aligned} \tag{2.5}$$

where

$$P=\int_{\Omega_1} pd\mu, \qquad W=\int_{\Omega_2} wd\nu$$
$$\overline{g}=\frac{1}{P}\int_{\Omega_1} pgd\mu, \qquad \overline{h}=\frac{1}{W}\int_{\Omega_2} whd\nu.$$

The above inequalities are sharp.

Proof. One-dimensional Jensen's inequality (2.1) gives us

$$\varphi\left(g,\overline{h}\right)\le\frac{1}{W}\int_{\Omega_2} w\varphi\left(g,h\right)d\nu, \tag{2.6}$$

$$\varphi\left(\overline{g},h\right)\le\frac{1}{P}\int_{\Omega_1} p\varphi\left(g,h\right)d\mu. \tag{2.7}$$

Multiplying (2.6) and (2.7) respectively by p and w and integrating over corresponding sets we obtain

$$\begin{aligned}
\frac{1}{P}\int_{\Omega_1} p\varphi\left(g,\overline{h}\right)d\mu &\le \frac{1}{PW}\int_{\Omega_1}\int_{\Omega_2} pw\varphi\left(g,h\right)d\mu d\nu,\\
\frac{1}{W}\int_{\Omega_2} w\varphi\left(\overline{g},h\right)d\nu &\le \frac{1}{PW}\int_{\Omega_1}\int_{\Omega_2} pw\varphi\left(g,h\right)d\mu d\nu,
\end{aligned}$$

from which we can get the right hand inequality in (2.5).

The left hand inequality in (2.5) is a simple consequence of Jensen's one-dimensional inequality (2.1), that is

$$\begin{aligned}
\varphi\left(\overline{g},\overline{h}\right) &\le \frac{1}{P}\int_{\Omega_1} p\varphi\left(g,\overline{h}\right)d\mu,\\
\varphi\left(\overline{g},\overline{h}\right) &\le \frac{1}{W}\int_{\Omega_2} w\varphi\left(\overline{g},h\right)d\nu.
\end{aligned}$$

If in (2.5) we choose $g(x)=h(x)=x$ for all $x\in\Omega_1=\Omega_2$, $p(x)=w(x)=1$ for all $x\in\Omega_1=\Omega_2$ and $\varphi(x,y)=xy$, then (2.5) becomes an equality, which shows that inequalities (2.5) are sharp.

This completes the proof. ■

Remark 1. If μ and ν are Lebesgue measures, $\Omega_1=[a,b]$, $\Omega_2=[c,d]$, $g(x)=x$ for all $x\in[a,b]$, $h(x)=x$ for all $x\in[c,d]$, $p(x)=1$ for all $x\in[a,b]$ and $w(x)=1$ for all $x\in[c,d]$, then Theorem 1 gives us the first and the second inequality in Dragomir's result (1.2).

Theorem 2. *Let φ be convex on the co-ordinates on $I \times J \subseteq \mathbb{R}^2$. If $\boldsymbol{x}$ is an $n-$tuple in I, $\boldsymbol{y}$ an $m-$tuple in J, $\boldsymbol{p}$ a nonnegative $n-$tuple such that $P_n = \sum_{i=1}^n p_i \neq 0$ and $\boldsymbol{w}$ a nonnegative $m-$tuple such that $W_m = \sum_{j=1}^m w_j \neq 0$, then*

$$\begin{aligned}
&\varphi\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i, \frac{1}{W_m}\sum_{j=1}^m w_j y_j\right) \\
&\leq \frac{1}{2}\left\{\frac{1}{P_n}\sum_{i=1}^n p_i \varphi\left(x_i, \overline{y}\right) + \frac{1}{W_m}\sum_{j=1}^m w_j \varphi\left(\overline{x}, y_j\right)\right\} \\
&\leq \frac{1}{P_n W_m}\sum_{i=1}^n \sum_{j=1}^m p_i w_j \varphi\left(x_i, y_j\right),
\end{aligned} \tag{2.8}$$

where

$$\overline{x} = \frac{1}{P_n}\sum_{i=1}^n p_i x_i, \qquad \overline{y} = \frac{1}{W_m}\sum_{j=1}^m w_j y_j.$$

Proof. Directly from Theorem 1. We simply choose $\Omega_1 = \{1, 2, ..., n\}$, $g(i) = x_i$, $p(i) = 1$, $\mu(\{i\}) = p_i$ for $i = 1, 2, ..., n$ and $\Omega_2 = \{1, 2, ..., m\}$, $h(j) = y_j$, $w(j) = 1$ and $\nu(\{j\}) = w_j$ for $j = 1, 2, ..., m$. ∎

Assume now that $I = [m, M]$ and $J = [n, N]$, where $-\infty < m < M < \infty$ and $-\infty < n < N < \infty$. The following result is valid.

Theorem 3. *Let functions p and w be as the above and let functions $g : \Omega_1 \to I$, $h : \Omega_2 \to J$ be measurable. If function $\varphi : I \times J \to \mathbb{R}$ is continuous and convex on the co-ordinates on $I \times J$, then we have the following inequalities:*

$$\begin{aligned}
&\frac{1}{PW}\int_{\Omega_1}\int_{\Omega_2} pw\varphi(g,h)\, d\mu d\nu \\
&\leq \frac{1}{2}\left\{\frac{N-\overline{h}}{N-n}\frac{1}{P}\int_{\Omega_1} p\varphi(g,n)\, d\mu + \frac{\overline{h}-n}{N-n}\frac{1}{P}\int_{\Omega_1} p\varphi(g,N)\, d\mu\right. \\
&\left.+\frac{M-\overline{g}}{M-m}\frac{1}{W}\int_{\Omega_2} w\varphi(m,h)\, d\nu + \frac{\overline{g}-m}{M-m}\frac{1}{W}\int_{\Omega_2} w\varphi(M,h)\, d\nu\right\} \\
&\leq \frac{M-\overline{g}}{M-m}\frac{N-\overline{h}}{N-n}\varphi(m,n) + \frac{\overline{g}-m}{M-m}\frac{N-\overline{h}}{N-n}\varphi(M,n) \\
&+\frac{M-\overline{g}}{M-m}\frac{\overline{h}-n}{N-n}\varphi(m,N) + \frac{\overline{g}-m}{M-m}\frac{\overline{h}-n}{N-n}\varphi(M,N).
\end{aligned} \tag{2.9}$$

These inequalities are sharp.

Proof. We can write $g(x)$ and $h(y)$ as

$$\begin{aligned} g(x) &= \frac{M-g(x)}{M-m}m + \frac{g(x)-m}{M-m}M, \\ h(y) &= \frac{N-h(y)}{N-n}n + \frac{h(y)-n}{N-n}N, \end{aligned}$$

so using the convexity of the function φ on the co-ordinates we get

$$\begin{aligned} \varphi(g(x), h(y)) &= \varphi\left(\frac{M-g(x)}{M-m}m + \frac{g(x)-m}{M-m}M, h(y)\right) \\ &\leq \frac{M-g(x)}{M-m}\varphi(m, h(y)) + \frac{g(x)-m}{M-m}\varphi(M, h(y)), \end{aligned}$$

$$\begin{aligned} \varphi(g(x), h(y)) &= \varphi\left(g(x), \frac{N-h(y)}{N-n}n + \frac{h(y)-n}{N-n}N\right) \\ &\leq \frac{N-h(y)}{N-n}\varphi(g(x), n) + \frac{h(y)-n}{N-n}\varphi(g(x), N). \end{aligned}$$

A simple calculation shows us that

$$\begin{aligned} \varphi(g(x), h(y)) \leq \frac{1}{2}\Bigg[&\frac{M-g(x)}{M-m}\varphi(m, h(y)) + \frac{g(x)-m}{M-m}\varphi(M, h(y)) \\ &+ \frac{N-h(y)}{N-n}\varphi(g(x), n) + \frac{h(y)-n}{N-n}\varphi(g(x), N)\Bigg]. \end{aligned} \tag{2.10}$$

Multiplying (2.10) by pw, dividing by PW and integrating over $\Omega_1 \times \Omega_2$ we obtain

$$\begin{aligned} &\frac{1}{PW}\int_{\Omega_1}\int_{\Omega_2} pw\varphi(g, h)\, d\mu d\nu \\ &\leq \frac{1}{2}\Bigg[\frac{N-\overline{h}}{N-n}\frac{1}{P}\int_{\Omega_1} p\varphi(g, n)\, d\mu + \frac{\overline{h}-n}{N-n}\frac{1}{P}\int_{\Omega_1} p\varphi(g, N)\, d\mu \\ &\quad \frac{M-\overline{g}}{M-m}\frac{1}{W}\int_{\Omega_2} w\varphi(m, h)\, d\nu + \frac{\overline{g}-m}{M-m}\frac{1}{W}\int_{\Omega_2} w\varphi(M, h)\, d\nu\Bigg], \end{aligned}$$

which is the left hand side of (2.9).

Now we can use again the convexity of the function φ on the co-ordinates, so we get for instance

$$\begin{aligned}
&\frac{M-\overline{g}}{M-m}\frac{1}{W}\int_{\Omega_2} w\varphi\left(m,h\right)d\nu \\
&= \frac{M-\overline{g}}{M-m}\frac{1}{W}\int_{\Omega_2} w\varphi\left(m,\frac{N-h}{N-n}n+\frac{h-n}{N-n}N\right)d\nu \\
&\leq \frac{M-\overline{g}}{M-m}\left[\frac{N-\overline{h}}{N-n}\varphi\left(m,n\right)+\frac{\overline{h}-n}{N-n}\varphi\left(m,N\right)\right].
\end{aligned}$$

If we sum all inequalities obtained in this way, a simple calculation gives us the right hand side of (2.9).

The sharpness can be proved similarly as in Theorem 1. ■

Remark 2. If μ and ν are Lebesgue measures, $\Omega_1 = [a,b]$, $\Omega_2 = [c,d]$, $g\left(x\right) = x$ for all $x \in [a,b]$, $h\left(x\right) = x$ for all $x \in [c,d]$, $p\left(x\right) = 1$ for all $x \in [a,b]$ and $w\left(x\right) = 1$ for all $x \in [c,d]$, then Theorem 3 gives us the third and the fourth inequality in Dragomir's result (1.2).

Theorem 4. *Let φ be convex on the co-ordinates on $I \times J = [m,M] \times [n,N] \subseteq \mathbb{R}^2$. If $\boldsymbol{x}$ is a $k-$tuple in I, $\boldsymbol{y}$ an $l-$tuple in J, $\boldsymbol{p}$ a nonnegative $k-$tuple such that $\sum_{i=1}^{k} p_i \neq 0$ and $\boldsymbol{w}$ a nonnegative $l-$tuple such that $\sum_{j=1}^{l} w_j \neq 0$, then*

$$\begin{aligned}
&\frac{1}{P_k W_l}\sum_{i=1}^{k}\sum_{j=1}^{l} p_i w_j \varphi\left(x_i,y_j\right) \\
&\leq \frac{1}{2}\left[\frac{N-\overline{y}}{N-n}\frac{1}{P_k}\sum_{i=1}^{k} p_i\varphi\left(x_i,n\right)+\frac{\overline{y}-n}{N-n}\frac{1}{P_k}\sum_{i=1}^{k} p_i\varphi\left(x_i,N\right)\right. \\
&\left.\quad+\frac{M-\overline{x}}{M-M}\frac{1}{W_l}\sum_{j=1}^{l} w_j\varphi\left(m,y_j\right)+\frac{\overline{x}-m}{M-m}\frac{1}{W_l}\sum_{j=1}^{l} w_j\varphi\left(M,y_j\right)\right] \\
&\leq \frac{M-\overline{x}}{M-m}\frac{N-\overline{y}}{N-n}\varphi\left(m,n\right)+\frac{\overline{x}-m}{M-m}\frac{N-\overline{y}}{N-n}\varphi\left(M,n\right) \\
&\quad+\frac{M-\overline{x}}{M-m}\frac{\overline{y}-n}{N-n}\varphi\left(m,N\right)+\frac{\overline{x}-m}{M-m}\frac{\overline{y}-n}{N-n}\varphi\left(M,N\right).
\end{aligned} \tag{2.11}$$

Proof. Similarly as in Theorem 2. ■

To prove our next result, we shall need the following theorem (see [4, p. 101] or [3, p. 10]).

Theorem B. *Let $\varphi : [m, M] \to \mathbb{R}$ be a continuous convex function on $[m, M]$, $g : \Omega \to [m, M]$ a measurable function and $p : \Omega \to \mathbb{R}$, $p \in L^1(\mu)$, a nonnegative function such that $P = \int_\Omega p d\mu \neq 0$. Let $F : T^2 \to \mathbb{R}$ be a nondecreasing function on its first coordinate, where $\varphi([m, M]) \subset T$. Then the inequality*

$$\begin{aligned} & F\left(\frac{1}{P}\int_\Omega p\varphi(g)\,d\mu, \varphi\left(\frac{1}{P}\int_\Omega pgd\mu\right)\right) \\ \leq & \max_{x\in[m,M]} F\left(\frac{M-x}{M-m}\varphi(m) + \frac{x-m}{M-m}\varphi(M), \varphi(x)\right) \end{aligned}$$

holds. The right side of the above inequality is nondecreasing function of M and nonincreasing function of m.

Theorem 5. *Let φ, g, h, p and w be defined as in Theorem 3, and let $F : T^2 \to \mathbb{R}$ be a function nondecreasing in its first variable (T is an interval in $\mathbb{R}$ such that $\varphi(I \times J) \subseteq T$). Then*

$$\begin{aligned} & F\left(\frac{1}{PW}\int_{\Omega_1}\int_{\Omega_2} pw\varphi(g,h)\,d\mu d\nu, \varphi\left(\frac{1}{P}\int_{\Omega_1} pgd\mu, \frac{1}{W}\int_{\Omega_2} whd\nu\right)\right) \\ \leq & \max_{(x,y)\in I\times J} F\left(\frac{M-x}{M-m}\frac{N-y}{N-n}\varphi(m,n) + \frac{x-m}{M-m}\frac{N-y}{N-n}\varphi(M,n)\right. \\ & \left. + \frac{M-x}{M-m}\frac{y-n}{N-n}\varphi(m,N) + \frac{x-m}{M-m}\frac{y-n}{N-n}\varphi(M,N), \varphi(x,y)\right), \end{aligned} \tag{2.12}$$

The right hand side of (2.12) is nondecreasing function of M and N and nonincreasing function of m and n.

Proof. By Theorem 3 and the monotonicity of F in the first variable we have

$$\begin{aligned} & F\left(\frac{1}{PW}\int_{\Omega_1}\int_{\Omega_2} pw\varphi(g,h)\,d\mu d\nu, \varphi\left(\frac{1}{P}\int_{\Omega_1} pgd\mu, \frac{1}{W}\int_{\Omega_2} whd\nu\right)\right) \\ \leq & F\left(\frac{M-\overline{g}}{M-m}\frac{N-\overline{h}}{N-n}\varphi(m,n) + \frac{\overline{g}-m}{M-m}\frac{N-\overline{h}}{N-n}\varphi(M,n)\right. \\ & \left. + \frac{M-\overline{g}}{M-m}\frac{\overline{h}-n}{N-n}\varphi(m,N) + \frac{\overline{g}-m}{M-m}\frac{\overline{h}-n}{N-n}\varphi(M,N), \varphi\left(\overline{g},\overline{h}\right)\right) \\ \leq & \max_{(x,y)\in I\times J} F\left(\frac{M-x}{M-m}\frac{N-y}{N-n}\varphi(m,n) + \frac{x-m}{M-m}\frac{N-y}{N-n}\varphi(M,n)\right. \\ & \left. + \frac{M-x}{M-m}\frac{y-n}{N-n}\varphi(m,N) + \frac{x-m}{M-m}\frac{y-n}{N-n}\varphi(M,N), \varphi(x,y)\right). \end{aligned}$$

Let $x, M' \in I,\ m < M' < M$, and $y, N' \in J,\ n < N' < N$. Since the function φ is convex on the co-ordinates and $\frac{N-y}{N-n} \geq 0$ for any $y \in [n, N]$, we can deduce that for any $y \in [n, N]$ function $\psi_y : [m, M] \to \mathbb{R}$ defined as

$$\psi_y(x) = \frac{N-y}{N-n}\varphi(x, n)$$

is convex on $[m, M]$. In that case we know from Theorem B that for any $x \in I$

$$\begin{aligned}
&\frac{M'-x}{M'-m}\psi_y(m) + \frac{x-m}{M'-m}\psi_y\left(M'\right) \\
&\leq \frac{M-x}{M-m}\psi_y(m) + \frac{x-m}{M-m}\psi_y(M),
\end{aligned}$$

i.e., for any $x \in I$ and $y \in J$

$$\begin{aligned}
&\frac{M'-x}{M'-m}\frac{N-y}{N-n}\varphi(m, n) + \frac{x-m}{M'-m}\frac{N-y}{N-n}\varphi\left(M', n\right) \\
&\leq \frac{M-x}{M-m}\frac{N-y}{N-n}\varphi(m, n) + \frac{x-m}{M-m}\frac{N-y}{N-n}\varphi(M, n).
\end{aligned} \tag{2.13}$$

Analogously, for any $y \in [n, N]$ function $\eta_y : [m, M] \to \mathbb{R}$ defined as

$$\eta_y(x) = \frac{y-n}{N-n}\varphi(x, N)$$

is convex on $[m, M]$. Again, for any $x \in I$ and $y \in J$ we obtain

$$\begin{aligned}
&\frac{M'-x}{M'-m}\frac{y-n}{N-n}\varphi(m, n) + \frac{x-m}{M'-m}\frac{y-n}{N-n}\varphi\left(M', n\right) \\
&\leq \frac{M-x}{M-m}\frac{y-n}{N-n}\varphi(m, n) + \frac{x-m}{M-m}\frac{y-n}{N-n}\varphi(M, n).
\end{aligned} \tag{2.14}$$

For the simplicity, let us denote

$$\begin{aligned}
&D\{(x, y), (m, M), (n, N); \varphi\} \\
&= F\Bigg(\frac{M-x}{M-m}\frac{N-y}{N-n}\varphi(m, n) + \frac{x-m}{M-m}\frac{N-y}{N-n}\varphi(M, n) \\
&\quad + \frac{M-x}{M-m}\frac{y-n}{N-n}\varphi(m, N) + \frac{x-m}{M-m}\frac{y-n}{N-n}\varphi(M, N), \\
&\quad \varphi(g(x), h(y))\Bigg).
\end{aligned}$$

We know that the function F is nondecreasing in its first variable, so using that fact and the inequalities (2.13), (2.14) we can deduce

$$\begin{aligned} & D\left\{(x,y),\left(m,M'\right),(n,N);\varphi\right\} \\ \leq\ & D\left\{(x,y),(m,M),(n,N);\varphi\right\}. \end{aligned}$$

In a similar way we can obtain

$$\begin{aligned} & D\left\{(x,y),(m,M),\left(n,N'\right);\varphi\right\} \\ \leq\ & D\left\{(x,y),(m,M),(n,N);\varphi\right\}. \end{aligned}$$

It can be easily seen that if we change first M into M', and than N into N', we can obtain

$$\begin{aligned} & D\left\{(x,y),(m,M'),(n,N');\varphi\right\} \\ \leq\ & D\left\{(x,y),(m,M'),(n,N);\varphi\right\} \\ \leq\ & D\left\{(x,y),(m,M),(n,N);\varphi\right\}. \end{aligned} \tag{2.15}$$

Since $m < M' < M$ and $n < N' < N$, we have $[m,M'] \times [n,N'] \subset [m,M] \times [n,N]$, so from the inequality (2.15) we obtain

$$\begin{aligned} & \max_{(x,y)\in[m,M]\times[n,N]} D\left\{(x,y),(m,M),(n,N);\varphi\right\} \\ \geq\ & \max_{(x,y)\in[m,M]\times[n,N]} D\left\{(x,y),\left(m,M'\right),\left(n,N'\right);\varphi\right\} \\ \geq\ & \max_{(x,y)\in[m,M']\times[n,N']} D\left\{(x,y),\left(m,M'\right),\left(n,N'\right);\varphi\right\}, \end{aligned}$$

from which we can deduce that the right side of (2.12) is nondecreasing function of M and N.

Similarly, for $m < m' < M$ and $n < n' < N$ we have

$$\begin{aligned} & \max_{(x,y)\in[m,M]\times[n,N]} D\left\{(x,y),(m,M),(n,N);\varphi\right\} \\ \geq\ & \max_{(x,y)\in[m',M]\times[n',N]} D\left\{(x,y),\left(m',M\right),\left(n',N\right);\varphi\right\}, \end{aligned}$$

which shows that the right side of (2.12) is nonincreasing function of m and n. This completes the proof. ∎

3. Giaccardi's and Petrović's Inequalities

In this section we will use the conversion of Jensen's inequality in its discrete case in order to obtain a generalization of the Giaccardi's inequality (see for example [3, p. 11]).

Theorem 6. *Let $\boldsymbol{p}$ be a nonnegative n-tuple, $\boldsymbol{w}$ an nonnegative m-tuple, $\boldsymbol{x} \in I^n$, $\boldsymbol{y} \in J^m$, $x_0, \widehat{x} = \sum_{i=1}^{n} p_i x_i \in I$ and $y_0, \widehat{y} = \sum_{j=1}^{m} w_j y_j, \in J$ such that*

$$\begin{aligned}(x_i - x_0)(\widehat{x} - x_i) &\geq 0 \quad (i = 1, ..., n)\\ \widehat{x} &\neq x_0\\ (y_j - y_0)(\widehat{y} - y_j) &\geq 0 \quad (j = 1, ..., m)\\ \widehat{y} &\neq y_0.\end{aligned} \tag{3.1}$$

If function φ is convex on the co-ordinates on $I \times J$, then

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \varphi(x_i, y_j)\\ &\leq AC(P_n - 1)(W_m - 1)\varphi(x_0, y_0) + BC(W_m - 1)\varphi(\widehat{x}, y_0)\\ &\quad + AD(P_n - 1)\varphi(x_0, \widehat{y}) + BD\varphi(\widehat{x}, \widehat{y})\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}A &= \frac{\widehat{x}}{\widehat{x} - x_0}, \qquad B = \frac{\sum_{i=1}^{n} p_i (x_i - x_0)}{\widehat{x} - x_0},\\ C &= \frac{\widehat{y}}{\widehat{y} - y_0}, \qquad D = \frac{\sum_{j=1}^{m} w_j (y_j - y_0)}{\widehat{y} - y_0}.\end{aligned}$$

Proof. Since we know that $x_0, \widehat{x} \in I$ and $y_0, \widehat{y} \in J$ holds, we can consider restriction of the function φ on $[x_0, \widehat{x}] \times [y_0, \widehat{y}]$ (for $x_0 < \widehat{x}$ and $y_0 < \widehat{y}$) or $[\widehat{x}, x_0] \times [\widehat{y}, y_0]$ (for $\widehat{x} < x_0$ and $\widehat{y} < y_0$) or similarly on $[x_0, \widehat{x}] \times [\widehat{y}, y_0]$ or $[\widehat{x}, x_0] \times [y_0, \widehat{y}]$. Let us consider the first case. The conditions (3.1) provide us

$$\begin{aligned}x_i &\in [x_0, \widehat{x}] \quad (i = 1, 2, ..., n),\\ y_j &\in [y_0, \widehat{y}] \quad (j = 1, 2, ..., m),\end{aligned}$$

so in case

$$\begin{aligned} m &= x_0, \quad M = \widehat{x}, \\ n &= y_0, \quad N = \widehat{y}, \end{aligned}$$

all conditions of Theorem 4 are satisfied. From the inequality (2.11), after multiplying by $P_n W_m$, we get

$$\begin{aligned} &\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \varphi\left(x_i, y_j\right) \\ \leq\ & \frac{1}{2}\left(\frac{W_m\widehat{y}-\widehat{y}}{\widehat{y}-y_0}\sum_{i=1}^{n} p_i\varphi\left(x_i, y_0\right) + \frac{\widehat{y}-W_m y_0}{\widehat{y}-y_0}\sum_{i=1}^{n} p_i\varphi\left(x_i, \widehat{y}\right)\right. \\ &\left. + \frac{P_n\widehat{x}-\widehat{x}}{\widehat{x}-x_0}\sum_{j=1}^{m} w_j\varphi\left(x_0, y_j\right) + \frac{\widehat{x}-P_n x_0}{\widehat{x}-x_0}\sum_{j=1}^{m} w_j\varphi\left(\widehat{x}, y_j\right)\right) \\ \leq\ & \frac{P_n\widehat{x}-\widehat{x}}{\widehat{x}-x_0}\frac{W_m\widehat{y}-\widehat{y}}{\widehat{y}-y_0}\varphi\left(x_0, y_0\right) + \frac{\widehat{x}-P_n x_0}{\widehat{x}-x_0}\frac{W_m\widehat{y}-\widehat{y}}{\widehat{y}-y_0}\varphi\left(\widehat{x}, y_0\right) \\ &+ \frac{P_n\widehat{x}-\widehat{x}}{\widehat{x}-x_0}\frac{\widehat{y}-W_m y_0}{\widehat{y}-y_0}\varphi\left(x_0, \widehat{y}\right) + \frac{\widehat{x}-P_n x_0}{\widehat{x}-x_0}\frac{\widehat{y}-W_m y_0}{\widehat{y}-y_0}\varphi\left(\widehat{x}, \widehat{y}\right). \end{aligned}$$

For A, B, C, D defined as the above, a simple calculus gives us

$$\begin{aligned} &\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \varphi\left(x_i, y_j\right) \\ \leq\ & \frac{1}{2}\left[\left(W_m - 1\right) C\sum_{i=1}^{n} p_i\varphi\left(x_i, y_0\right) + D\sum_{i=1}^{n} p_i\varphi\left(x_i, \widehat{y}\right)\right. \\ &\left. + \left(P_n - 1\right) A\sum_{j=1}^{m} w_j\varphi\left(x_0, y_j\right) + B\sum_{j=1}^{m} w_j\varphi\left(\widehat{x}, y_j\right)\right] \\ \leq\ & AC\left(P_n - 1\right)\left(W_m - 1\right)\varphi\left(x_0, y_0\right) + BC\left(W_m - 1\right)\varphi\left(\widehat{x}, y_0\right) \\ &+ AD\left(P_n - 1\right)\varphi\left(x_0, \widehat{y}\right) + BD\varphi\left(\widehat{x}, \widehat{y}\right) \end{aligned}$$

which is the desired inequality (3.2). In the second case $\widehat{x} < x_0$ and $\widehat{y} < y_0$ we define

$$\begin{aligned} m &= \widehat{x}, \quad M = x_0, \\ n &= \widehat{y}, \quad N = y_0, \end{aligned}$$

and the rest of the proof is similar. ∎

From the Giaccardi's type inequality (3.2) we can obtain a Petrović's type inequality if we choose

$$x_0 = 0, \quad y_0 = 0,$$

which implies

$$A = B = C = D = 1.$$

In this case, the conditions (3.1) become

$$0 \le x_i \le \widehat{x}, \quad i = 1, ..., n,$$
$$0 \le y_j \le \widehat{y}, \quad j = 1, ..., m,$$

and from (3.2) we obtain

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \varphi(x_i, y_j) \\
\le\ & \frac{1}{2}\Bigg[(W_m - 1)\sum_{i=1}^{n} p_i \varphi(x_i, 0) + \sum_{i=1}^{n} p_i \varphi(x_i, \widehat{y}) \\
& (P_n - 1)\sum_{j=1}^{m} w_j \varphi(0, y_j) + \sum_{j=1}^{m} w_j \varphi(\widehat{x}, y_j)\Bigg] \\
\le\ & (P_n - 1)(W_m - 1)\varphi(0,0) + (W_m - 1)\varphi(\widehat{x}, 0) + (P_n - 1)\varphi(0, \widehat{y}) + \varphi(\widehat{x}, \widehat{y}).
\end{aligned}$$

Remark 3. The above results show that a number of one-dimensional results can be similarly extended on bidimensional functions.

4. Some Functions in Connection to Jensen's Inequality

For a function φ defined as in Section 2, we can define a new function $H : [0,1]^2 \to \mathbb{R}$ as

$$H(t,s) = \frac{1}{PW}\int_{\Omega_1}\int_{\Omega_2} pw\varphi\left(tg + (1-t)\overline{g}, sh + (1-s)\overline{h}\right) d\mu d\nu.$$

This function, which is strongly connected to the integral Jensen's inequality (2.2) has the following properties:

Theorem 7. *Let φ, p, w, g, h be as in Theorem 1. Then:*

(i) *The function* H *defined as the above is convex on the co-ordinates on* $[0,1]^2$*;*

(ii) *We have the bounds:*

$$\sup_{(t,s)\in[0,1]^2} H(t,s) = H(1,1) = \frac{1}{PW}\int_{\Omega_1}\int_{\Omega_2} pw\varphi(g,h)\,d\mu d\nu,$$
$$\inf_{(t,s)\in[0,1]^2} H(t,s) = H(0,0) = \varphi\left(\overline{g},\overline{h}\right),$$

where $P, W, \overline{g}$ *and* $\overline{h}$ *are defined as in Theorem 1;*

(iii) *The function* H *is nondecreasing on the co-ordinates.*

Proof. (i) Fix $s\in[0,1]$. Then for all $\lambda,\overline{\lambda}\geq 0$ such that $\lambda+\overline{\lambda}=1$ and $t_1,t_2\in[0,1]$ we have

$$\begin{aligned}
&H\left(\lambda t_1+\overline{\lambda}t_2, s\right)\\
&= \frac{1}{PW}\int_{\Omega_1}\int_{\Omega_2} pw\varphi\left(\left(\lambda t_1+\overline{\lambda}t_2\right)g+\left[1-\left(\lambda t_1+\overline{\lambda}t_2\right)\right]\overline{g}, sh+(1-s)\overline{h}\right)d\mu d\nu\\
&= \frac{1}{PW}\int_{\Omega_1}\int_{\Omega_2} pw\\
&\quad\times\varphi\left(\lambda\left[t_1 g+(1-t_1)\overline{g}\right]+\overline{\lambda}\left[t_2 g+(1-t_2)\overline{g}\right], sh+(1-s)\overline{h}\right)d\mu d\nu\\
&\leq \frac{\lambda}{PW}\int_{\Omega_1}\int_{\Omega_2} pw\varphi\left(t_1 g+(1-t_1)\overline{g}, sh+(1-s)\overline{h}\right)d\mu d\nu\\
&\quad+\frac{\overline{\lambda}}{PW}\int_{\Omega_1}\int_{\Omega_2} pw\varphi\left(t_2 g+(1-t_2)\overline{g}, sh+(1-s)\overline{h}\right)d\mu d\nu\\
&= \lambda H(t_1,s)+\overline{\lambda}H(t_2,s).
\end{aligned}$$

In the same way we can prove that for $t\in[0,1]$ fixed, $\lambda,\overline{\lambda}\geq 0$ such that $\lambda+\overline{\lambda}=1$ and any $s_1,s_2\in[0,1]$

$$H\left(t,\lambda s_1+\overline{\lambda}s_2\right)=\lambda H(t,s_1)+\overline{\lambda}H(t,s_2),$$

which means that the function H is convex on the co-ordinates

(ii) Let $(t,s)\in[0,1]^2$. Using the integral Jensen's inequality (2.1) on the co-ordinates, we obtain

$$\begin{aligned}
H(t,s) &= \frac{1}{P}\int_{\Omega_1} p\left[\frac{1}{W}\int_{\Omega_2} w\varphi\left(tg+(1-t)\overline{g}, sh+(1-s)\overline{h}\right)d\nu\right]d\mu\\
&\geq \frac{1}{P}\int_{\Omega_1} p\varphi\left(tg+(1-t)\overline{g}, \frac{1}{W}\int_{\Omega_2} w\left[sh+(1-s)\overline{h}\right]d\nu\right)d\mu\\
&= \frac{1}{P}\int_{\Omega_1} p\varphi\left(tg+(1-t)\overline{g},\overline{h}\right)d\mu
\end{aligned}$$

$$\geq \varphi\left(\frac{1}{P}\int_{\Omega_1} p\left(gt+(1-t)\,\overline{g}\right)d\mu, \overline{h}\right)$$

$$= \varphi\left(\overline{g}, \overline{h}\right) = H\left(0,0\right),$$

which means that

$$\inf_{(t,s)\in[0,1]^2} H\left(t,s\right) = H\left(0,0\right).$$

On the other hand, since φ is convex on the co-ordinates, for all $(t,s) \in [0,1]^2$ we have

$$\begin{aligned}
H\left(t,s\right) \leq\ & \frac{1}{P}\int_{\Omega_1} p\left[\frac{s}{W}\int_{\Omega_2} w\varphi\left(tg+(1-t)\,\overline{g}, h\right)d\nu\right. \\
& \left. + \frac{1-s}{W}\int_{\Omega_2} w\varphi\left(tg+(1-t)\,\overline{g}, \overline{h}\right)d\nu\right] d\mu \\
\leq\ & \frac{s}{W}\int_{\Omega_2} w\left[\frac{t}{P}\int_{\Omega_1} p\varphi\left(g,h\right)d\mu + \frac{1-t}{P}\int_{\Omega_1} p\varphi\left(\overline{g},h\right)d\mu\right] d\nu \\
& + \frac{1-s}{W}\int_{\Omega_2} w\left[\frac{t}{P}\int_{\Omega_1} p\varphi\left(g,\overline{h}\right)d\mu + \frac{1-t}{P}\int_{\Omega_1} p\varphi\left(\overline{g},\overline{h}\right)d\mu\right] d\nu.
\end{aligned}$$

Jensen's inequality (2.1) gives us

$$\begin{aligned}
\varphi\left(\overline{g},h\right) &\leq \frac{1}{P}\int_{\Omega_1} p\varphi\left(g,h\right)d\mu, \\
\varphi\left(g,\overline{h}\right) &\leq \frac{1}{W}\int_{\Omega_2} w\varphi\left(g,h\right)d\nu,
\end{aligned}$$

thus, by integration, we obtain

$$\begin{aligned}
\frac{1}{W}\int_{\Omega_2} w\varphi\left(\overline{g},h\right)d\nu &\leq \frac{1}{PW}\int_{\Omega_1}\int_{\Omega_2} pw\varphi\left(g,h\right)d\mu d\nu, \\
\frac{1}{P}\int_{\Omega_1} p\varphi\left(g,\overline{h}\right)d\mu &\leq \frac{1}{PW}\int_{\Omega_1}\int_{\Omega_2} pw\varphi\left(g,h\right)d\mu d\nu.
\end{aligned}$$

From this, we can easily deduce that

$$\begin{aligned}
H\left(t,s\right) \leq\ & \left[st+s\left(1-t\right)+\left(1-s\right)t+\left(1-s\right)\left(1-t\right)\right] \\
& \times\frac{1}{PW}\int_{\Omega_1}\int_{\Omega_2} pw\varphi\left(g,h\right)d\mu d\nu \\
=\ & \frac{1}{PW}\int_{\Omega_1}\int_{\Omega_2} pw\varphi\left(g,h\right)d\mu d\nu = H\left(1,1\right),
\end{aligned}$$

and this gives us

$$\sup_{(t,s)\in[0,1]^2} H(t,s) = H(1,1).$$

(iii) By the Jensen's inequality (2.1) we have

$$\begin{aligned} H(t,s) &\geq \frac{1}{W}\int_{\Omega_2} w\varphi\left(\frac{1}{P}\int_{\Omega_1} p\left[tg+(1-t)\overline{g}\right]d\mu, sh+(1-s)\overline{h}\right)d\nu \\ &= \frac{1}{W}\int_{\Omega_2} w\varphi\left(\overline{g}, sh+(1-s)\overline{h}\right)d\nu = H(0,s), \end{aligned}$$

for all $(t,s)\in[0,1]^2$.

Now let $0<t_1<t_2\leq 1$. Since H is convex on the co-ordinates and the above inequality holds, for all $s\in[0,1]$ we have

$$\frac{H(t_2,s)-H(t_1,s)}{t_2-t_1} \geq \frac{H(t_1,s)-H(0,s)}{t_1} \geq 0,$$

which implicates

$$H(t_1,s) \leq H(t_2,s). \tag{4.1}$$

If $t_1=0$, then $H(t_1,s)-H(0,s)=0$, so (4.1) remains true.

In the same way, for $t\in[0,1]$ and $0\leq s_1<s_2\leq 1$ we obtain

$$H(t,s_1) \leq H(t,s_2),$$

so the function H is nondecreasing on the co-ordinates.

The proof is complete. ∎

Theorem 8. *Let function* $\varphi: I\times J\to\mathbb{R}$ *be convex on* $I\times J$ *and let functions* g, h, p *and* w *be as in Theorem 7. Then:*

(i) *Function* H *is convex on* $[0,1]^2$;

(ii) *Define the function* $G:[0,1]\to\mathbb{R}$ *with* $G(t)=H(t,t)$. *Then* G *is convex, monotonic nondecreasing on* $[0,1]$ *and one has the bounds:*

$$\begin{aligned} \sup_{t\in[0,1]} G(t) &= G(1) = \frac{1}{PW}\int_{\Omega_1}\int_{\Omega_2} pw\varphi(g,h)\,d\mu d\nu, \\ \inf_{t\in[0,1]} G(t) &= G(0) = \varphi\left(\overline{g},\overline{h}\right). \end{aligned}$$

Proof. (i) Let $(t_1, s_1), (t_2, s_2) \in [0,1]^2$ and $\lambda, \overline{\lambda} \geq 0$ with $\lambda + \overline{\lambda} = 1$. Since φ is convex on $I \times J$ we have

$$\begin{aligned}
& H\left(\lambda (t_1, s_1) + \overline{\lambda} (t_2, s_2)\right) \\
= \; & H\left(\lambda t_1 + \overline{\lambda} t_2, \lambda s_1 + \overline{\lambda} s_2\right) \\
= \; & \frac{1}{PW} \int_{\Omega_1} \int_{\Omega_2} pw\varphi \left[\lambda \left(t_1 u + (1-t_1)\overline{u}, s_1 h + (1-s_1)\overline{h}\right)\right. \\
& \left. + \overline{\lambda} \left(t_2 u + (1-t_2)\overline{u}, s_2 h + (1-s_2)\overline{h}\right)\right] d\mu d\nu \\
\leq \; & \frac{\lambda}{PW} \int_{\Omega_1} \int_{\Omega_2} pw\varphi \left[t_1 u + (1-t_1)\overline{u}, s_1 h + (1-s_1)\overline{h}\right] \\
& + \frac{\overline{\lambda}}{PW} \int_{\Omega_1} \int_{\Omega_2} pw\varphi \left[t_1 u + (1-t_1)\overline{u}, s_1 h + (1-s_1)\overline{h}\right] \\
= \; & \lambda H(t_1, s_1) + \overline{\lambda} H(t_2, s_2),
\end{aligned}$$

so H is convex on $[0,1]^2$.

(ii) Let $t_1, t_2 \in [0,1]$ and $\lambda, \overline{\lambda} \geq 0$ with $\lambda + \overline{\lambda} = 1$. Then, using the convexity of the function H, we obtain

$$\begin{aligned}
G\left(\lambda t_1 + \overline{\lambda} t_2\right) &= H\left(\lambda t_1 + \overline{\lambda} t_2, \lambda t_1 + \overline{\lambda} t_2\right) \\
& H\left(\lambda (t_1, t_1) + \overline{\lambda} (t_2, t_2)\right) \\
&\leq \lambda H(t_1, t_1) + \overline{\lambda} H(t_2, t_2) \\
&= \lambda G(t_1) + \overline{\lambda} G(t_2)
\end{aligned}$$

which shows the convexity of G on $[0,1]$.

By Theorem 7, for $t \in [0,1]$ we have

$$\begin{aligned}
G(t) &= H(t,t) \leq H(1,1), \\
G(t) &= H(t,t) \geq H(0,0),
\end{aligned}$$

from which we deduce

$$\begin{aligned}
\sup_{t \in [0,1]} G(t) &= G(1) = \frac{1}{PW} \int_{\Omega_1} \int_{\Omega_2} pw\varphi(g,h)\, d\mu d\nu, \\
\inf_{t \in [0,1]} G(t) &= G(0) = \varphi\left(\overline{g}, \overline{h}\right).
\end{aligned}$$

In the end, for $0 \leq t_1 < t_2 \leq 1$ similarly as in Theorem 7 we can deduce

$$G(t_1) \leq G(t_2),$$

which means that the function G is nondecreasing on $[0,1]$.

This completes the proof. ∎

Some results related to those given in this section can be found in [1].

5. Some Related Results

Let us consider functions φ, p, w, g and h defined as in Section 2. If we change the condition on the function φ in the way that we demand that the second derivative φ_{xx} is convex on the second co-ordinate, or that the second derivative φ_{yy} is convex on the first co-ordinate, we obtain some other inequalities of Jensen's type related to those given in Section 2.

Theorem 9. *Let $\varphi : I \times J \to \mathbb{R}$ be a function, and let p, w, g and h be as in Theorem 1. If φ_{xx} is convex on the second co-ordinate, or if φ_{yy} is convex on the first co-ordinate, then*

$$\int_{\Omega_1}\int_{\Omega_2} pw\varphi(g,h)\,d\mu d\nu + PW\varphi\left(\overline{g},\overline{h}\right) \tag{5.1}$$

$$\geq P\int_{\Omega_2} w\varphi(\overline{g},h)\,d\nu + W\int_{\Omega_1} p\varphi\left(g,\overline{h}\right)d\mu. \tag{5.1}$$

Proof. Let φ_{xx} be convex on the second co-ordinate. We define function $F_{\overline{h}} : I \to \mathbb{R}$ as

$$F_{\overline{h}}(x) = \int_{\Omega_2} w\varphi(x,h)\,d\nu - W\varphi\left(x,\overline{h}\right).$$

We have

$$F''_{\overline{h}}(x) = \int_{\Omega_2} w\varphi_{xx}(x,h)\,d\nu - W\varphi_{xx}\left(x,\overline{h}\right),$$

so by the convexity of φ_{xx} and the integral Jensen's inequality (2.1) we may conclude that $F''_{\overline{h}}(x) \geq 0$ for all $x \in I$. In other words, if the function φ_{xx} is convex on the second co-ordinate, then the function $F_{\overline{h}}$ is convex on I. This means that we can apply the integral Jensen's inequality (2.1) on the function $F_{\overline{h}}$ to obtain

$$PF_{\overline{h}}(\overline{g}) \leq \int_{\Omega_1} pF_{\overline{h}}(g)\,d\mu.$$

This can be rewritten as

$$\begin{aligned} &P\left[\int_{\Omega_2} w\varphi(\overline{g},h)\,d\nu - W\varphi\left(\overline{g},\overline{h}\right)\right] \\ \leq\ &\int_{\Omega_1} p\left[\int_{\Omega_2} w\varphi(g,h)\,d\nu - W\varphi\left(g,\overline{h}\right)\right]d\mu, \end{aligned}$$

from which we can easily obtain (5.1). If the function φ_{yy} is convex on the first co-ordinate, we proceed analogously. ∎

Theorem 10. *Let $\varphi : [m, M] \times [n, N] \to \mathbb{R}$ be a function, and let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}$ and $\boldsymbol{w}$ be as in Theorem 2. If φ_{xx} is convex on the second co-ordinate, or if φ_{yy} is convex on the first co-ordinate, then*

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \varphi(x_i, y_j) + P_n W_m \varphi(\overline{x}, \overline{y})\\ \geq\ & P_n \sum_{j=1}^{m} w_j \varphi(\overline{x}, y_j) + W_m \sum_{i=1}^{n} p_i \varphi(x_i, \overline{y}).\end{aligned}$$

Proof. Directly from Theorem 9. ∎

Theorem 11. *Let $\varphi : [m, M] \times [n, N] \to \mathbb{R}$ be a function, and let p, w, g and h be as in Theorem 1. If φ_{xx} is convex on the second co-ordinate, or if φ_{yy} is convex on the first co-ordinate, then*

$$\begin{aligned}&\frac{M-\overline{g}}{M-m}\frac{N-\overline{h}}{N-n}\varphi(m,n) + \frac{\overline{g}-m}{M-m}\frac{N-\overline{h}}{N-n}\varphi(M,n)\\ &+\frac{M-\overline{g}}{M-m}\frac{\overline{h}-n}{N-n}\varphi(m,N) + \frac{\overline{g}-m}{M-m}\frac{\overline{h}-n}{N-n}\varphi(M,N)\\ &+\frac{1}{PW}\int_{\Omega_1}\int_{\Omega_2} pw\varphi(g,h)\,d\mu d\nu\\ \geq\ & \frac{1}{W}\left[\frac{M-\overline{g}}{M-m}\int_{\Omega_2} w\varphi(m,h)\,d\nu + \frac{\overline{g}-m}{M-m}\int_{\Omega_2} w\varphi(M,h)\,d\nu\right]\\ &+\frac{1}{P}\left[\frac{N-\overline{h}}{N-n}\int_{\Omega_1} p\varphi(g,n)\,d\mu + \frac{\overline{h}-n}{N-n}\int_{\Omega_1} p\varphi(g,N)\,d\mu\right].\end{aligned} \tag{5.2}$$

Proof. Let φ_{xx} be convex on the second co-ordinate. We define function $F_{\overline{h}} : [m, M] \to \mathbb{R}$ as

$$F_{\overline{h}}(x) = \frac{N-\overline{h}}{N-n}\varphi(x,n) + \frac{\overline{h}-n}{N-n}\varphi(x,N) - \frac{1}{W}\int_{\Omega_2} w\varphi(x,h)\,d\nu.$$

We have

$$F''_{\overline{h}}(x) = \frac{N-\overline{h}}{N-n}\varphi_{xx}(x,n) + \frac{\overline{h}-n}{N-n}\varphi_{xx}(x,N) - \frac{1}{W}\int_{\Omega_2} w\varphi_{xx}(x,h)\,d\nu,$$

so by the convexity of φ_{xx} and the converse integral Jensen's inequality (2.4) we may conclude that $F''_{\overline{h}}(x) \geq 0$ for all $x \in I$. This means that we can apply the inequality (2.4) on the function $F_{\overline{h}}$ to obtain

$$\frac{M-\overline{g}}{M-m}F_{\overline{h}}(m)+\frac{\overline{g}-m}{M-m}F_{\overline{h}}(M) \geq \frac{1}{P}\int_{\Omega_1} pF_{\overline{h}}(g)\,d\mu,$$

and from this we can easily obtain (5.2). If the function φ_{yy} is convex on the first co-ordinate, we proceed analogously. ∎

Theorem 12. *Let $\varphi : [m, M] \times [n, N] \to \mathbb{R}$ be a function, and let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}$ and $\boldsymbol{w}$ be as in Theorem 2. If φ_{xx} is convex on the second co-ordinate, or if φ_{yy} is convex on the first co-ordinate, then*

$$\begin{aligned}
&\frac{M-\overline{x}}{M-m}\frac{N-\overline{y}}{N-n}\varphi(m,n)+\frac{\overline{x}-m}{M-m}\frac{N-\overline{y}}{N-n}\varphi(M,n)+\frac{M-\overline{x}}{M-m}\frac{\overline{y}-n}{N-n}\varphi(m,N)\\
&+\frac{\overline{x}-m}{M-m}\frac{\overline{y}-n}{N-n}\varphi(M,N)+\frac{1}{P_nW_m}\sum_{i=1}^{n}\sum_{j=1}^{m}p_iw_j\varphi(x_i,y_j)\\
\geq\;&\frac{1}{W_m}\left[\frac{M-\overline{x}}{M-m}\sum_{j=1}^{m}w_j\varphi(m,y_j)+\frac{\overline{x}-m}{M-m}\sum_{j=1}^{m}w_j\varphi(M,y_j)\right]\\
&+\frac{1}{P_n}\left[\frac{N-\overline{y}}{N-n}\sum_{i=1}^{n}p_i\varphi(x_i,n)+\frac{\overline{y}-n}{N-n}\sum_{i=1}^{n}p_i\varphi(x_i,N)\right].
\end{aligned}$$

Proof. Directly from Theorem 11. ∎

Let $g : [a, b] \to (m, M)$, where $-\infty < a < b < \infty$ and $-\infty \leq m < M \leq \infty$, be a continuous and monotonic function and let $\lambda : [a, b] \to \mathbb{R}$, where $\lambda(a) \leq \lambda(t) \leq \lambda(b)$, $\forall t \in [a, b]$ and $\lambda(b) - \lambda(a) > 0$, be a continuous function or a function of bounded variation on $[a, b]$. If function $\varphi : (m, M) \to \mathbb{R}$ is convex on (m, M), then the inequality

$$\varphi\left(\frac{\int_a^b g(u)\,d\lambda(u)}{\int_a^b d\lambda(u)}\right) \leq \frac{1}{\int_a^b d\lambda(u)}\int_a^b \varphi(g(u))\,d\lambda(u), \tag{5.3}$$

holds. Inequality (5.3) is the well-known Jensen-Steffensen's inequality (see for example [4, p. 59]).

In the following theorem we show that a similar result to the one given in Theorem 9 can be obtained if we, instead of the integral Jensen's inequality, consider the integral Jensen-Steffensen's inequality.

Theorem 13. *Let $\varphi : (m, M) \times (n, N) \to \mathbb{R}$ be a function, let $g : [a, b] \to (m, M)$ and $h : [c, d] \to (n, N)$, where $-\infty < a < b < \infty$, $-\infty \leq m < M \leq \infty$, $-\infty < c < d < \infty$ and $-\infty \leq n < N \leq \infty$, be continuous and monotonic functions, and let $\lambda : [a, b] \to \mathbb{R}$ and $\rho : [c, d] \to \mathbb{R}$ be continuous functions or functions of bounded variation such that*

$$\lambda(a) \leq \lambda(s) \leq \lambda(b), \ \forall s \in [a, b]; \ \lambda(b) - \lambda(a) > 0,$$
$$\rho(c) \leq \rho(t) \leq \rho(d), \ \forall t \in [c, d]; \ \rho(d) - \rho(c) > 0.$$

If φ_{xx} is convex on the second co-ordinate, or if φ_{yy} is convex on the first co-ordinate, then the inequality

$$\int_a^b \int_c^d \varphi(g(u), h(v))\, d\lambda(u)\, d\rho(v) + LR\varphi\left(\overline{g}, \overline{h}\right)$$
$$\geq R \int_a^b \varphi\left(g(u), \overline{h}\right) d\lambda(u) + L \int_c^d \varphi(\overline{g}, h(v))\, d\rho(v)$$

holds, where

$$L = \int_a^b d\lambda(u), \quad R = \int_c^d d\rho(v)$$
$$\overline{g} = \frac{1}{L} \int_a^b g(u)\, d\lambda(u), \quad \overline{h} = \frac{1}{R} \int_c^d h(v)\, d\rho(v)$$

Proof. Let φ_{xx} be convex on the second co-ordinate. We define function $F_{\overline{h}} : (m, M) \to \mathbb{R}$ as

$$F_{\overline{h}}(x) = \int_c^d \varphi(x, h(v))\, d\rho(v) - R\varphi\left(x, \overline{h}\right).$$

Afte this we proceed analogously as in Theorem 9, but using (5.3) instead of (2.1). ∎

Remark 4. A discrete version of Theorem 13 can be obtained in a similar way if instead of inequality (5.3) we use discrete version of Jensen-Steffensen's inequality (see for example [3, p. 6] or [4, p. 57]).

Analogous results to those given in Theorem 9 and Theorem 13 can be obtained if we (instead of Jensen's or Jensen-Steffensen's inequality) use the Majorization theorem (see for example [4, p. 319, 325]) or the Popoviciu's inequality (see [4, p. 171]).

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M. Klaričić Bakula
Department of Mathematics,
Faculty of Natural Sciences,
Mathematics and Education,
University of Split,
Teslina 12, 21000 Split,
Croatia
E-mail: milica@pmfst.hr

J. Pečarić
Faculty of Textile Technology,
University of Zagreb,
Pierottijeva 6, 10000 Zagreb,
Croatia
E-mail: pecaric@hazu.hr