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ON SOME SUFFICIENT CONDITIONS FOR STARLIKENESS OF ORDER α IN C^n

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Abstract. In this paper, we obtain some new sufficient conditions for starlikeness of order α of biholomorphic mappings on the unit ball in C^n or a complex Hilbert space X by using differential inequalities. We also obtain a distortion theorem and a covering theorem. As their special case, we obtain some sufficient conditions for starlikeness of order α of analytic functions on the unit disc in the complex plane C, which generalize some results of P. T. Mocanu and G. Oros.

1. Introduction

Let H be the class of functions of the form

$$f(z) = z + \sum_{k=2}^{+\infty} a_k z^k$$

which are analytic on the unit disk $U = \{z \in C;\ |z| < 1\}$. By $S^*(\alpha)$ we denote the class of starlike functions of order α in U, where $0 \leq \alpha < 1$. It is obvious that $f \in S^*(\alpha)$ if and only if $f(z) \in H$ satisfies

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > \alpha, \quad \text{for all } z \in U.$$

Suppose that n, m, j, k and l are positive integers, and let C^n be the space of n complex variables $z = (z_1, z_2, \cdots, z_n)$ with the usual inner product $\langle z, w \rangle =$

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$\sum_{j=1}^{n} z_j \overline{w_j}$ and Euclidian norm $\|z\| = \sqrt{\langle z, z\rangle}$. Let $N(B^n)$ be the class of mappings $f(z) = (f_1(z), \cdots, f_n(z)), z = (z_1, \cdots, z_n) \in C^n$, which are holomorphic on the unit ball $B^n = \{z \in C^n : \|z\| < 1\}$ with values in C^n. A mapping $f \in N(B^n)$ is said to be locally biholomorphic on B^n if f has a locally inverse at each point $z \in B^n$ or, equivalently, if the first Fréchet derivative

$$Df(z) = \left(\frac{\partial f_j(z)}{\partial z_k}\right)_{1\le j,k\le n}$$

is nonsingular at each point in B^n.

The second Fréchet derivative of a mapping $f \in N(B^n)$ is a symmetric bilinear operator $D^2 f(z)(\cdot,\cdot)$ on $C^n \times C^n$, and $D^2 f(z)(z,\cdot)$ is the linear operator obtained by restricting $D^2 f(z)$ to $\{z\} \times C^n$. The matrix representation of $D^2 f(z)(b,\cdot)$ is

$$D^2 f(z)(b,\cdot) = \left(\sum_{l=1}^{n} \frac{\partial^2 f_j(z)}{\partial z_k \partial z_l} b_l\right)_{1\le j,k\le n},$$

where $f(z) = (f_1(z), \cdots, f_n(z)), b = (b_1, \cdots, b_n) \in C^n$. The norm of $n \times n$ complex matrix A is defined by

$$\|A\| = \sup_{\|z\|\le 1} \|Az\|.$$

If $f \in N(B^n)$, then for every $k = 1, 2, \cdots$, there exists a bounded symmetric k-linear map $D^k f(0) : C^n \times C^n \times \cdots \times C^n \to C^n$ such that $f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} D^k f(0)(z^k)$ for $z \in B^n$, where $D^0 f(0)(z^0) = f(0)$ and $D^k f(0)(z^k) = D^k f(0)(z, z, \cdots, z)$.

Let $H_m(B^n)$ denote the subclass of $N(B^n)$ consisting of mappings f, which are local biholomorphic and $f(z) = z + \sum_{k=m+1}^{\infty} \frac{1}{k!} D^k f(0)(z^k)$. $H_m(B^1)$ is denoted by $H_m(\Delta)$.

The class of biholomorphic starlike mappings f on B^n with $f(0) = 0$ is denoted by $S^*(B^n)$. Then $f \in S^*(B^n)$ if and only if f is local biholomorphic such that

$$\mathrm{Re}\langle Df(z)^{-1} f(z), z\rangle > 0$$

for all $z \in B^n - \{0\}$(see [8, Theorem 1]).

We now define

$$S^*(\alpha, B^n) = \left\{ f \in H_1(B^n) : \left| \frac{1}{\|z\|^2} \langle Df(z)^{-1} f(z), z\rangle - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha} \quad \text{for all } z \in B^n - \{0\} \right\}$$

for $0 < \alpha < 1$ and $S^*(0, B^n) \equiv S^*(B^n)$. P. Curt [1] and G. Kohr [2] called the biholomorphic mapping $f \in S^*(\alpha, B^n)$ starlike of order α. Let $S_m^*(\alpha, B^n) \equiv$

$S^*(\alpha, B^n) \cap H_m(B^n)$ for $0 \leq \alpha < 1$. It is obvious that $S^*(\alpha, B^1) \equiv S^*(\alpha)$ and $S^*_m(\alpha, B^n) \subset S^*(\alpha, B^n) \equiv S^*_1(\alpha, B^n) \subset S^*(B^n)$ for $0 \leq \alpha < 1$.

In order to derive our main results, we need the following lemma.

Lemma 1. *Suppose that $w : B^n(r) \to C^n$ is a holomorphic mapping with $w(z) = \sum_{k=m+1}^{\infty} \frac{1}{k!} D^k w(0)(z^k)$. If the point $z_0 \in B^n(r) - \{0\}$ satisfies*

$$\|w(z_0)\| = \max_{\|z\| \leq \|z_0\| < r} \|w(z)\|,$$

then there exists a real number $t \geq m+1$ such that

$$\langle Dw(z_0)(z_0), w(z_0)\rangle = t\|w(z_0)\|^2. \tag{1.1}$$

Proof. Let $\psi(\xi) = \langle w(\frac{\xi}{\|z_0\|} z_0), w(z_0)\rangle, \xi \in \mathbf{C}$, then $\psi(\xi) = \sum_{k=m+1}^{\infty} a_k \xi^k$ is analytic on the disc $U = \{\xi : |\xi| < r\}$ and

$$|\psi(\|z_0\|)| = \max_{|\xi| \leq \|z_0\|} |\psi(\xi)|.$$

By Lemma A of [5], we obtain that there exists a real number $t \geq m+1$ such that

$$\|z_0\| \psi'(\|z_0\|) = t\psi(\|z_0\|).$$

Since

$$\psi'(\|z_0\|) = \left\langle Dw(z_0)(\frac{z_0}{\|z_0\|}), w(z_0)\right\rangle \quad \text{and} \quad \psi(\|z_0\|) = \|w(z_0)\|^2,$$

hence (1.1) holds, and the proof is complete.

Remark 1. In the case $r = 1$ and $m = 0$, the result of Lemma 1 was obtained by P. Liczberski [3].

2. Main Results

Theorem 1. *Suppose that Re$\lambda < m+1$ or Im$\lambda \neq 0$ and $\alpha \in [0, 1)$, and let*

$$R(\lambda) = \begin{cases} |m+1-\lambda|, & \text{Re}\lambda < m+1, \\ |\text{Im}\lambda|, & \text{Re}\lambda \geq m+1, \text{Im}\lambda \neq 0, \end{cases} \tag{2.1}$$

and

$$N = N(\lambda, \alpha) = \begin{cases} \dfrac{\sqrt{1-2\alpha}}{\sqrt{(|\lambda|+R(\lambda))^2+1-2\alpha}}, & 0 \leq \alpha \leq \dfrac{1}{|\lambda|+R(\lambda)+2}, \\ \dfrac{1-\alpha}{|\lambda|+R(\lambda)+\alpha}, & \dfrac{1}{|\lambda|+R(\lambda)+2} < \alpha < 1. \end{cases} \tag{2.2}$$

If $f \in H_m(B^n)$ satisfies the inequality

$$\| \|z\|^2 (Df(z)(u) - u) - \lambda \langle u, z \rangle (f(z) - z) \| \leq M \|z\|^2 \tag{2.3}$$

for all $z \in B^n$ and all $||u|| = 1$, where $M = R(\lambda)N(\lambda, \alpha)$, then $f \in S_m^*(\alpha, B^n)$.

Proof. Let $q(z) = f(z) - z$. Then $q(z) = \sum_{k=m+1}^{\infty} \frac{1}{k!} D^k q(0)(z^k) \in N(B^n)$ and

$$Df(z)(z) - \lambda f(z) + (\lambda - 1)z = Dq(z)(z) - \lambda q(z). \tag{2.4}$$

Setting $u = \frac{z}{||z||}$ in (2.3) for $z \in B - \{0\}$, using (2.4) and noting $q(0) = 0$, we have

$$\|Dq(z)(z) - \lambda q(z)\| \leq M\|z\|. \tag{2.5}$$

for all $z \in B^n$.

Now we prove that $\|q(z)\| < N$ for all $z \in B^n$.

If it is not true, then there exists a point $z_0 \in B^n - \{0\}$ such that

$$N = \|q(z_0)\| = \max_{\|z\| \leq \|z_0\| < 1} \|q(z)\|. \tag{2.6}$$

Since

$$\langle Dq(z_0)(z_0) - \lambda q(z_0), q(z_0) \rangle = \langle Dq(z_0)(z_0), q(z_0) \rangle - \lambda \|q(z_0)\|^2, \tag{2.7}$$

according to Lemma 1 and (2.5)–(2.7), there exists a real number $t \geq m + 1$ such that

$$\sqrt{(t - \mathrm{Re}\lambda)^2 + (\mathrm{Im}\lambda)^2} N^2 = |t - \lambda| N^2 \leq \|Dq(z_0)(z_0) - \lambda q(z_0)\| \|q(z_0)\| \leq MN\|z_0\|.$$

When $\mathrm{Re}\lambda < m + 1$, we obtain

$$\sqrt{(t - \mathrm{Re}\lambda)^2 + (\mathrm{Im}\lambda)^2} \geq \sqrt{(m + 1 - \mathrm{Re}\lambda)^2 + (\mathrm{Im}\lambda)^2} = |m + 1 - \lambda| \tag{2.8}$$

for $t \geq m + 1$.

When $\mathrm{Re}\lambda \geq m + 1$ and $\mathrm{Im}\lambda \neq 0$, we obtain

$$\sqrt{(t - \mathrm{Re}\lambda)^2 + (\mathrm{Im}\lambda)^2} \geq |\mathrm{Im}\lambda| \tag{2.9}$$

for $t \geq m + 1$.

From (2.1), (2.8) and (2.9), we have

$$R(\lambda)N^2 \leq MN\|z_0\| < MN. \tag{2.10}$$

This leads to $M = R(\lambda)N < M$, which is a contradiction. Hence we conclude that $\|q(z)\| < N$ for all $z \in B^n$. According to Schwarz's Lemma, we have

$$\|q(z)\| \leq N\|z\|^{m+1} \quad \text{for all} \quad z \in B^n. \tag{2.11}$$

From (2.3), we have

$$\|\|z\|^2 Dq(z)(u) - \lambda\langle u, z\rangle q(z)\| \leq M||z||^2 \quad \text{for} \quad z \in B^n, \quad \|u\| = 1.$$

It follows that

$$\begin{aligned} \|Dq(z)\| &\leq \sup_{\|u\|\leq 1} \{\|Dq(z)(u)\|\} \\ &\leq \sup_{\|u\|\leq 1} \left\{ \left\| Dq(z)(u) - \lambda\langle u, z\rangle \frac{q(z)}{||z||^2} \right\| + |\lambda| \frac{\|q(z)\|}{||z||^2} |\langle u, z\rangle| \right\} \\ &\leq M + |\lambda|N||z||^m \leq M + |\lambda|N = M_1, \end{aligned} \tag{2.12}$$

where $M_1 = (|\lambda| + R(\lambda))N$. Let $w(z) = Df(z)^{-1}f(z)$. Then by (2.12), we have

$$\begin{aligned} \|q(z) + z - w(z)\| &= \|Df(z)w(z) - w(z)\| = \|Dq(z)w(z)\| \\ &\leq \|Dq(z)\|\|w(z)\| \leq M_1\|w(z)\| \end{aligned} \tag{2.13}$$

for all $z \in B$.

In the following, we split into two cases to prove.

Case 1. When $\alpha = 0$,

$$N = N(\lambda, 0) = \frac{1}{\sqrt{(|\lambda| + R(\lambda))^2 + 1}}. \tag{2.14}$$

Suppose that f is not in $S^*(0, B^n) = S^*(B^n)$, then there exists a point $z_1 \in B^n - \{0\}$ such that $\text{Re}\langle w(z_1), z_1\rangle = 0$. From (2.13), we have

$$\|q(z_1) + z_1 - w(z_1)\| \leq M_1\|w(z_1)\|. \tag{2.15}$$

Claim 1.

$$\|z_1 - w(z_1)\| - N\|z_1\| \geq M_1\|w(z_1)\|. \tag{2.16}$$

It is equivalent to

$$\|z_1\|^2 + \|w(z_1)\|^2 = \|z_1 - w(z_1)\|^2 \geq [N\|z_1\| + M_1\|w(z_1)\|]^2. \tag{2.17}$$

From (2.17), we obtain

$$(1-N^2)\|z_1\|^2+[1-M_1^2]\|w(z_1)\|^2-2M_1N\|z_1\|\|w(z_1)\|\geq 0. \tag{2.18}$$

Note that $N^2+M_1^2=1$, the inequality (2.16) is equivalent to

$$M_1^2\|z_1\|^2+N^2\|w(z_1)\|^2-2M_1N\|z_1\|\|w(z_1)\|=[M_1\|z_1\|-N\|w(z_1)\|]^2\geq 0.$$

Hence the claim (2.16) is established.

Using (2.16) and (2.11), we obtain

$$\|q(z_1)+z_1-w(z_1)\|\geq\|z_1-w(z_1)\|-N\|z_1\|^{m+1}>\|z_1-w(z_1)\|-N\|z_1\|\geq M_1\|w(z_1)\|,$$

which contradicts (2.15). Hence $f\in S_m^*(B^n)$.

Case 2. When $0<\alpha<1$. Let $h(z)=2\alpha Df(z)^{-1}f(z)-z$, We shall prove that $\|h(z)\|<\|z\|$ for all $z\in B^n-\{0\}$. If not, then there exists a point $z_2\in B^n$ such that $\|h(z_2)\|=\|z_2\|$, it follows that

$$\operatorname{Re}\langle w(z_2),z_2\rangle=\alpha\|w(z_2)\|^2 \quad\text{and}\quad \|w(z_2)\|\leq\frac{1}{\alpha}\|z_2\|. \tag{2.19}$$

Claim 2.

$$\|z_2-w(z_2)\|-N\|z_2\|\geq M_1\|w(z_2)\|. \tag{2.20}$$

This inequality is equivalent to

$$(1-N^2)\|z_2\|^2+[1-2\alpha-M_1^2]\|w(z_2)\|^2-2M_1N\|z_2\|\|w(z_2)\|\geq 0. \tag{2.21}$$

If $\|w(z_2)\|=0$, then the inequality holds. If $\|w(z_2)\|>0$, then from (2.19), we have $\frac{\|z_2\|}{\|w(z_2)\|}\geq\alpha$. According to (2.21), we have

$$x^2+1-2\alpha\geq[M_1+Nx]^2,$$

where $x=\frac{\|z_2\|}{\|w(z_2)\|}$. Hence the inequality (2.20) is equivalent to

$$N\leq\frac{\sqrt{x^2+1-2\alpha}}{|\lambda|+R(\lambda)+x}. \tag{2.22}$$

for $x\geq\alpha$.

Let $\varphi(x)=\frac{\sqrt{x^2+1-2\alpha}}{|\lambda|+R(\lambda)+x}$ for $x\geq\alpha$. Then

$$\varphi'(x)=\frac{(|\lambda|+R(\lambda))x-1+2\alpha}{\sqrt{x^2+1-2\alpha}(|\lambda|+R(\lambda)+x)^2}. \tag{2.23}$$

Taking $\varphi'(x) = 0$, we conclude that $x_0 = \frac{1-2\alpha}{|\lambda|+R(\lambda)}$.

If $0 \leq \alpha \leq \frac{1}{|\lambda|+R(\lambda)+2}$, then $x_0 \geq \alpha$. Therefore

$$\min_{x\geq\alpha} \varphi(x) = \varphi(x_0) = \frac{\sqrt{1-2\alpha}}{\sqrt{(|\lambda| + R(\lambda))^2 + 1 - 2\alpha}} = N(\lambda, \alpha). \tag{2.24}$$

If $\frac{1}{|\lambda|+R(\lambda)+2} < \alpha < 1$, then $x_0 < \alpha$. Therefore

$$\min_{x\geq\alpha} \varphi(x) = \varphi(\alpha) = \frac{1-\alpha}{|\lambda| + R(\lambda) + \alpha} = N(\lambda, \alpha). \tag{2.25}$$

Hence the claim (2.20) is established.

Using (2.20) and (2.11), we obtain

$$\begin{aligned} &\|q(z_2) + z_2 - w(z_2)\| \geq \|z_2 - w(z_2)\| - N\|z_2\|^{m+1} \\ &> \|z_2 - w(z_2)\| - N\|z_2\| \geq M_1\|w(z_2)\|, \end{aligned}$$

which contradicts (2.13). Hence $\|2\alpha Df(z)^{-1}f(z) - z\| < \|z\|$ for all $z \in B^n - \{0\}$. Thus we conclude that

$$\begin{aligned} \left| \frac{1}{\|z\|^2}\langle Df(z)^{-1}f(z), z\rangle - \frac{1}{2\alpha} \right| &= \frac{1}{2\alpha\|z\|^2}\left|\langle 2\alpha Df(z)^{-1}f(z) - z, z\rangle\right| \\ &\leq \frac{1}{2\alpha\|z\|^2}\|2\alpha Df(z)^{-1}f(z) - z\| \cdot \|z\| < \frac{1}{2\alpha} \end{aligned}$$

for all $z \in B^n - \{0\}$. Hence we obtain that $f(z) \in S_m^*(\alpha, B^n)$, and the proof is complete.

Setting $n = 1$ in Theorem 1, we obtain the following corollary.

Corollary 1. *Suppose that* $Re\lambda < m + 1$ *or* $Im\lambda \neq 0$, $\alpha \in [0, 1)$ *and* $M = R(\lambda)N(\lambda, \alpha)$, *where* $R(\lambda)$ *and* $N = N(\lambda, \alpha)$ *are defined by* (2.1) *and* (2.2), *respectively. If* $f \in H_m(\Delta)$ *satisfies the inequality*

$$\left| f'(z) - \lambda\frac{f(z)}{z} + \lambda - 1 \right| \leq M$$

for all $z \in U$, *then* $f \in S^*(\alpha)$.

Remark 2. Corollary 1 generalizes Theorem 2.1 in [7] and Theorem 2.2 in [4], where λ is a real number in Theorem 2.1 of [7] and Theorem 2.2 of [4].

Setting $\lambda = 0$ in Theorem 1, we have the following corollary.

Corollary 2. *Let* $\alpha \in [0,1)$ *and*

$$N_m(\alpha) = \begin{cases} \dfrac{\sqrt{1-2\alpha}}{\sqrt{(m+1)^2+1-2\alpha}}, & 0 \le \alpha \le \dfrac{1}{m+3}, \\ \dfrac{1-\alpha}{m+1+\alpha}, & \dfrac{1}{m+3} < \alpha < 1. \end{cases}$$

If $f \in H_m(B^n)$ *satisfies the following inequality*

$$\|Df(z) - I\| \le M \equiv (m+1)N_m(\alpha)$$

for all $z \in B^n$*, then* $f \in S_m^*(\alpha, B^n)$*.*

Remark 3. Setting $n = 1, \alpha = 0$ in Corollary 2, we get the result obtained by Mocanu [6]. Setting $n = 1$ in Corollary 2, we get a result, which is better than Corollary 2.2 in [7].

Example 1. Suppose that A is a bounded symmetric $(m+1)-$linear operator from $C^n \times C^n \times \cdots \times C^n$ to C^n with $\|A\| \le \frac{M}{m+1+|\lambda|}$, where $M = R(\lambda)N(\lambda,\alpha)$ is defined in Theorem 1. Let $f(z) = z + A(z^{m+1}), \quad z \in C^n$. Then $f \in S_m^*(\alpha, B^n)$.

Proof. Some direct computations yield the relations

$$Df(z) = I + (m+1)A(z^m, \cdot)$$

for $z \in B^n$. It implies that

$$\begin{aligned} \| \|z\|^2(Df(z)(u)-u) - \lambda\langle u,z\rangle(f(z)-z)\| &= \|A(z^m, (m+1)\|z\|^2 u + \lambda\langle u,z\rangle z)\| \\ &\le \|A\|\|z\|^m\|(m+1)\|z\|^2 u + \lambda\langle u,z\rangle z\| \\ &\le (m+1+|\lambda|)\|A\|\|z\|^2 \le M\|z\|^2 \end{aligned}$$

for all $z \in B^n$ and all $u \in C^n$ with $\||u|| = 1$. Hence by Theorem 1, we obtain that $f \in S_m^*(\alpha, B^n)$.

In particular, let

$$A(z_1, z_2, \cdots, z_{m+1}) = a < z_1, u >< z_2, u > \cdots < z_{m+1}, u > v,$$

wherwe $u, v \in C^n$ with $\|u\| = \|v\| = 1$ and $a \in C$. Then A is a bounded symmetric $(m+1)-$linear operator from $C^n \times C^n \times \cdots \times C^n$ to C^n with $\|A\| = |a|$. If

$$f(z) = z + a[< z, u >]^{m+1} v$$

and $|a| \le \frac{m}{m+1+|\lambda|}$, where $M = R(\lambda)N(\lambda,\alpha)$ is defined in Theorem 1, then $f \in S_m^*(\alpha, B^n)$.

Theorem 2. *Suppose that Re*$\lambda < m+1$ *or Im*$\lambda \neq 0$ *and* $0 < R \leq \frac{R(\lambda)}{\sqrt{(|\lambda|+R(\lambda))^2+1}}$*, where* $R(\lambda)$ *is defined by (2.1). If* $f \in H_m(B^n)$ *satisfies the inequality*

$$\||\|z\|^2(Df(z)(u)-u)-\lambda\langle u,z\rangle(f(z)-z)\| \leq R\|z\|^2 \tag{2.26}$$

for all $z \in B^n$ *and all* $\|u\|=1$*, then* $f \in S_m^*(\beta, B^n)$*, where*

$$\beta = \begin{cases} \dfrac{R(\lambda)(1-R)-|\lambda|R}{R+R(\lambda)}, & 0 < R < \dfrac{R(\lambda)}{|\lambda|+R(\lambda)+1}, \\ \dfrac{1}{2}+\dfrac{R^2(|\lambda|+R(\lambda))^2}{2(R^2-R(\lambda)^2)}, & \dfrac{R(\lambda)}{|\lambda|+R(\lambda)+1} \leq R \leq \dfrac{R(\lambda)}{\sqrt{(|\lambda|+R(\lambda))^2+1}}. \end{cases} \tag{2.27}$$

Proof.

Case 1. When $0 < R < \frac{R(\lambda)}{|\lambda|+R(\lambda)+1}$, we have

$$\frac{1}{|\lambda|+R(\lambda)+2} < \beta = \frac{R(\lambda)(1-R)-|\lambda|R}{R+R(\lambda)} < 1,$$

it is equivalent to

$$0 < R = \frac{R(\lambda)(1-\beta)}{|\lambda|+R(\lambda)+\beta} < \frac{R(\lambda)}{|\lambda|+R(\lambda)+1}.$$

Hence by Theorem 1, we have $f \in S_m^*(\beta, B^n)$.

Case 2. When $\frac{R(\lambda)}{|\lambda|+R(\lambda)+1} \leq R \leq \frac{R(\lambda)}{\sqrt{(|\lambda|+R(\lambda))^2+1}}$, we have

$$0 \leq \beta = \frac{1}{2}+\frac{R^2(|\lambda|+R(\lambda))^2}{2(R^2-R(\lambda)^2)} \leq \frac{1}{|\lambda|+R(\lambda)+2},$$

it is equivalent to

$$\frac{R(\lambda)}{|\lambda|+R(\lambda)+1} \leq R = \frac{R(\lambda)\sqrt{1-2\beta}}{\sqrt{(|\lambda|+R(\lambda))^2+1-2\beta}} \leq \frac{R(\lambda)}{\sqrt{(|\lambda|+R(\lambda))^2+1}}.$$

Hence by Theorem 1, we have $f \in S_m^*(\beta, B^n)$, and the proof is complete.

Setting $n=1$ in Theorem 2, we obtain the following corollary.

Corollary 3. *Suppose that* Re$\lambda < m+1$ *or* Im$\lambda \neq 0$ *and* $0 < R \leq \frac{R(\lambda)}{\sqrt{(|\lambda|+R(\lambda))^2+1}}$*, where* $R(\lambda)$ *is defined by* (2.1). *If* $f \in H_m(\Delta)$ *satisfies the inequality*

$$\left|f'(z)-\lambda\frac{f(z)}{z}+\lambda-1\right| \leq R$$

for all $z \in U$, *then* $f \in S^*(\beta)$, *where* β *is defined by* (2.27).

Theorem 3. *Suppose that* $\mathrm{Re}\lambda < m+1$ *or* $\mathrm{Im}\lambda \neq 0$ *and* $\alpha \in [0,1)$, $R(\lambda)$ *is defined by* (2.1) *and* $N = N(\lambda, \alpha)$ *is defined by* (2.2). *If* $f \in H_m(B^n)$ *satisfies the inequality*

$$\|\|z\|^2(Df(z)(u) - u) - \lambda\langle u, z\rangle(f(z) - z)\| \leq M\|z\|^2$$

for all $z \in B^n$ *and all* $\|u\| = 1$, *where* $M = R(\lambda)N(\lambda, \alpha)$, *then*

$$\|z\| - N\|z\|^{m+1} \leq \|f(z)\| \leq \|z\| + N\|z\|^{m+1}, \tag{2.28}$$

and

$$1 - (|\lambda| + R(\lambda))N\|z\|^m \leq \|Df(z)\| \leq 1 + (|\lambda| + R(\lambda))N\|z\|^m$$

for $z \in B^n$.

Proof. From the proof of Theorem 1, we obtain

$$\|f(z) - z\| \leq N\|z\|^{m+1}.$$

Hence we have

$$\begin{aligned}\|z\| - N\|z\|^{m+1} &\leq \|z\| - \|f(z) - z\| \leq \|f(z)\| \\ &= \|[f(z) - z] + z\| \leq \|f(z) - z\| + \|z\| \leq \|z\| + N\|z\|^{m+1}\end{aligned}$$

for $z \in B^n$. From (2.12) and $Dq(z)(u) = \sum_{m+1}^{\infty} \frac{kD^kq(0)}{k!}(z^{k-1}, u)$, where $q(z) = f(z) - z$, by Schwarz's Lemma, we obtain

$$\|Dq(z)\| \leq (|\lambda| + R(\lambda))N\|z\|^m$$

for $z \in B^n$. Hence we have

$$\|Df(z) - I\| \leq (|\lambda| + R(\lambda))N\|z\|^m$$

for $z \in B^n$. It follows that

$$1 - (|\lambda| + R(\lambda))N\|z\|^m \leq \|Df(z)\| \leq 1 + (|\lambda| + R(\lambda))N\|z\|^m$$

for $z \in B^n$. Hence the proof is complete.

Corollary 4. [Covering Theorem] *Suppose that Re*$\lambda < m+1$ *or Im*$\lambda \neq 0$ *and* $\alpha \in [0,1)$, $R(\lambda)$ *is defined by* (2.1) *and* $N = N(\lambda, \alpha)$ *is defined by* (2.2). *If* $f \in H_m(B^n)$ *satisfies the inequality*

$$\|\|z\|^2(Df(z)(u) - u) - \lambda\langle u, z\rangle(f(z) - z)\| \leq M\|z\|^2$$

for all $z \in B^n$ *and all* $\|u\| = 1$*, where* $M = R(\lambda)N(\lambda, \alpha)$*, then*

$$f(B^n) \supset (1 - N)B^n.$$

Theorem 4. *Suppose that Re*$\mu < m$ *or* $\mathrm{Im}\mu \neq 0$ *and* $\alpha \in [0, 1)$*, and let*

$$T(\mu) = \begin{cases} |m - \mu|, & \mathrm{Re}\mu < m, \\ |\mathrm{Im}\mu|, & \mathrm{Re}\mu \geq m, \mathrm{Im}\mu \neq 0, \end{cases} \tag{2.29}$$

and

$$S = S_m(\mu, \alpha) = \begin{cases} \dfrac{T(\mu)(m+1)\sqrt{1-2\alpha}}{\sqrt{(m+1)^2 + 1 - 2\alpha}}, & 0 \leq \alpha \leq \dfrac{1}{m+3}, \\ \dfrac{T(\mu)(m+1)(1-\alpha)}{m+1+\alpha}, & \dfrac{1}{m+3} < \alpha < 1. \end{cases} \tag{2.30}$$

If $f \in H_m(B^n)$ *satisfies the inequality*

$$\|D^2 f(z)(z, \cdot) - \mu Df(z) + \mu I\| < S \tag{2.31}$$

for all $z \in B^n$*, then* $f \in S_m^*(\alpha, B^n)$*.*

Proof. Let $u \in B^n - \{0\}$ and fix it. Set $w(z) = Df(z)(u) - u$, then $w(z) \in N(B^n)$ with $w(z) = \sum_{m+1}^{\infty} \frac{kD^k f(0)}{k!}(z^{k-1}, u)$ and $w(0) = 0$.

Now we verify that $\|w(z)\| < S_1 = \frac{S}{T(\mu)}\|u\|$ for all $z \in B^n$. If not, then there exists a point $z_3 \in B^n$ such that

$$S_1 = \|w(z_3)\| = \max_{\|z\| \leq \|z_3\|} \|w(z)\|.$$

By Lemma 1, there exists a real number $t \geq m$ such that

$$\langle Dw(z_3)(z_3), w(z_3)\rangle = t\|w(z_3)\|^2. \tag{2.32}$$

Then by a simple computation, from (2.31), we obtain

$$\|Dw(z_3)(z_3) - \mu w(z_3)\| < S\|u\|. \tag{2.33}$$

It follows from (2.32) and (2.33) that

$$|t - \mu|\|w(z_3)\|^2 \leq |\langle Dw(z_3)(z_3) - \mu w(z_3), w(z_3)\rangle| < S\|u\|\|w(z_3)\|.$$

When Re$\mu < m$, we obtain

$$|t - \mu| = \sqrt{(t - \mathrm{Re}\mu)^2 + (\mathrm{Im}\mu)^2} \geq \sqrt{(m - \mathrm{Re}\mu)^2 + (\mathrm{Im}\mu)^2} = |m - \mu| \tag{2.34}$$

for $t \geq m$.

When Re$\mu \geq m$ and Im$\mu \neq 0$, we obtain

$$|t-\mu| = \sqrt{(t-\text{Re}\mu)^2 + (\text{Im}\mu)^2} \geq |\text{Im}\mu|. \tag{2.35}$$

From (2.29), (2.34) and (2.35), we have

$$T(\mu)\|w(z_3)\|^2 \leq |\langle Dw(z_3)(z_3) - \mu w(z_3), w(z_3)\rangle| < S\|u\|\|w(z_3)\|.$$

Therefore $\|w(z_3)\| < \frac{S}{T(\mu)}\|u\| = S_1$, which contradicts $\|w(z_3)\| = S_1$. Hence we obtain

$$\|Df(z)(u) - u\| \leq \frac{S}{T(\mu)}\|u\|$$

for all $\|u\| = 1$. From this, we conclude that

$$\|Df(z) - I\| \leq \frac{S}{T(\mu)} = (m+1)N_m(\alpha),$$

for all $z \in B^n$. By Corollary 2, we obtain that $f(z) \in S_m^*(\alpha, B^n)$ and the proof is complete.

Remark 4. Suppose that X is a complex Hilbert space with product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$, and $B = \{z \in X : \|z\| < 1\}$ is the unit ball in X.

Similarly, $f \in S_m^*(\alpha, B)$ if and only if $f(z) = z + \sum\limits_{k=m+1}^{+\infty} \frac{1}{k!} D^k f(0)(z^k)$ is a locally biholomorphic mapping on B and satisfies the following inequalities

$$\left| \frac{1}{\|z\|^2} \langle Df(z)^{-1} f(z), z\rangle - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}, \quad z \in B - \{0\}$$

for $0 < \alpha < 1$ and

$$\text{Re}\langle Df(z)^{-1} f(z), z\rangle > 0, \quad z \in B - \{0\}$$

for $\alpha = 0$. We call the biholomorphic mapping $f \in S_m^*(\alpha, B)$ starlike of order α.

Recently, we discover that if we let X instead of C^n and $f : B \to X$ is a locally biholomorphic mapping (see [8], p. 146-147), then the results of Lemma 1 and Theorem 1-4 still hold. The proofs are similar. For example, we state two results as follows and omit their proofs.

Theorem 1′. *Suppose that* $\alpha \in [0,1), f(z) = z + \sum\limits_{k=m+1}^{+\infty} \frac{1}{k!} D^k f(0)(z^k)$: $B \to X$ *is a locally biholomorphic mapping on* B *and* $R(\lambda)$ *is defined by* (2.1), $N = N(\lambda, \alpha)$ *is defined by* (2.2). *If* $f(z)$ *satisfies the inequality*

$$\|\|z\|^2(Df(z)(u) - u) - \lambda\langle u, z\rangle(f(z) - z)\| \leq M\|z\|^2$$

for all $z \in B$ *and all* $u \in X$ *with* $||u|| = 1$, *where* $M = R(\lambda)N(\lambda, \alpha)$, *then* $f \in S_m^*(\alpha, B)$.

Theorem 4′. *Suppose that* $\alpha \in [0, 1), f(z) = z + \sum_{k=m+1}^{+\infty} \frac{1}{k!} D^k f(0)(z^k) : B \to X$ *is a locally biholomorphic mapping on* B *and* $T(\mu)$ *is defined by* (2.29), $S = S_m(\mu, \alpha)$ *is defined by* (2.30). *If* $f(z)$ *satisfies the inequality*

$$\|D^2 f(z)(z, \cdot) - \mu Df(z) + \mu I\| < S$$

for all $z \in B$, *then* $f \in S_m^*(\alpha, B)$.

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