

PARTITION OF A SET OF INTEGERS INTO SUBSETS WITH PRESCRIBED SUMS

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Abstract. A nonincreasing sequence of positive integers $\langle m_1, m_2, \dots, m_k \rangle$ is said to be *n-realizable* if the set $I_n = \{1, 2, \dots, n\}$ can be partitioned into k mutually disjoint subsets S_1, S_2, \dots, S_k such that $\sum_{x \in S_i} x = m_i$ for each $1 \leq i \leq k$. In this paper, we will prove that a nonincreasing sequence of positive integers $\langle m_1, m_2, \dots, m_k \rangle$ is *n-realizable* under the conditions that $\sum_{i=1}^k m_i = \binom{n+1}{2}$ and $m_{k-1} \geq n$.

1. INTRODUCTION

Given a set $I_n = \{1, 2, \dots, n\}$, it is interesting to know whether it can be partitioned into k mutually disjoint subsets S_1, S_2, \dots, S_k such that the sums of elements in S_i 's are prescribed values. For example, can we partition I_{10} into five sets of sum 11? Of course, the answer is affirmative. However, partitioning the set I_{10} into 5 sets with prescribed sums $\langle 20, 17, 14, 2, 2 \rangle$ is impossible. Now, a question is posed naturally: Under what conditions the set I_n has such a prescribed partition? And if so, how to find it? This question has received considerable attention and some effective partitioning methods are proposed in [2,4].

Now, we introduce some definitions and then give a brief retrospect to this issue. A nonincreasing sequence of positive integers $\langle m_1, m_2, \dots, m_k \rangle$ is said to be *n-realizable* if I_n can be partitioned into k mutually disjoint subsets S_1, S_2, \dots, S_k such that $\sum_{x \in S_i} x = m_i$ for each $1 \leq i \leq k$. Obviously, if $\langle m_1, m_2, \dots, m_k \rangle$ is *n-realizable*, then $\sum_{i=1}^k m_i = \binom{n+1}{2}$, and the converse may be not true. Some sequences of total sum $\binom{n+1}{2}$ are shown to be *n-realizable* which are listed below:

1. $\langle m, m, \dots, m \rangle$, where $m \geq n$;

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2. $\langle m, m, \dots, m, l \rangle$, where $m \geq n$ and $l \leq m$ is an arbitrary positive integer (see [2.4]);
3. $\langle m+1, \dots, m+1, m, m, \dots, m \rangle$, where $m \geq n$ (see [4]);
4. $\langle m+k-2, m+k-3, \dots, m+1, m, l \rangle$, where $m \geq n$ and l is an arbitrary positive integer ([4]);
5. $\langle m_1, m_2, \dots, m_k \rangle$, where $m_k \geq n$ (see [5]).

The definition of n -realizability is closely related to the concept of ascending subgraph decomposition of a graph. A graph G with $\binom{n+1}{2}$ edges is said to have an ascending subgraph decomposition (ASD) if the edge set of G , $E(G)$, can be partitioned into n sets E_1, E_2, \dots, E_n which induce n graphs G_1, G_2, \dots, G_n such that $|E_i| = i$ for $i = 1, 2, \dots, n$ and G_i is isomorphic to a subgraph of G_{i+1} for $i = 1, 2, \dots, n-1$. Clearly, it is easy to see that a graph with small size has an ASD. In [1,3], it was conjectured by Alavi et al that every graph of size $\binom{n+1}{2}$ has an ASD. Also in [1,3], they pose the following ASD conjecture on star forests.

Conjecture (Star forest) If G is a star forest with $\binom{n+1}{2}$ edges and each component has at least n edges, then G has an ASD such that each member of the decomposition is a star.

This conjecture is equivalent to prove that Sequence 5 mentioned above is n -realizable, and it is proved in [5] by Ma et al. In this paper, we shall prove the following more general conclusion.

Theorem 1.1. *Let $\langle m_1, m_2, \dots, m_k \rangle$ be a nonincreasing sequence of positive integers such that $\sum_{i=1}^k m_i = \binom{n+1}{2}$ and $m_{k-1} \geq n$. Then $\langle m_1, m_2, \dots, m_k \rangle$ is n -realizable.*

The conclusion in this theorem is sharp in the sense that the condition $m_{k-1} \geq n$ can not be replaced by $m_{k-2} \geq n$ since $\langle 7, 6, 1, 1 \rangle$ and $\langle 6, 5, 2, 2 \rangle$ are not 5-realizable. A partition algorithm will also be given in this paper.

Note that sets used in this paper are the sets in the sense that their any repeated integers are taken as different elements. Also, the union of two multisets is also a multiset. For example, $\{10, 10, 12\} \cup \{10, 12, 13\} = \{10, 10, 10, 12, 12, 13\}$.

2. DECOMPOSITION ALGORITHM

In this section, we would give a partition algorithm to prove that a nonincreasing sequence of positive integers $\langle m_1, m_2, \dots, m_k \rangle$ of total sum $\binom{n+1}{2}$ with $m_{k-1} \geq n$ is n -realizable. We begin with some observations on m_k and m_{k-1} . If $m_k \geq n$, then we have done by [5]. So we may assume that $m_k < n$. If $m_{k-1} = n$, then we can partition I_{n-1} into $k-1$ sets having sums $m_1, m_2, \dots, m_{k-2}, m_k$ by the

inductive hypothesis, and so we get the desired partition by combining these subsets and setting $S_k = \{n\}$. Therefore, we may assume that $m_{k-1} \geq n + 1$ which implies that $m_i \geq n + 1$ for all $1 \leq i \leq k - 1$. Define

$$l = n - 2(k - 1), N = 2k + 2l - 1, M = m_1.$$

Since

$$\frac{n(n + 1)}{2} = m_1 + m_2 + \cdots + m_k \geq (n + 1)(k - 1) + m_k > (n + 1)(k - 1),$$

we have $n > 2(k - 1)$. Therefore, $l \geq 1$ and $N = n + l + 1 \geq n + 2$.

Let

$$W = \begin{bmatrix} u_1 & u_2 & \cdots & u_{M-k-l+1} \\ v_1 & v_2 & \cdots & v_{M-k-l+1} \end{bmatrix}$$

be a $2 \times (M - k - l + 1)$ array, where $u_i = N - M - 1 + i$ and $v_i = M + 1 - i$, for $1 \leq i \leq M - k - l + 1$. That is,

$$W = \begin{bmatrix} N-M & N-M+1 & \cdots & -1 & 0 & 1 & \cdots & k+l-2 & k+l-1 \\ M & M-1 & \cdots & N+1 & N & N-1 & \cdots & k+l+1 & k+l \end{bmatrix}.$$

It is easy to see that $u_i + v_j = N + i - j$ for $1 \leq i, j \leq M - k - l + 1$. The array W plays an important role in our partition algorithm. From now on, we will take each number in W as a vertex of a bipartite graph.

Our algorithm will repeatedly construct a directed bipartite graph G_t on the vertices of W to obtain a desired partition of I_n . First, we define the following three types of arcs: An arc (u_i, v_i) for $1 \leq i \leq M - k - l + 1$ is called a vertical arc; an arc (u_i, v_j) with $i > j$ is called a *down oblique arc*; an arc (v_j, u_i) with $i < j$ is called an *up oblique arc*. For $i > j$, we call vertices $v_{j+1}, v_{j+2}, \dots, v_i$ the *down shadow vertices* corresponding to the down oblique arc (u_i, v_j) and for $i < j$, we call the vertices $u_{i+1}, u_{i+2}, \dots, u_j$ the *up shadow vertices* corresponding to the up oblique arc (v_j, u_i) , respectively. At any stage of the algorithm, a vertex which is not incident with any arc is said to be unsaturated. On the other hand, a vertex which is incident with an arc is said to be saturated. We define the following multisets corresponding to the set A_t of integers:

- (i) $A_t^0 = \{x \in A_t \mid x = N\}$,
- (ii) $A_t^+ = \{x \in A_t \mid x > N\}$,
- (iii) $A_t^- = \{x \in A_t \mid x < N\}$.

The basic idea of our algorithm is as follows. First, set $\overline{A}_{-1} = \emptyset$ and $A_0 = A = \{m_1, m_2, \dots, m_{k-1}\}$, compute A_0^0, A_0^+, A_0^- , respectively. Then construct a bipartite

graph G_0 on the vertices of W and compute \overline{A}_0 which will be defined in Step 7 of the following algorithm. If $\overline{A}_0 = \overline{A}_{-1}$, then each number m_r ($1 \leq r \leq k-1$) corresponds to an arc (u_i, v_j) or (v_j, u_i) in G_0 in the sense that $u_i + v_j = m_r$, we obtain the desired partition and the algorithm terminates; otherwise, set $A_1 = A \cup \overline{A}_0$, repeat this process until $\overline{A}_{t+1} = \overline{A}_t$.

Now, we are ready to describe our algorithm formally.

Algorithm 2.1.

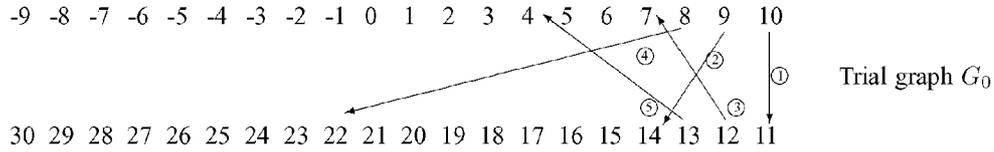
- Step 0.** Let $A = \{m_1, m_2, \dots, m_{k-1}\}$, $\overline{A}_{-1} = \emptyset$, $A_0 = A$, $t = 0$.
- Step 1.** If $|A_t^0| = s \geq 1$, then add s vertical arcs (u_i, v_i) from the right-hand side.
- Step 2.** If $A_t^+ \neq \emptyset$, then find the rightmost unsaturated vertex u_i , add the arc (u_i, v_j) such that $i - j = \min\{y \mid y \in A_t^+\}$, determine the set of down shadow vertices corresponding to (u_i, v_j) and delete $i - j$ from A_t^+ .
- Step 3.** If $A_t^- \neq \emptyset$, then find the rightmost unsaturated vertex v_j and add an arc (v_j, u_i) such that $i - j = \max\{y \mid y \in A_t^-\}$, determine the set of up shadow vertices corresponding to (v_j, u_i) and delete $i - j$ from A_t^- .
- Step 4.** Repeat the following procedure until $A_t^+ = \emptyset$ or any up shadow vertex is saturated. Find the rightmost unsaturated vertex u_i and add an arc (u_i, v_j) such that $i - j = \min\{y \mid y \in A_t^+\}$, determine the set of down shadow vertices corresponding to (u_i, v_j) and delete $i - j$ from A_t^+ .
- Step 5.** If $A_t^- \neq \emptyset$ and there exists an unsaturated down shadow vertex, then find the rightmost unsaturated vertex v_j , add an arc (v_j, u_i) such that $i - j = \max\{y \mid y \in A_t^-\}$ and delete $i - j$ from A_t^- , goto Step 4. Otherwise, goto Step 6.
- Step 6.** If $A_t^+ \cup A_t^- = \phi$, then we obtain a bipartite graph G_t , goto step 7; otherwise, if $A_t^+ \neq \phi$, goto Step 2, if $A_t^- \neq \phi$, goto Step 3.
- Step 7.** Compute the set $\overline{A}_t = \{v_j \mid v_j > n \text{ and } (u_i, v_j) \in G_t\}$.
- Step 8.** If $\overline{A}_t \neq \overline{A}_{t-1}$, then set $A_{t+1} = A \cup \overline{A}_t$, $t = t + 1$, and goto Step 1.
- Step 9.** If $v_j \leq n$ for each arc (u_i, v_j) corresponding to an element of A , then stop. Otherwise, find the corresponding arc of v_j in graph G_{t-1} .

It is not difficult to see that the algorithm is well defined, that is, the algorithm can be executed. In Section 3, we will show that the algorithm will terminate after finite iterations. For clearness, we also include an example here.

Example: $\langle 30, 23, 21, 19, 17, 10 \rangle$ is 15-realizable. Let $A = \{30, 23, 21, 19, 17\}$.

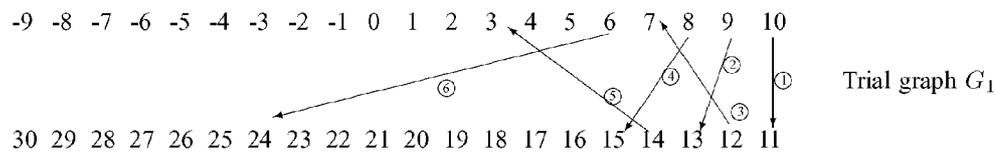
Trial-decomp

Input: $A_0 = A \cup \bar{A}_{-1} = A$. Then $n = 15, k = 6, l = 5, N = 21$ and $M = 30$.



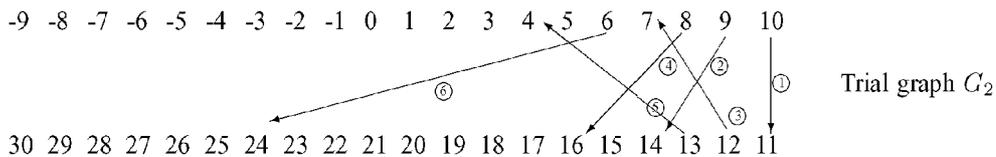
Output: $\bar{A}_0 = \{22\}$ ($\bar{A}_0 \neq \bar{A}_{-1} = \emptyset$).

Input: $A_1 = A \cup \bar{A}_0 = \{30, 23, 22, 21, 19, 17\}$.



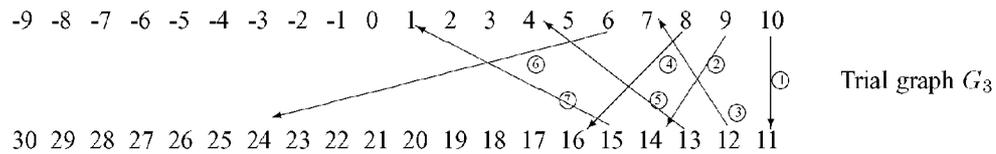
Output: $\bar{A}_1 = \{24\}$.

Input: $A_2 = A \cup \bar{A}_1 = \{30, 24, 23, 21, 19, 17\}$.



Output: $\bar{A}_2 = \{24, 16\}$.

Input: $A_3 = A \cup \bar{A}_2 = \{30, 24, 23, 21, 19, 17, 16\}$.



Output: $\bar{A}_3 = \{24, 16\}$.

$\bar{A}_3 = \bar{A}_2$ (While loop stops.).

Final-decomp

$30 \in A \subseteq A_3$ (In G_3).

$30 = 6 + 24, 24 > 15$ and $24 \in \bar{A}_3 = \bar{A}_2 \subseteq A_3$.

Hence, $24 = 8 + 16, 16 > 15$ and $16 \in \bar{A}_3 = \bar{A}_2 \subseteq A_3$.

Therefore, $16 = 1 + 15$. This implies that $S_1 = \{6, 8, 1, 15\}$ and $\sum_{x \in S_1} x = 30$.

Similarly, $S_2 = \{9, 14\}$, $S_3 = \{10, 11\}$, $S_4 = \{7, 12\}$, $S_5 = \{4, 13\}$ and $S_6 = \{2, 3, 5\}$.

3. PROOF OF MAIN RESULT

It is easy to see that each vertex in G_t has the degree less than 1 and Algorithm 2.1 terminates whenever $\bar{A}_t = \bar{A}_{t-1}$. Therefore, in order to obtain the desired partition, we have to make sure that after finite iterations, this happens. To prove this, we give the following definition.

Definition 3.1. Let $A = \{a_1, a_2, \dots, a_p\}$ and $B = \{b_1, b_2, \dots, b_q\}$ be two sets of integers such that all of their elements are arranged in a nonincreasing order, we say that A dominates B if (i) $p \geq q$ and (ii) $a_i \geq b_i$, for $1 \leq i \leq q$. In this case, we denote $A \succ B$ (or $B \prec A$).

From this definition, we have the following result immediately.

Lemma 3.1. Let A, \bar{A}_s and \bar{A}_t be three sets of integers. If $\bar{A}_s \succ \bar{A}_t$, then $\bar{A}_s \cup A \succ \bar{A}_t \cup A$.

Now, we explore the dominative relation between the sets \bar{A}_t 's. For convenience, we assume that all the elements in A_t and A_t^+ are arranged in a non-increasing order.

Lemma 3.2. At the s -th and t -th iteration of Algorithm 2.1, if $A_s^+ \succ A_t^+$, $A_s^0 = A_t^0$ and $A_s^- = A_t^-$, then $\bar{A}_s \succ \bar{A}_t$.

Proof. First, we give some notions. Let $A_s^+ = \{x_1, x_2, \dots, x_p\}$, $A_t^+ = \{y_1, y_2, \dots, y_q\}$ and $A_s^- = A_t^- = \{w_1, w_2, \dots, w_r\}$. Since $A_s^+ \succ A_t^+$, it holds that $p \geq q$ and $x_i \geq y_i$ for $1 \leq i \leq q$. For $1 \leq i \leq p$ and $1 \leq j \leq q$, denote the arcs in G_s and G_t corresponding to x_i and y_j by (u_{x_i}, v_{x_i}) and (u_{y_j}, v_{y_j}) , respectively. For $1 \leq i \leq r$, denote the arcs in G_s and G_t corresponding to w_i by $a_s(w_i)$ and $a_t(w_i)$, respectively.

Now, we consider the case that $p = q$. In this case, since $A_s^+ \succ A_t^+$, from Steps from 1 to 6, we know that for $1 \leq i \leq r$, the arcs $a_s(w_i)$ and $a_t(w_i)$ either have the same position on the array W or $a_s(w_i)$ is to the left of $a_t(w_i)$ which in turn deduces that either v_{x_i}, v_{y_i} have the same position or the vertex v_{x_i} is to the left of v_{y_i} for $1 \leq i \leq p$. Hence, $v_{x_i} \geq v_{y_i}$ for $1 \leq i \leq p$ and thus $\bar{A}_s \succ \bar{A}_t$.

Next, we consider the case that $p > q$. In this case, we construct a bipartite graph $G_{s'}$ and the set $\bar{A}_{s'}$ using Steps from 1 to 6 by inputting $A_{s'}^+ = \{x_1, x_2, \dots, x_q\}$, $A_{s'}^0 = A_s^0$ and $A_{s'}^- = A_s^-$. By the proved assertion for $p = q$, we know that $\bar{A}_{s'} \succ \bar{A}_t$. Since $A_s^+ = A_{s'}^+ \cup \{x_{q+1}, \dots, x_p\}$, and $A_s^0 = A_{s'}^0$, $A_s^- = A_{s'}^-$, we can easily deduce that $\bar{A}_s \succ \bar{A}_{s'}$. So $\bar{A}_s \succ \bar{A}_t$. ■

Similarly, we have the following result.

Lemma 3.3. *At the s -th and t -th iteration of Algorithm 2.1, if $A_s^+ = A_t^+$, $A_s^0 = A_t^0$ and $A_s^- \succ A_t^-$, then $\overline{A}_s \succ \overline{A}_t$.*

Using the lemmas above, we discuss the monotonicity of $\{\overline{A}_t\}$.

Lemma 3.4. *At the s -th and t -th iteration of Algorithm 2.1, if $A_s \succ A_t$, then $\overline{A}_s \succ \overline{A}_t$.*

Proof. Let $A_s = \{a_1, a_2, \dots, a_p\}$ and $A_t = \{b_1, b_2, \dots, b_q\}$. Since $A_s \succ A_t$, one has $p \geq q$ and $a_i \geq b_i$ for $1 \leq i \leq q$.

First, we consider the following two special cases.

Case 1. $p = q + 1$ and $a_i = b_i$ for $1 \leq i \leq q$. If $a_p < N$, then $A_s^+ = A_t^+$, $A_s^0 = A_t^0$ and $A_s^- \setminus \{a_p - N\} = A_t^-$. By Lemma 3.3, since $A_s^- \succ A_t^-$, it holds that $\overline{A}_s \succ \overline{A}_t$. On the other hand, if $a_p > N$, then $A_s^- = A_t^-$, $A_s^0 = A_t^0$ and $A_s^+ \succ A_t^+$. By Lemma 3.2, it follows that $\overline{A}_s \succ \overline{A}_t$. Finally, if $a_p = N$, then $A_s^+ = A_t^+$, $A_s^- = A_t^-$ and $A_s^0 \succ A_t^0$. In fact, G_s has one more vertical arc than G_t . Thus, every oblique arc in G_s can be obtained by shifting each oblique arc in G_t one unit to the left-hand side which shows that $\overline{A}_s \succ \overline{A}_t$.

Case 2. $p = q$ and there exists i_0 such that $a_{i_0} = b_{i_0} + 1$ and $a_i = b_i$ for each $i \in \{1, 2, \dots, q\} \setminus \{i_0\}$. We consider the following four subcases:

(i) $b_{i_0} - N > 0$.

Then $A_s^- = A_t^-$, $A_s^0 = A_t^0$ and $A_s^+ \succ A_t^+$. By Lemma 3.2, we have $\overline{A}_s \succ \overline{A}_t$.

(ii) $b_{i_0} + 1 - N < 0$.

Then $A_s^- \succ A_t^-$, $A_s^0 = A_t^0$ and $A_s^+ = A_t^+$. By Lemma 3.3, we have $\overline{A}_s \succ \overline{A}_t$.

(iii) $b_{i_0} - N = 0$.

Then $A_s^- = A_t^-$, $A_s^0 = A_t^0 \setminus \{0\}$ and $A_s^+ \setminus \{1\} = A_t^+$. Now, G_t has one more vertical arc compared with G_s , and G_s has one more down oblique arc compared with G_t . From the procedure of Algorithm 2.1, this difference makes each up oblique arc in G_s either have the same position as the corresponding up oblique arc in G_t or is on the right-hand side of the corresponding up oblique arc in G_t , which in turn implies that each down oblique arc in G_s either have the same position as the corresponding down oblique arc in G_t or is on the left-hand side of the corresponding down oblique arc in G_t using the fact that G_s has one more down oblique arc than G_t . This shows that $\overline{A}_s \succ \overline{A}_t$. By the way, the arc in G_s produced by a_i is said to be corresponded to the arc in G_t produced by b_i for $i \in \{1, 2, \dots, q\} \setminus \{i_0\}$.

(iv) $b_{i_0} + 1 - N = 0$.

Then $A_s^- = A_t^- \setminus \{-1\}$, $A_s^0 \setminus \{0\} = A_t^0$ and $A_s^+ = A_t^+$. By similar discussion as in (iii), we can show that $\overline{A}_s \succ \overline{A}_t$.

Now, let us turn to our proof of this lemma. Using the notation at the beginning of the proof, we know that there exists a series of sets of integers $\{A'_i\}_{1 \leq i \leq r}$ such that the relation between any two neighboring sets of $A_s, A'_1, A'_2, \dots, A'_r, A_t$ falls into cases 1) or 2) and $A_s \succ A'_1 \succ A'_2 \succ \dots \succ A'_r \succ A_t$. Thus the desired result is obtained using the transitive relation of domination. ■

Lemma 3.5. *At t -th iteration of Algorithm 2.1, $|\overline{A}_t| \leq l - 1$, where $l = n - 2(k - 1)$.*

Proof. By Lemmas 3.1 to 3.4 and by the induction, we can easily show that $\overline{A}_{-1} \prec \overline{A}_0 \prec \overline{A}_1 \prec \dots$. If $\overline{A}_0 = \emptyset$ then the algorithm stops when $t = 1$. Thus the lemma holds, so, without loss of generality, we assume that $\overline{A}_0 \neq \emptyset$.

Assume the assertion is not true, then there exists an integer $t \geq 1$ such that $|\overline{A}_t| \leq l - 1$ and $|\overline{A}_{t+1}| \geq l$. Suppose $\overline{A}_{t+1} = \{b_1, b_2, \dots, b_v\}$, where $v \geq l$. Set $\overline{A}'_{t+1} = \{b_1, b_2, \dots, b_{l-1}\}$ and $A'_{t+2} = A \cup \overline{A}'_{t+1}$. Construct a bipartite graph G'_{t+2} using Steps from 1 to 6 by inputting A'_{t+2} , we get $\overline{A}'_{t+2} = \{c_1, c_2, \dots, c_{v'}\}$. Since $\overline{A}'_{t+1} \succ \overline{A}_t$, so $A'_{t+2} \succ A_{t+1}$. Using Lemma 3.4, we know that $\overline{A}'_{t+2} \succ \overline{A}_{t+1}$, which in turn leads to that $\overline{A}'_{t+2} \succ \overline{A}_{t+1} \succ \overline{A}'_{t+1}$ and $v' \geq v$.

For bipartite graph G'_{t+2} , since A'_{t+2} contains $(k - 1) + (l - 1)$ elements and \overline{A}'_{t+2} contains v' elements, the number of saturated vertices in the vertex set $\{n, n - 1, \dots, k + l\}$ of G'_{t+2} is $(k - 1) + (l - 1) - v'$, so the number of unsaturated vertices in $\{n, n - 1, \dots, k + l\}$ of G'_{t+2} is $(n - (k + l) + 1) - ((k - 1) + (l - 1) - v')$, i.e., $v' - (l - 1)$, which is a positive number. So, there exists at least one unsaturated vertex in $\{n, n - 1, \dots, k + l\}$. For the leftmost up oblique arc (v_{i_0}, u_{j_0}) , since each element A'_{t+2} is strictly greater than n and $v_{i_0} < n$, it holds that $u_{j_0} > 1$. Hence, the vertex corresponding to 1 is an unsaturated vertex in G'_{t+2} . Moreover, since $|A'_{t+2}| = (k - 1) + (l - 1)$, there exists $(k + l - 1) - ((k - 1) + (l - 1))$, i.e., 1, unsaturated vertex in $\{1, 2, \dots, k + l - 1\}$, so 1 is the unique unsaturated vertex in this set. That is, each vertex in $\{N - M, N - M + 1, \dots, 0, 1\}$ are unsaturated.

Denote the set composed by the unsaturated vertex in $\{n, n - 1, \dots, k + l\}$ by $\{y_1, y_2, \dots, y_{v'-(l-1)}\}$, then

$$\begin{aligned} & m_1 + m_2 + \dots + m_{k-1} \\ &= (1 + 2 + \dots + n) - 1 - (y_1 + y_2 + \dots + y_{v'-(l-1)}) + (c_1 + c_2 + \dots + c_{v'}) \\ & \quad - (b_1 + b_2 + \dots + b_{l-1}) \\ &= \binom{n+1}{2} - 1 - (y_1 + \dots + y_{v'-(l-1)}) + (c_1 - b_1) + \dots + (c_{l-1} - b_{l-1}) \\ & \quad + (c_l + c_{l+1} + \dots + c_{v'}) \\ &\geq \binom{n+1}{2} - 1 - (y_1 + \dots + y_{v'-(l-1)}) + (c_l + c_{l+1} + \dots + c_{v'}) \end{aligned}$$

$$\begin{aligned}
 &\geq \binom{n+1}{2} - 1 - n(v' - l + 1) + (n + 1)(v' - l + 1) \\
 &= \binom{n+1}{2} + v' - l \\
 &> \binom{n+1}{2} - m_k \\
 &= m_1 + m_2 + \cdots + m_{k-1}.
 \end{aligned}$$

Where, the first equality follows from the definition of array W and the graph G'_{t+1} ; the first inequality follows from Lemma 3.4; the second inequality follows from the fact that $y_i \leq n$, for $1 \leq i \leq v' - (l - 1)$, and $c_i \geq n + 1$, for $l \leq i \leq v'$; the strict inequality follows from $v' - l \geq 0 > -m_k$. Thus, a contradiction can be derived and the desired result is obtained. ■

Lemma 3.6. For each $x \in \overline{A}_t$, $n \leq x \leq M - 2$.

Proof. Consider the bipartite graph G_t . By Lemma 3.5, we know that $|A_t| \leq (k - 1) + (l - 1)$. Since $\{1, 2, \dots, k + l - 1\}$ has the size of $k + l - 1$, by Steps from 1 to 6, we know that the starting vertex of the leftmost down oblique arc is not less than 2 and its ending vertex is not greater than $M - 2$, this completes the proof. ■

Now, we are ready to give the proof of our main result.

Proof of Theorem 1.1. As mentioned earlier in Section 2, we only need to prove the assertion for the case that $m_{k-1} > n$ and $m_k < n$. By Lemmas 3.4, 3.5 and 3.6, there exists a positive number t_0 such that $\overline{A}_{t_0} = \overline{A}_{t_0+1}$ and Step 9 is to be done. For $1 \leq i \leq k - 1$, m_i corresponds to an arc $(u_{i'}, v_{j'})$ or $(v_{j'}, u_{i'})$ in G_{t_0+1} in the sense that $m_i = u_{i'} + v_{j'}$. If $v_{j'} \leq n$, then $S_i = \{u_{i'}, v_{j'}\}$. Otherwise, $v_{j'}$ corresponds to another arc $(u_{i''}, v_{j''})$ or $(v_{j''}, u_{i''})$ in both G_{t_0} and G_{t_0+1} in the sense that $v_{j'} = u_{i''} + v_{j''}$ since $v_{j'} \in \overline{A}_{t_0}$. Continuing this process until each summand is not greater than n , then S_i can be obtained such that $\sum_{x \in S_i} x = m_i$. Since each vertex in the graph has either indegree or outdegree at most 1, we are able to get $k - 1$ mutually disjoint subsets of $\{1, 2, \dots, n\}$. Finally, letting S_k be the set of unsaturated vertices in $\{1, 2, \dots, n\}$, we obtain the desired partition and complete the proof. ■

4. CONCLUDING REMARKS

Theorem 1.1 can be restated in another equivalent form: Let m_1, m_2, \dots, m_k be positive integers such that $m_i \geq n$ for $1 \leq i \leq k$ and $\sum_{i=1}^k m_i \leq \binom{n+1}{2}$, then there exist k mutually disjoint subsets S_1, S_2, \dots, S_k of I_n such that $m_i = \sum_{x \in S_i} x$ for $1 \leq i \leq k$.

Obviously, the result obtained in this paper is more general than that of [5], but it is still very far from characterizing the n -realizable sequence $\langle m_1, m_2, \dots, m_k \rangle$. Indeed, it is an interesting topic for further research.

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