TAIWANESE JOURNAL OF MATHEMATICS
Vol. 9, No. 4, pp. 595-616, December 2005
This paper is available online at http://www.math.nthu.edu.tw/tjm/

JORDAN ϵ-HOMOMORPHISMS AND JORDAN ϵ-DERIVATIONS

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Abstract. Herstein's theorems on Jordan homomorphisms and Jordan derivations on prime associative algebras are extended to graded prime associative algebras.

1. Introduction

Let $\mathcal{A}$ be an associative algebra over a field Φ. Introducing a new product in $\mathcal{A}$, the so called Jordan product (resp. Lie product), by $x \circ y = xy + yx$ (resp. $[x, y] = xy - yx$), $\mathcal{A}$ becomes a Jordan algebra (resp. Lie algebra). In the 1950's Herstein initiated the study of the relationship between the associative and the Jordan and Lie structure of associative rings (see e.g. [9, 10]). Over the recent years, a number of Herstein's results were extended to associative superalgebras (i.e. $\mathbb{Z}_2$-graded associative algebras) [1,3-6, 11], and also to more general graded associative algebras [2, 12, 14]. In particular, a superalgebra version of Herstein's theorem on Jordan homomorphisms (resp. Jordan derivations) was proved recently in [1] by Beidar, Brešar and Chebotar (resp. in [3] by the present author). On the other hand, Bergen and Grzeszczuk [2] extended Herstein's theorems on Jordan ideals to the context of associative algebras graded by an arbitrary abelian group equipped with a bicharacter ϵ. This gives rise to a question whether the results from [1] and [3] can be extended to the more general graded context. The goal of this paper is to show that this is indeed possible.

In Section 2 we shall recall all the necessary definitions and state some preliminary results. Section 3 is devoted to Jordan ϵ-homomorphisms, and Section 4 is devoted to Jordan ϵ-derivations.

We shall make use, without explicit mention, several ideas from [1] and [3]. On the other hand, there are several problems that are trivial in the superalgebra setting,

Received December 17, 2003; accepted January 8, 2004.
Communicated by Pjek-Hwee Lee.
2000 *Mathematics Subject Classification*: 16W50, 16W10, 16W20, 16W25, 17C50.
Key words and phrases: Prime associative algebra, Graded prime associative algebra, Jordan homomorphism, Jordan derivation, ϵ-homomorphism, ϵ-derivation, Jordan ϵ-homomorphism, Jordan ϵ-derivation.

that now appear in the more general situation studied in the present paper. It should also be mentioned that the concepts of the proofs of the main results, Theorem 3.4 (on Jordan ϵ-homomorphisms) and Theorem 4.3 (on Jordan ϵ-derivations), are similar. Anyway, in some parts of the proofs there are some rather significant differences, and therefore we shall give details of both proofs.

2. Preliminaries

Throughout the paper, Φ will be a field of characteristic not 2, and by an algebra we shall always mean an algebra over Φ.

Let $\mathcal{A}$ and $\mathcal{B}$ be associative algebras. Recall that a linear map $\varphi : \mathcal{B} \to \mathcal{A}$ is called a *Jordan homomorphism* if

$$\varphi(x \circ y) = \varphi(x) \circ \varphi(y)$$

for all $x, y \in \mathcal{B}$. A *Jordan derivation* on $\mathcal{A}$ is a linear map $D : \mathcal{A} \to \mathcal{A}$ satisfying

$$D(x \circ y) = D(x) \circ y + x \circ D(y)$$

for all $x, y \in \mathcal{A}$. Let us now state the classical Herstein's theorems on Jordan homomorphisms and Jordan derivations. To be precise, we shall state their simplified versions since Herstein proved these theorems for rings and not algebras over fields. On the other hand, Herstein obtained the first theorem under the additional assumption that the characteristic of rings is different from 3. This assumption was later removed by Smiley [13] who also simplified the proof.

Theorem 2.1. [7, 13] *A Jordan homomorphism from an arbitrary associative algebra onto a prime associative algebra is either a homomorphism or an antihomomorphism.*

Theorem 2.2. [8] *A Jordan derivation on a prime associative algebra is a derivation.*

Let G be an abelian group and let $\mathcal{A}$ be a G-graded associative algebra. That is, there are subspaces $\mathcal{A}_g$, $g \in G$, of $\mathcal{A}$ such that $\mathcal{A} = \oplus_{g \in G} \mathcal{A}_g$ and $\mathcal{A}_g \mathcal{A}_h \subseteq \mathcal{A}_{gh}$ for all $g, h \in G$. We say that an element $a \in \mathcal{A}$ is *homogeneous* if $a \in \mathcal{A}_g$ for some $g \in G$. The set of all homogeneous elements in $\mathcal{A}$ will be denoted by $\mathcal{H}(\mathcal{A})$, i.e. $\mathcal{H}(\mathcal{A}) = \cup_{g \in G} \mathcal{A}_g$. A subspace $\mathcal{S}$ of $\mathcal{A}$ is said to be *graded* if $\mathcal{S} = \oplus_{g \in G} \mathcal{S} \cap \mathcal{A}_g$. We say that $\mathcal{A}$ is *graded prime* if the product of two nonzero graded ideals of $\mathcal{A}$ is always nonzero.

Let $\Phi^* = \Phi \setminus \{0\}$ and let $\epsilon : G \times G \to \Phi^*$ be a fixed anti-symmetric bicharacter. That is, ϵ is a homomorphism in each argument and $\epsilon(g, h) = \epsilon(h, g)^{-1}$ for all $g, h \in$

G. We shall use the same symbol, ϵ, to denote the map from $\mathcal{H}(\mathcal{A}) \times \mathcal{H}(\mathcal{A})$ to Φ^* defined by $\epsilon(x, y) = \epsilon(g, h)$ where $x \in \mathcal{A}_g$ and $y \in \mathcal{A}_h$. We shall also write $\epsilon(g, y)$ for $\epsilon(g, h)$ where $g \in G$ and $y \in \mathcal{A}_h$. Clearly, ϵ satisfies $\epsilon(xy, z) = \epsilon(x, z)\epsilon(y, z)$ and $\epsilon(x, y) = \epsilon(y, x)^{-1}$ for all $x, y, z \in \mathcal{H}(\mathcal{A})$. Consequently, $\epsilon(x, x) = \pm 1$ for every $x \in \mathcal{H}(\mathcal{A})$. Set $G_+ = \{g \in G \mid \epsilon(g, g) = 1\}$ and $G_- = \{g \in G \mid \epsilon(g, g) = -1\}$. We define $\mathcal{A}_+ = \oplus_{g \in G_+} \mathcal{A}_g$ and $\mathcal{A}_- = \oplus_{g \in G_-} \mathcal{A}_g$. Of course, $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$.

Introducing a new product in $\mathcal{A}$ by

$$x \circ_\epsilon y = xy + \epsilon(x, y)yx$$

(resp. $[x, y]_\epsilon = xy - \epsilon(x, y)yx$) for all $x, y \in \mathcal{H}(\mathcal{A})$, $\mathcal{A}$ becomes a *Jordan* (resp. *Lie*) *color algebra* (see [2] and [12] for details). The *ϵ-center* of $\mathcal{A}$ is $\mathcal{Z}_\epsilon(\mathcal{A}) = \{x \in \mathcal{A} \mid [y, x]_\epsilon = 0 \text{ for all } y \in \mathcal{A}\}$. Note that $\mathcal{Z}_\epsilon(\mathcal{A})$ is a graded subspace.

Given another associative G-graded algebra $\mathcal{B}$, we shall say that a linear map $\varphi : \mathcal{B} \to \mathcal{A}$ is a *Jordan ϵ-homomorphism* if it is homogeneous (that is, $\varphi(\mathcal{B}_g) \subseteq \mathcal{A}_g$ for all $g \in G$) and satisfies

$$\varphi(x \circ_\epsilon y) = \varphi(x) \circ_\epsilon \varphi(y)$$

for all $x, y \in \mathcal{H}(\mathcal{B})$. A homogeneous linear map $\varphi : \mathcal{B} \to \mathcal{A}$ will be called an *ϵ-homomorphism* (resp. *ϵ-antihomomorphism*) if $\varphi(xy) = \varphi(x)\varphi(y)$ (resp. $\varphi(xy) = \epsilon(x, y)\varphi(y)\varphi(x)$) for all $x, y \in \mathcal{H}(\mathcal{B})$. Clearly, ϵ-homomorphisms and ϵ-antihomomorphisms are examples of Jordan ϵ-homomorphisms, and of course one might ask when these are in fact the only possible examples.

In the case where $G = \{1, -1\} \cong \mathbb{Z}_2$ we call a G-graded algebra a *superalgebra*. In this context we consider the bicharacter defined by $\epsilon(-1, -1) = -1$ (and of course $\epsilon(i, j) = 1$ if one of i, j is 1). Accordingly, $\mathcal{A}_+ = \mathcal{A}_1$, which is called an even part of $\mathcal{A}$, and $\mathcal{A}_- = \mathcal{A}_{-1}$, an odd part of $\mathcal{A}$. Following [1] we shall use the term *superhomomorphism* instead of an ϵ-homomorphism in this context. Let us now state the main result of [1].

Theorem 2.3. [2] *A Jordan superhomomorphism from an arbitrary associative superalgebra onto a prime associative superalgebra $\mathcal{A}$ such that $[\mathcal{A}_1, \mathcal{A}_1] \neq 0$ is either a superhomomorphism or a superantihomomorphism.*

In [1, Examples 4 and 5] it is shown that superalgebras with commutative even part must really be excluded.

In Theorem 3.4 we shall generalize Theorem 2.3 to the case where G is an arbitrary abelian group.

Let $k \in G$. We shall say that a linear map $D_k : \mathcal{A} \to \mathcal{A}$ is a *Jordan ϵ-derivation of degree k* (resp. an *ϵ-derivation of degree k*) if $D_k(\mathcal{A}_g) \subseteq \mathcal{A}_{kg}$ and

$$D_k(x \circ_\epsilon y) = D_k(x) \circ_\epsilon y + \epsilon(k, x) x \circ_\epsilon D_k(y)$$

(resp. $D_k(xy) = D_k(x)y + \epsilon(k,x)xD_k(y)$) for all $x, y \in \mathcal{H}(\mathcal{A})$. We define a Jordan ϵ-derivation (resp. an ϵ-derivation) as a finite sum of Jordan ϵ-derivations (resp. ϵ-derivations) of different degrees. Clearly, every ϵ-derivation is also a Jordan ϵ-derivation. The question that appears is when the converse is true. For superalgebras this question was answered in [3]. We now state the main result from this paper (the terminology used should be self-explanatory).

Theorem 2.4. [3] *A Jordan superderivation on a prime associative superalgebra $\mathcal{A}$ such that $[\mathcal{A}_1, \mathcal{A}_1] \neq 0$ is a superderivation.*

The case when the even part is commutative is indeed exceptional, as shown in [3, Examples 3.3, 3.4 and 4.3].

We shall generalize Theorem 2.4 to the graded context in Theorem 4.3.

We continue by pointing out a few useful observations on graded prime associative algebras which shall be needed later. It is well-known that the primeness of an algebra $\mathcal{A}$ can be characterized by the condition that $a\mathcal{A}b = 0$, where $a, b \in \mathcal{A}$, implies $a = 0$ or $b = 0$. The first lemma is an analogous result for graded algebras, which can be proved by making some obvious modifications in the argument:

Theorem 2.5. *$\mathcal{A}$ is graded prime if and only if $a\mathcal{A}b = 0$, where $a, b \in \mathcal{H}(\mathcal{A})$, implies $a = 0$ or $b = 0$.*

Theorem 2.6. *Let $\mathcal{A}$ be graded prime and let $a \in \mathcal{H}(\mathcal{A}_+)$, $b \in \mathcal{H}(\mathcal{A}_-)$ and $c \in \mathcal{H}(\mathcal{A})$.*

(i) *If $a\mathcal{A}_+a = 0$, then $a = 0$.*

(ii) *If $c\mathcal{A}_- = 0$ (or $\mathcal{A}_-c = 0$), then $c = 0$ or $\mathcal{A}_- = 0$.*

(iii) *If $a\mathcal{A}_-b = b\mathcal{A}_-a = 0$, then $a = 0$ or $b = 0$.*

(iv) *If $a\mathcal{A}_+b = b\mathcal{A}_+a = 0$, then $a = 0$ or $b = 0$.*

(v) *If $a\mathcal{A}_-c = 0$ (or $c\mathcal{A}_-a = 0$), then $c = 0$ or $a\mathcal{A}_-a = 0$.*

(vi) *If $a\mathcal{A}_+c = 0$ (or $c\mathcal{A}_+a = 0$), then $c = 0$ or $a\mathcal{A}_-a = 0$.*

Proof. Pick any $x \in \mathcal{A}_-$. Then we have $(axa)y(axa) \in a\mathcal{A}_+a\mathcal{A}_- = 0$ for all $y \in \mathcal{A}_-$ and $(axa)z(axa) = 0$ for all $z \in \mathcal{A}_+$. Therefore $(axa)\mathcal{A}(axa) = 0$ which yields $axa = 0$ for all $x \in \mathcal{H}(\mathcal{A}_-)$ by Lemma 2.5. Thus $a\mathcal{A}_-a = 0$, which together with $a\mathcal{A}_+a = 0$ implies $a\mathcal{A}a = 0$. Hence $a = 0$ by the primeness of $\mathcal{A}$. Therefore we proved (i).

Let $c\mathcal{A}_- = 0$. Hence $c\mathcal{A}_+\mathcal{A}_- \subseteq c\mathcal{A}_- = 0$ and also $c\mathcal{A}_-\mathcal{A}_- = 0$. Therefore $c\mathcal{A}\mathcal{A}_- = 0$. By Lemma 2.5 we get $c = 0$ or $\mathcal{A}_- = 0$. In a similar fashion we can prove the same if $\mathcal{A}_-c = 0$.

Assume that $a\mathcal{A}_-b = b\mathcal{A}_-a = 0$. Whence $(axb)y(axb) = 0$ for all $x \in \mathcal{A}_+$ and $y \in \mathcal{A}_-$. On the other hand, we have $(axb)z(axb) \in a\mathcal{A}_-b = 0$ for all $x, z \in \mathcal{A}_+$. Consequently $(axb)\mathcal{A}(axb) = 0$ for all $x \in \mathcal{H}(\mathcal{A}_+)$. By Lemma 2.5 it follows that $a\mathcal{A}_+b = 0$. According to our assumption we get $a\mathcal{A}b = 0$. Again using Lemma 2.5 the result follows. Analogously we can show that (iv) holds true.

Let $a\mathcal{A}_-c = 0$. Therefore $axa\mathcal{A}_-c = 0$ for all $x \in \mathcal{A}_-$. On the other hand, we have $axa\mathcal{A}_+c \subseteq a\mathcal{A}_-c = 0$ for all $x \in \mathcal{A}_-$. These two relations yield $(axa)\mathcal{A}c = 0$ for all $x \in \mathcal{A}_-$. Since $\mathcal{A}$ is prime it follows that $a\mathcal{A}_-a = 0$ or $c = 0$. The same is true if $c\mathcal{A}_-a = 0$.

Suppose that $a\mathcal{A}_+c = 0$. Hence $axa\mathcal{A}_+c = 0$ for all $x \in \mathcal{A}_-$. We also have $axa\mathcal{A}_-c \subseteq a\mathcal{A}_+c = 0$ for all $x \in \mathcal{A}_-$. Consequently $axa\mathcal{A}c = 0$ which yields $a\mathcal{A}_-a = 0$ or $c = 0$ by the primeness of $\mathcal{A}$. Similarly we can show that the same is true if $c\mathcal{A}_+a = 0$. ■

Finally, we state three important identities, which hold true in any graded algebra $\mathcal{A}$. One can check them directly.

(1) $$[xy, z]_\epsilon = x[y, z]_\epsilon + \epsilon(y, z)[x, z]_\epsilon y, \quad x, y \in \mathcal{H}(\mathcal{A}),$$

(2) $$[[x, y]_\epsilon, z]_\epsilon = x \circ_\epsilon (y \circ_\epsilon z) - \epsilon(x, y)y \circ_\epsilon (x \circ_\epsilon z), \quad x, y \in \mathcal{H}(\mathcal{A}),$$

(3) $$2yxy = (y \circ_\epsilon x) \circ_\epsilon y - \epsilon(y, x)x \circ_\epsilon y^2, \; x \in \mathcal{H}(\mathcal{A}), \; y \in \mathcal{H}(\mathcal{A}_+).$$

3. Jordan ϵ-Homomorphisms

Throughout this section, $\mathcal{A}$ and $\mathcal{B}$ will be associative algebras graded by G, and ϵ will be a fixed bicharacter for G. Further, by φ we denote a Jordan ϵ-homomorphism from $\mathcal{B}$ onto $\mathcal{A}$. We set

$$\begin{aligned} \tau(x, y) &= \varphi(xy) - \varphi(x)\varphi(y), \\ \omega(x, y) &= \varphi(xy) - \epsilon(x, y)\varphi(y)\varphi(x) \end{aligned}$$

for all $x, y \in \mathcal{H}(\mathcal{B})$. In the case when one of x or y lies in $\mathcal{B}_1$ it follows that $\omega(x, y) = \varphi(xy) - \varphi(y)\varphi(x)$. Of course φ is an ϵ-homomorphism (resp. an ϵ-antihomomorphism) if and only if $\tau(x, y) = 0$ (resp. $\omega(x, y) = 0$) for all $x, y \in \mathcal{H}(\mathcal{B})$.

A straightforward calculation shows us that for all $x, y, z \in \mathcal{H}(\mathcal{B})$ we have

(4) $$\tau(x, y) = -\epsilon(x, y)\tau(y, x),$$

(5) $$\omega(x, y) = -\epsilon(x, y)\omega(y, x),$$

(6) $$\tau(xy, z) - \tau(x, yz) = \varphi(x)\tau(y, z) - \tau(x, y)\varphi(z),$$

(7) $$\omega(xy, z) - \omega(x, yz) = \epsilon(x, yz)\omega(y, z)\varphi(x) - \epsilon(xy, z)\varphi(z)\omega(x, y).$$

Note that (4) (resp. (5)) implies that for any $g, h \in G$, $\tau(\mathcal{A}_g, \mathcal{A}_h) = 0$ (resp. $\omega(\mathcal{A}_g, \mathcal{A}_h) = 0$) if and only if $\tau(\mathcal{A}_h, \mathcal{A}_g) = 0$ (resp. $\omega(\mathcal{A}_h, \mathcal{A}_g) = 0$). Using (2) it follows immediately from the definition of a Jordan ϵ-homomorphism that

$$\varphi([[x, y]_\epsilon, z]_\epsilon) = [[\varphi(x), \varphi(y)]_\epsilon, \varphi(z)]_\epsilon. \tag{8}$$

Lemma 3.1. *Let* $p, q, r, s \in G$.

(i) *If* $\tau(\mathcal{B}_{pq}, \mathcal{B}_{rs}) = 0$, *then* $[\tau(\mathcal{B}_p, \mathcal{B}_q), \mathcal{A}_{rs}]_\epsilon = [\mathcal{A}_{pq}, \tau(\mathcal{B}_r, \mathcal{B}_s)]_\epsilon = 0$.

(ii) *If* $\omega(\mathcal{B}_{pq}, \mathcal{B}_{rs}) = 0$, *then* $[\omega(\mathcal{B}_p, \mathcal{B}_q), \mathcal{A}_{rs}]_\epsilon = [\mathcal{A}_{pq}, \omega(\mathcal{B}_r, \mathcal{B}_s)]_\epsilon = 0$.

In particular, if $\tau(\mathcal{B}_{pq}, \mathcal{B}_{rs}) = 0$ *or* $\omega(\mathcal{B}_{pq}, \mathcal{B}_{rs}) = 0$ *then*

$$[\tau(\mathcal{B}_p, \mathcal{B}_q), \omega(\mathcal{B}_r, \mathcal{B}_s)]_\epsilon = 0.$$

Proof. Suppose that $\tau(\mathcal{B}_{pq}, \mathcal{B}_{rs}) = 0$. Note that $\tau(\mathcal{B}_{rs}, \mathcal{B}_{pq}) = 0$, as well. Pick $x \in \mathcal{B}_p$, $y \in \mathcal{B}_q$ and $z \in \mathcal{B}_{rs}$. Since $[x, y]_\epsilon \in \mathcal{B}_{pq}$ it follows that

$$\begin{aligned} \varphi([[x, y]_\epsilon, z]_\epsilon) &= \varphi([x, y]_\epsilon z) - \epsilon(xy, z)\varphi(z[x, y]_\epsilon) \\ &= \varphi([x, y]_\epsilon)\varphi(z) - \epsilon(xy, z)\varphi(z)\varphi([x, y]_\epsilon) \\ &= [\varphi([x, y]_\epsilon), \varphi(z)]_\epsilon. \end{aligned}$$

Comparing this relation with (8) we get $[\varphi([x, y]_\epsilon) - [\varphi(x), \varphi(y)]_\epsilon, \varphi(z)]_\epsilon = 0$. We have $\varphi(x \circ_\epsilon y) = \varphi(x) \circ_\epsilon \varphi(y)$ and so $[\tau(x, y), \varphi(z)]_\epsilon = 0$ for all $x \in \mathcal{B}_p$, $y \in \mathcal{B}_q$, $z \in \mathcal{B}_{rs}$. Thus $[\tau(\mathcal{B}_p, \mathcal{B}_q), \mathcal{A}_{rs}]_\epsilon = 0$. In particular, we have $[\tau(\mathcal{B}_p, \mathcal{B}_q), \omega(\mathcal{B}_r, \mathcal{B}_s)]_\epsilon \subseteq [\tau(\mathcal{B}_p, \mathcal{B}_q), \mathcal{A}_{rs}]_\epsilon = 0$. Analogously we can show that $[\mathcal{A}_{pq}, \tau(\mathcal{B}_r, \mathcal{B}_s)]_\epsilon = 0$.

Assume now that $\omega(\mathcal{B}_{pq}, \mathcal{B}_{rs}) = 0$. Since also $\omega(\mathcal{B}_{rs}, \mathcal{B}_{pq}) = 0$ it follows that

$$\begin{aligned} \varphi([[x, y]_\epsilon, z]_\epsilon) &= \epsilon(xy, z)\varphi(z)\varphi([x, y]_\epsilon) - \varphi([x, y]_\epsilon)\varphi(z) \\ &= -[\varphi([x, y]_\epsilon), \varphi(z)]_\epsilon \end{aligned}$$

for all $x \in \mathcal{B}_p$, $y \in \mathcal{B}_q$ and $z \in \mathcal{B}_{rs}$. Using (8) and the assumption that φ is a Jordan ϵ-homomorphism we get $[\omega(\mathcal{B}_p, \mathcal{B}_q), \mathcal{A}_{rs}]_\epsilon = 0$. In a similar fashion we show that $[\mathcal{A}_{pq}, \omega(\mathcal{B}_r, \mathcal{B}_s)]_\epsilon = 0$. Thereby $[\tau(\mathcal{B}_p, \mathcal{B}_q), \omega(\mathcal{B}_r, \mathcal{B}_s)]_\epsilon \subseteq [\mathcal{A}_{pq}, \omega(\mathcal{B}_r, \mathcal{B}_s),]_\epsilon = 0$ and the proof is complete. ∎

From now on we assume that $\mathcal{A}$ is graded prime and $[\mathcal{A}_1, \mathcal{A}_1] \neq 0$.

Lemma 3.2. *If* $\tau(\mathcal{B}_1, \mathcal{B}_+) = 0$, *then* φ *is an* ϵ*-homomorphism.*

Proof. Of course, $\tau(\mathcal{B}_+, \mathcal{B}_1) = 0$ as well. In particular, $\tau(\mathcal{B}_{pr}, \mathcal{B}_1) = 0$ for all $p, r \in G_-$. Using Lemma 3.1 (i) it follows that $[\tau(\mathcal{B}_p, \mathcal{B}_r), \mathcal{A}_1]_\epsilon = 0$ for all $p, r \in G_-$. Thus we have

$$[\tau(\mathcal{B}_-, \mathcal{B}_-), \mathcal{A}_1]_\epsilon = 0. \tag{9}$$

Pick any $x, y \in \mathcal{H}(\mathcal{B}_-)$ and $z \in \mathcal{B}_1$. From (6) it follows that the element $\varphi(x)\tau(y, z)$ ϵ-commutes with $\varphi(z)$ since $[\tau(x, y)\varphi(z), \varphi(z)]_\epsilon = 0$ by (1). Therefore

$$[\mathcal{A}_-\tau(\mathcal{B}_-, z), \varphi(z)]_\epsilon = 0 \tag{10}$$

for all $z \in \mathcal{B}_1$. Let $u \in \mathcal{H}(\mathcal{A}_+)$ and $v \in \mathcal{H}(\mathcal{A}_-)$. Hence $uv \in \mathcal{H}(\mathcal{A}_-)$. Next, apply (1) and (10) to derive that

$$0 = [uv\tau(y, x), \varphi(x)]_\epsilon - u[v\tau(y, x), \varphi(x)]_\epsilon = \epsilon(vyx, x)[u, \varphi(x)]_\epsilon v\tau(y, x)$$

for all $y \in \mathcal{H}(\mathcal{B}_-)$ and $x \in \mathcal{B}_1$. Consequently

$$[\mathcal{A}_+, \varphi(x)]_\epsilon \mathcal{A}_-\tau(\mathcal{B}_-, x) = 0 \tag{11}$$

for all $x \in \mathcal{B}_1$. Analogously, by (6) and (9) we get that $\tau(x, y)\varphi(z)$ ϵ-commutes with $\varphi(x)$ for all $x \in \mathcal{B}_1$, $y, z \in \mathcal{H}(\mathcal{B}_-)$, since $[\varphi(x)\tau(y, z), \varphi(x)]_\epsilon = 0$ by (1). Using (4) we also have that $\tau(y, x)\varphi(z)$ ϵ-commutes with $\varphi(x)$. That is

$$[\tau(\mathcal{B}_-, x)\mathcal{A}_-, \varphi(x)]_\epsilon = 0$$

for all $x \in \mathcal{B}_1$. Again using (1) and $vu \in \mathcal{H}(\mathcal{A}_-)$ it follows that

$$0 = [\tau(y, x)vu, \varphi(x)]_\epsilon - \epsilon(u, x)[\tau(y, x)v, \varphi(x)]_\epsilon u = \tau(y, x)v[u, \varphi(x)]_\epsilon$$

for all $y \in \mathcal{H}(\mathcal{B}_-)$ and $x \in \mathcal{B}_1$. Thus

$$\tau(\mathcal{B}_-, x)\mathcal{A}_-[\mathcal{A}_+, \varphi(x)]_\epsilon = 0 \tag{12}$$

for all $x \in \mathcal{B}_1$. Now compare (11) and (12) and note that Lemma 2.6 (iii) can be used. Whence it follows that for each $x \in \mathcal{B}_1$ either $\tau(\mathcal{B}_-, x) = 0$ or $[\mathcal{A}_+, \varphi(x)]_\epsilon = 0$. Using the fact a group $\mathcal{B}_1$ cannot be the union of its proper subgroups and the assumption $[\mathcal{A}_1, \mathcal{A}_1] \neq 0$ it follows that

$$\tau(\mathcal{B}_-, \mathcal{B}_1) = 0. \tag{13}$$

In particular, $\tau(\mathcal{B}_{pr}, \mathcal{B}_1) = 0$ for all $p \in G_-$ and $r \in G_+$. Using Lemma 3.1 (i) it follows that $[\tau(\mathcal{B}_p, \mathcal{B}_r), \mathcal{A}_1]_\epsilon = 0$. Consequently,

$$[\tau(\mathcal{B}_-, \mathcal{B}_+), \mathcal{A}_1]_\epsilon = 0. \tag{14}$$

Using (6), (13) and (14) we get $[\tau(x, y)\varphi(z), \mathcal{A}_1]_\epsilon = 0$ for all $x \in \mathcal{H}(\mathcal{B}_-)$, $y \in \mathcal{H}(\mathcal{B}_+)$ and $z \in \mathcal{B}_1$. By (1) and (14) we get $\tau(x, y)[\varphi(z), \mathcal{A}_1]_\epsilon = 0$. Using $[\mathcal{A}_1, \mathcal{A}_1] \subseteq \mathcal{A}_1$ and (14) we obtain

$$\tau(\mathcal{B}_-, \mathcal{B}_+)[\mathcal{A}_1, \mathcal{A}_1] = [\mathcal{A}_1, \mathcal{A}_1]\tau(\mathcal{B}_-, \mathcal{B}_+) = 0. \tag{15}$$

Pick any $x, z \in \mathcal{H}(\mathcal{B}_+)$ and $y \in \mathcal{H}(\mathcal{B}_-)$. In view of (6) one easily deduces from (4) and (15) that $\tau(x,y)\varphi(z)[\mathcal{A}_1,\mathcal{A}_1] = 0$ and $[\mathcal{A}_1,\mathcal{A}_1]\varphi(x)\tau(y,z) = 0$, which yields

$$\tau(\mathcal{B}_+,\mathcal{B}_-)\mathcal{A}_+[\mathcal{A}_1,\mathcal{A}_1] = [\mathcal{A}_1,\mathcal{A}_1]\mathcal{A}_+\tau(\mathcal{B}_+,\mathcal{B}_-) = 0.$$

Lemma 2.6 (iv) implies that $\tau(\mathcal{B}_+,\mathcal{B}_-) = 0$ or $[\mathcal{A}_1,\mathcal{A}_1] = 0$. According to our assumption it follows that

$$\tau(\mathcal{B}_+,\mathcal{B}_-) = 0. \tag{16}$$

Again making use of (6) we get $\tau(x,y)\varphi(z) = 0$ for all $x, y \in \mathcal{H}(\mathcal{B}_+)$ and $z \in \mathcal{H}(\mathcal{B}_-)$. Thus we have $\tau(\mathcal{B}_+,\mathcal{B}_+)\mathcal{A}_- = 0$. Using Lemma 2.6 (ii) it follows that $\mathcal{A}_- = 0$ or $\tau(\mathcal{B}_+,\mathcal{B}_+) = 0$. First assume that the second relation holds true,

$$\tau(\mathcal{B}_+,\mathcal{B}_+) = 0. \tag{17}$$

Hence by Lemma 3.1 (i) it follows that

$$[\tau(\mathcal{B}_-,\mathcal{B}_-),\mathcal{A}_+]_\epsilon = 0. \tag{18}$$

Let $x, y \in \mathcal{H}(\mathcal{B}_-)$ and $z \in \mathcal{H}(\mathcal{B}_+)$. From (6), (16), (17) and (18) we get $[\tau(x,y)\varphi(z),\mathcal{A}_+]_\epsilon = 0$. Using (1) and (18) we arrive at $\tau(x,y)[\varphi(z),\mathcal{A}_+]_\epsilon = 0$. Thus we have

$$\tau(\mathcal{B}_-,\mathcal{B}_-)[\mathcal{A}_+,\mathcal{A}_+]_\epsilon = 0. \tag{19}$$

By (18) we get

$$\tau(\mathcal{B}_-,\mathcal{B}_-)\mathcal{A}_+[\mathcal{A}_+,\mathcal{A}_+]_\epsilon = 0. \tag{20}$$

Pick any $x, y, z \in \mathcal{H}(\mathcal{B}_-)$. Multiply (6) on the right by $[\mathcal{A}_+,\mathcal{A}_+]_\epsilon$ and use (16) and (19) to get $\tau(x,y)\varphi(z)[\mathcal{A}_+,\mathcal{A}_+]_\epsilon = 0$. Therefore

$$\tau(\mathcal{B}_-,\mathcal{B}_-)\mathcal{A}_-[\mathcal{A}_+,\mathcal{A}_+]_\epsilon = 0$$

which together with (20) yields $\tau(\mathcal{B}_-,\mathcal{B}_-)\mathcal{A}[\mathcal{A}_+,\mathcal{A}_+]_\epsilon = 0$. Using the primeness of $\mathcal{A}$ and $[\mathcal{A}_1,\mathcal{A}_1] \neq 0$ it follows that $\tau(\mathcal{B}_-,\mathcal{B}_-) = 0$. Combining all our conclusions it follows that φ is an ϵ-homomorphism.

Assume now that $\mathcal{A}_- = 0$. This trivially yields $\tau(\mathcal{B}_-,\mathcal{B}_-) = 0$. From the assumption that $\tau(\mathcal{B}_+,\mathcal{B}_1) = 0$ we get

$$[\tau(\mathcal{B}_+,\mathcal{B}_+),\mathcal{A}_1]_\epsilon = 0 \tag{21}$$

by Lemma 3.1 (i). Using (6) we get $[\tau(x,y)\varphi(z),\mathcal{A}_1]_\epsilon = 0$ for all $x, y \in \mathcal{H}(\mathcal{B}_+)$ and $z \in \mathcal{B}_1$. Accordingly, $\tau(x,y)[\varphi(z),\mathcal{A}_1]_\epsilon = 0$ by (1) and (21). Therefore

$$\tau(\mathcal{B}_+,\mathcal{B}_+)[\mathcal{A}_1,\mathcal{A}_1] = 0. \tag{22}$$

Let $x, y, z \in \mathcal{H}(\mathcal{B}_+)$. Multiply (6) on the right by $[\mathcal{A}_1, \mathcal{A}_1]$ and use (22) to get $\tau(\mathcal{B}_+, \mathcal{B}_+)\mathcal{A}_+[\mathcal{A}_1, \mathcal{A}_1] = 0$. Since $\mathcal{A} = \mathcal{A}_+$ is a graded prime algebra and $[\mathcal{A}_1, \mathcal{A}_1] \neq 0$ it follows that $\tau(\mathcal{B}_+, \mathcal{B}_+) = 0$. ■

Lemma 3.3. *If $\omega(\mathcal{B}_1, \mathcal{B}_+) = 0$, then φ is an ϵ-antihomomorphism.*

Proof. The proof is a simple modification of that of Lemma 3.2, so we give only an outline. First we observe using Lemma 3.1 (ii) that

$$[\omega(\mathcal{B}_-, \mathcal{B}_-), \mathcal{A}_1]_\epsilon = 0.$$

Using (7) we derive $[\omega(y, z)\varphi(x), \varphi(z)]_\epsilon = 0$ for all $x, y \in \mathcal{H}(\mathcal{B}_-)$, $z \in \mathcal{A}_1$ and $[\varphi(z)\omega(x, y), \varphi(x)]_\epsilon = 0$ for all $x \in \mathcal{B}_1$, $y, z \in \mathcal{H}(\mathcal{B}_-)$. Therefore $[\omega(\mathcal{B}_-, x)\mathcal{A}_-, \varphi(x)]_\epsilon = [\mathcal{A}_-\omega(\mathcal{B}_-, x), \varphi(x)]_\epsilon = 0$ for all $x \in \mathcal{B}_1$. By (1) we arrive at

$$\omega(\mathcal{B}_-, x)\mathcal{A}_-[\mathcal{A}_+, \varphi(x)]_\epsilon = [\mathcal{A}_+, \varphi(x)]_\epsilon\mathcal{A}_-\omega(\mathcal{B}_-, x) = 0$$

for all $x \in \mathcal{B}_1$. Lemma 2.6 (iii) implies

$$\omega(\mathcal{B}_-, \mathcal{B}_1) = 0.$$

Again making use of Lemma 3.1 (ii) it follows that

$$[\omega(\mathcal{B}_-, \mathcal{B}_+), \mathcal{A}_1]_\epsilon = 0.$$

By (7) we arrive at $[\varphi(z)\omega(x, y), \mathcal{A}_1]_\epsilon = 0$ for all $x \in \mathcal{H}(\mathcal{B}_-)$, $y \in \mathcal{H}(\mathcal{B}_+)$ and $z \in \mathcal{B}_1$, and hence

$$\omega(\mathcal{B}_-, \mathcal{B}_+)[\mathcal{A}_1, \mathcal{A}_1] = [\mathcal{A}_1, \mathcal{A}_1]\omega(\mathcal{B}_-, \mathcal{B}_+) = 0$$

by (1). From (7) we derive that $\omega(y, z)\varphi(x)[\mathcal{A}_1, \mathcal{A}_1] = [\mathcal{A}_1, \mathcal{A}_1]\varphi(z)\omega(x, y) = 0$ for all $x, z \in \mathcal{H}(\mathcal{B}_+)$, $y \in \mathcal{H}(\mathcal{B}_-)$. Thus we have

$$\omega(\mathcal{B}_-, \mathcal{B}_+)\mathcal{A}_+[\mathcal{A}_1, \mathcal{A}_1] = [\mathcal{A}_1, \mathcal{A}_1]\mathcal{A}_+\omega(\mathcal{B}_-, \mathcal{B}_+) = 0.$$

In view of Lemma 2.6 (iv) we get

$$\omega(\mathcal{B}_-, \mathcal{B}_+) = 0.$$

Again using (7) we arrive at $\omega(y, z)\varphi(x) = 0$ for all $x \in \mathcal{H}(\mathcal{B}_-)$ and $y, z \in \mathcal{H}(\mathcal{B}_+)$. Hence $\omega(\mathcal{B}_+, \mathcal{B}_+)\mathcal{A}_- = 0$. Lemma 2.6 (ii) implies $\mathcal{A}_- = 0$ or $\omega(\mathcal{B}_+, \mathcal{B}_+) = 0$. Suppose that

$$\omega(\mathcal{B}_+, \mathcal{B}_+) = 0.$$

Using Lemma 3.1 (ii) we infer

$$[\omega(\mathcal{B}_-,\mathcal{B}_-),\mathcal{A}_+]_\epsilon = 0.$$

By (7) we get $[\varphi(z)\omega(x,y),\mathcal{A}_+]_\epsilon = 0$ for all $x,y\in\mathcal{H}(\mathcal{B}_-)$, $z\in\mathcal{H}(\mathcal{B}_+)$, which yields

$$\omega(\mathcal{B}_-,\mathcal{B}_-)[\mathcal{A}_+,\mathcal{A}_+]_\epsilon = 0.$$

Therefore also

$$\omega(\mathcal{B}_-,\mathcal{B}_-)\mathcal{A}_+[\mathcal{A}_+,\mathcal{A}_+]_\epsilon = 0.$$

Using (7) we get $\omega(y,z)\varphi(x)[\mathcal{A}_+,\mathcal{A}_+]_\epsilon = 0$ for all $x,y,z\in\mathcal{H}(\mathcal{B}_-)$. Hence

$$\omega(\mathcal{B}_-,\mathcal{B}_-)\mathcal{A}_-[\mathcal{A}_+,\mathcal{A}_+]_\epsilon = 0$$

which in turn implies $\omega(\mathcal{B}_-,\mathcal{B}_-)\mathcal{A}[\mathcal{A}_+,\mathcal{A}_+]_\epsilon = 0$. According to our assumption that $\mathcal{A}$ is prime and $[\mathcal{A}_1,\mathcal{A}_1]\neq 0$ it follows that $\omega(\mathcal{B}_-,\mathcal{B}_-) = 0$.

Finally, suppose that $\mathcal{A}_- = 0$. First, this trivially implies $\omega(\mathcal{B}_-,\mathcal{B}_-) = 0$. Since $\omega(\mathcal{B}_1,\mathcal{B}_+) = 0$ it follows from Lemma 3.1 (ii) that

$$[\omega(\mathcal{B}_+,\mathcal{B}_+),\mathcal{A}_1]_\epsilon = 0.$$

Therefore one can deduce from (7) that $[\varphi(z)\omega(x,y),\mathcal{A}_1]_\epsilon = 0$ for all $x,y\in\mathcal{H}(\mathcal{B}_+)$, $z\in\mathcal{B}_1$. Hence

$$[\mathcal{A}_1,\mathcal{A}_1]\omega(\mathcal{B}_+,\mathcal{B}_+) = 0.$$

Again making use of (7) we arrive at $[\mathcal{A}_1,\mathcal{A}_1]\varphi(z)\omega(x,y) = 0$ for all $x,y,z\in\mathcal{H}(\mathcal{B}_+)$. Thus $[\mathcal{A}_1,\mathcal{A}_1]\mathcal{A}_+\omega(\mathcal{B}_+,\mathcal{B}_+) = 0$. Since $\mathcal{A} = \mathcal{A}_+$ is a graded prime algebra it follows that $\omega(\mathcal{B}_+,\mathcal{B}_+) = 0$. Hence ω is an ϵ-antihomomorphism, as desired. ■

Theorem 3.4. *Let φ be a Jordan ϵ-homomorphism from an arbitrary graded associative algebra $\mathcal{B}$ onto a graded prime associative algebra $\mathcal{A}$ such that $[\mathcal{A}_1,\mathcal{A}_1]\neq 0$. Then φ is either an ϵ-homomorphism or an ϵ-antihomomorphism.*

Proof. Using (3) and the assumption that φ is a Jordan ϵ-homomorphism it follows that

$$\varphi(axa) = \varphi(a)\varphi(x)\varphi(a) \tag{23}$$

for all $a\in\mathcal{H}(\mathcal{B}_+)$ and $x\in\mathcal{H}(\mathcal{B})$. Linearizing (23) we get

$$\varphi(axb+bxa) = \varphi(a)\varphi(x)\varphi(b)+\varphi(b)\varphi(x)\varphi(a) \tag{24}$$

for all $a, b \in \mathcal{B}_g$, $g \in G_+$ and $x \in \mathcal{H}(\mathcal{B})$. Pick any $x \in \mathcal{H}(\mathcal{B})$ and $a, b \in \mathcal{H}(\mathcal{B}_+)$. Using (23) we get

$$\begin{aligned}\varphi(abxba + baxab) &= \varphi(a(bxb)a + b(axa)b)\\ &= \varphi(a)\varphi(bxb)\varphi(a) + \varphi(b)\varphi(axa)\varphi(b)\\ &= \varphi(a)\varphi(b)\varphi(x)\varphi(b)\varphi(a) + \varphi(b)\varphi(a)\varphi(x)\varphi(a)\varphi(b).\end{aligned}$$

On the other hand, using (24) we get

$$\begin{aligned}\varphi(abxba + baxab) &= \varphi((ab)x(ba) + (ba)x(ab))\\ &= \varphi(ab)\varphi(x)\varphi(ba) + \varphi(ba)\varphi(x)\varphi(ab).\end{aligned}$$

Comparing identities so obtained and using $\varphi(a \circ_\epsilon b) = \varphi(a) \circ_\epsilon \varphi(b)$ it follows that

$$\begin{aligned}0 &= \varphi(ab)\varphi(x)(-\epsilon(b,a)\varphi(ab) + \varphi(b)\varphi(a) + \epsilon(b,a)\varphi(a)\varphi(b))\\ &\quad +(-\epsilon(b,a)\varphi(ab) + \varphi(b)\varphi(a) + \epsilon(b,a)\varphi(a)\varphi(b))\varphi(x)\varphi(ab)\\ &\quad -\varphi(a)\varphi(b)\varphi(x)\varphi(b)\varphi(a) - \varphi(b)\varphi(a)\varphi(x)\varphi(a)\varphi(b)\\ &= \tau(a,b)\varphi(x)\varphi(b)\varphi(a) + \epsilon(b,a)\omega(a,b)\varphi(x)\varphi(a)\varphi(b)\\ &\quad -\epsilon(b,a)\tau(a,b)\varphi(x)\varphi(ab) - \epsilon(b,a)\omega(a,b)\varphi(x)\varphi(ab)\\ &= -\epsilon(b,a)(\tau(a,b)\varphi(x)\omega(a,b) + \omega(a,b)\varphi(x)\tau(a,b)).\end{aligned}$$

for all $x \in \mathcal{H}(\mathcal{B})$ and $a, b \in \mathcal{H}(\mathcal{B}_+)$. Hence

$$\tau(a,b)\varphi(x)\omega(a,b) + \omega(a,b)\varphi(x)\tau(a,b) = 0 \tag{25}$$

for all $x \in \mathcal{H}(\mathcal{B})$ and $a, b \in \mathcal{H}(\mathcal{B}_+)$. Let $a, b \in \mathcal{H}(\mathcal{B}_+)$ and write $\tau = \tau(a,b)$ and $\omega = \omega(a,b)$ for brevity. Using that φ is onto it follows that (25) can be written as $\tau y\omega + \omega y\tau = 0$ for all $y \in \mathcal{A}$. Therefore $\tau y(\omega z\tau) = -\tau(y\tau z)\omega = (\omega y\tau)z\tau = -\tau y\omega z\tau$ for all $y, z \in \mathcal{A}$. Hence $\tau\mathcal{A}\omega\mathcal{A}\tau = 0$. Since $\mathcal{A}$ is prime it follows by Lemma 2.5 that $\tau = 0$ or $\omega = 0$. Let $g, h \in G_+$ and let us show that one of these two conditions is fulfilled for all $a \in \mathcal{B}_h$ and $b \in \mathcal{B}_g$. For any fixed $a \in \mathcal{B}_h$ the sets $\{b \in \mathcal{B}_g \mid \tau(a,b) = 0\}$ and $\{b \in \mathcal{B}_g \mid \omega(a,b) = 0\}$ are additive subgroups of $\mathcal{B}_g$ whose union is, by what we proved, equal to $\mathcal{B}_g$. Since a group cannot be the union of its proper subgroups, it follows that either $\tau(a, \mathcal{B}_g) = 0$ or $\omega(a, \mathcal{B}_g) = 0$. Therefore $\mathcal{B}_h$ is the union of its additive subgroups $\{a \in \mathcal{B}_h \mid \tau(a, \mathcal{B}_g) = 0\}$ and $\{a \in \mathcal{B}_h \mid \omega(a, \mathcal{B}_g) = 0\}$, and so one of them equals $\mathcal{B}_h$. Consequently, for each pair $g, h \in G_+$ either $\tau(\mathcal{B}_h, \mathcal{B}_g) = 0$ or $\omega(\mathcal{B}_h, \mathcal{B}_g) = 0$.

Let us show that either $\tau(\mathcal{B}_1, \mathcal{B}_g) = 0$ for all $g \in G_+$ or $\omega(\mathcal{B}_1, \mathcal{B}_g) = 0$ for all $g \in G_+$. Assume that $\omega(\mathcal{B}_1, \mathcal{B}_g) \neq 0$ and $\tau(\mathcal{B}_1, \mathcal{B}_h) \neq 0$ for some $g, h \in G_+$. Therefore, by what we proved,

$$\tau(\mathcal{B}_1, \mathcal{B}_g) = 0 \text{ and } \omega(\mathcal{B}_1, \mathcal{B}_h) = 0. \tag{26}$$

Further, we have $\tau(\mathcal{B}_1,\mathcal{B}_1)=0$ or $\omega(\mathcal{B}_1,\mathcal{B}_1)=0$. If $\tau(\mathcal{B}_1,\mathcal{B}_1)=\omega(\mathcal{B}_1,\mathcal{B}_1)=0$ then $[\mathcal{A}_1,\mathcal{A}_1]=0$, a contradiction. Now we may assume that $\tau(\mathcal{B}_1,\mathcal{B}_1)\neq 0$. Namely, $\tau(\mathcal{B}_1,\mathcal{B}_1)=0$ implies $\omega(\mathcal{B}_1,\mathcal{B}_1)\neq 0$ and the remaining proof is similar, so we omit it. We have $\tau(\mathcal{B}_1,\mathcal{B}_{gpr})=0$ or $\omega(\mathcal{B}_1,\mathcal{B}_{gpr})=0$ and also $\tau(\mathcal{B}_1,\mathcal{B}_{pr})=0$ or $\omega(\mathcal{B}_1,\mathcal{B}_{pr})=0$ for all $p,r\in G_-$. Since $\mathcal{B}_i\mathcal{B}_j\subseteq\mathcal{B}_{ij}$ for all $i,j\in G$ it follows by Lemma 3.1 that

$$[\tau(\mathcal{B}_1,\mathcal{B}_1),\omega(\mathcal{B}_g\mathcal{B}_p,\mathcal{B}_r)]_\epsilon=0, \tag{27}$$

$$[\tau(\mathcal{B}_1,\mathcal{B}_1),\omega(\mathcal{B}_g,\mathcal{B}_p\mathcal{B}_r)]_\epsilon=0, \tag{28}$$

$$[\tau(\mathcal{B}_1,\mathcal{B}_1),\omega(\mathcal{B}_p,\mathcal{B}_r)]_\epsilon=0. \tag{29}$$

Since $\tau(\mathcal{B}_1,\mathcal{B}_g)=0$ we get $[\tau(\mathcal{B}_1,\mathcal{B}_1),\mathcal{A}_g]_\epsilon=0$ by Lemma 3.1 (i). By (1) and (29) we arrive at

$$\begin{aligned}[\tau(\mathcal{B}_1,\mathcal{B}_1),\omega(\mathcal{B}_p,\mathcal{B}_r)\mathcal{A}_g]_\epsilon &\subseteq [\tau(\mathcal{B}_1,\mathcal{B}_1),\omega(\mathcal{B}_p,\mathcal{B}_r)]_\epsilon\mathcal{A}_g\\ &\quad+\omega(\mathcal{B}_p,\mathcal{B}_r)[\tau(\mathcal{B}_1,\mathcal{B}_1),\mathcal{A}_g]_\epsilon=0.\end{aligned} \tag{30}$$

Pick any $x\in\mathcal{B}_g$, $y\in\mathcal{B}_p$ and $z\in\mathcal{B}_r$. Using (7), (27), (28) and (30) it follows that $[\tau(\mathcal{B}_1,\mathcal{B}_1),\varphi(z)\omega(x,y)]_\epsilon=0$. Hence $[\tau(\mathcal{B}_1,\mathcal{B}_1),\mathcal{A}_r\omega(\mathcal{B}_g,\mathcal{B}_p)]_\epsilon=0$ for all $p,r\in G_-$. Using (1) it follows that

$$\begin{aligned}[\tau(\mathcal{B}_1,\mathcal{B}_1),\mathcal{A}_s]_\epsilon\mathcal{A}_r\omega(\mathcal{B}_g,\mathcal{B}_p) &\subseteq [\tau(\mathcal{B}_1,\mathcal{B}_1),\mathcal{A}_s\mathcal{A}_r\omega(\mathcal{B}_g,\mathcal{B}_p)]_\epsilon\\ &\quad+\mathcal{A}_s[\tau(\mathcal{B}_1,\mathcal{B}_1),\mathcal{A}_r\omega(\mathcal{B}_g,\mathcal{B}_p)]_\epsilon=0\end{aligned}$$

for all $s\in G_+$ and $p,r\in G_-$. Thus we have

$$[\tau(\mathcal{B}_1,\mathcal{B}_1),\mathcal{A}_+]_\epsilon\mathcal{A}_-\omega(\mathcal{B}_g,\mathcal{B}_-)=0. \tag{31}$$

In a similar fashion, by considering the condition that $\tau(\mathcal{B}_1,\mathcal{B}_{rpg})=0$ or $\omega(\mathcal{B}_1,\mathcal{B}_{rpg})=0$, we get $[\tau(\mathcal{B}_1,\mathcal{B}_1),\omega(\mathcal{B}_g,\mathcal{B}_p)\mathcal{A}_r]_\epsilon=0$ which yields

$$\omega(\mathcal{B}_g,\mathcal{B}_-)\mathcal{A}_-[\tau(\mathcal{B}_1,\mathcal{B}_1),\mathcal{A}_+]_\epsilon=0. \tag{32}$$

Compare (31) and (32) and note that Lemma 2.6 (iii) can be used. Thus $\omega(\mathcal{B}_g,\mathcal{B}_-)=0$ or $[\tau(\mathcal{B}_1,\mathcal{B}_1),\mathcal{A}_+]_\epsilon=0$.

Suppose that

$$[\tau(\mathcal{B}_1,\mathcal{B}_1),\mathcal{A}_+]_\epsilon=0. \tag{33}$$

It is easy to see that $\omega(a,b)-\tau(a,b)=[\varphi(a),\varphi(b)]_\epsilon$ for all $a,b\in\mathcal{H}(\mathcal{B})$. Therefore, given $a,b\in\mathcal{B}_1$ we have $\omega(a,b)=0$ since $\omega(\mathcal{B}_1,\mathcal{B}_1)=0$. Hence $\tau(a,b)=$

$-[\varphi(a),\varphi(b)]$. Consequently $[[\varphi(a),\varphi(b)],\mathcal{A}_+]_\epsilon = 0$ by (33). Since φ is onto it follows that $[[x,y],\mathcal{A}_+]_\epsilon = 0$ for all $x,y \in \mathcal{A}_1$. In particular, $[[x^2,y],z]_\epsilon = 0$ for all $z \in \mathcal{H}(\mathcal{A}_+)$. Using (1) we see that $[x[x,y]+[x,y]x,z]_\epsilon = [2x[x,y],z]_\epsilon = 0$. which in turn implies $[x,z]_\epsilon[x,y] = 0$. Since $[x,y]\epsilon$-commutes with $\mathcal{A}_+$ it follows that $[x,z]_\epsilon\mathcal{A}_+[x,y]_\epsilon = 0$ for all $z \in \mathcal{H}(\mathcal{A}_+)$. Therefore $[x,y]\mathcal{A}_+[x,y] = 0$ for all $x,y \in A_1$. Consequently, by Lemma 2.6 (i) we get $[x,y] = 0$ for all $x,y \in \mathcal{A}_1$, a contradiction.

Therefore $\omega(\mathcal{B}_g,\mathcal{B}_-) = 0$. Lemma 3.1 (ii) implies

$$[\omega(\mathcal{B}_g,\mathcal{B}_1),\mathcal{A}_-]_\epsilon = 0. \tag{34}$$

Pick $x \in \mathcal{B}_g$, $y \in \mathcal{B}_1$, $z \in \mathcal{H}(\mathcal{A}_+)$ and $w \in \mathcal{H}(\mathcal{A}_-)$. Using (1) it follows that

$$[\omega(x,y),z]_\epsilon w \in [\omega(\mathcal{B}_g,\mathcal{B}_1),\mathcal{A}_+\mathcal{A}_-]_\epsilon + \mathcal{A}_+[\omega(\mathcal{B}_g,\mathcal{B}_1),\mathcal{A}_-]_\epsilon = 0,$$

which yields $[\omega(\mathcal{B}_g,\mathcal{B}_1),\mathcal{A}_+]_\epsilon\mathcal{A}_- = 0$. Hence $[\omega(\mathcal{B}_g,\mathcal{B}_1),\mathcal{A}_+]_\epsilon = 0$ or $\mathcal{A}_- = 0$ by Lemma 2.6 (ii). Suppose that the first relation holds true. Together with (34) this yields

$$\omega(\mathcal{B}_g,\mathcal{B}_1) \subseteq \mathcal{Z}_\epsilon(\mathcal{A}).$$

By what we proved we have $\omega(\mathcal{B}_g,\mathcal{B}_h) = 0$ or $\tau(\mathcal{B}_g,\mathcal{B}_h) = 0$. Let us show that in both cases we get

$$\omega(\mathcal{B}_1,\mathcal{B}_g)\tau(\mathcal{B}_1,\mathcal{B}_h) = 0. \tag{35}$$

Suppose first that $\omega(\mathcal{B}_g,\mathcal{B}_h) = 0$. Hence $\omega(\mathcal{B}_h,\mathcal{B}_g) = 0$, as well. Therefore, given $x \in \mathcal{B}_h$, $y \in \mathcal{B}_1$ and $z \in \mathcal{B}_g$ we have $\omega(y,z)\varphi(x) = 0$ by (7) and (26). Thus we have $\omega(\mathcal{B}_1,\mathcal{B}_g)\mathcal{A}_h = 0$, which implies $\omega(\mathcal{B}_1,\mathcal{B}_g)\tau(\mathcal{B}_1,\mathcal{B}_h) \subseteq \omega(\mathcal{B}_1,\mathcal{B}_g)\mathcal{A}_h = 0$.

Suppose now that $\tau(\mathcal{B}_g,\mathcal{B}_h) = 0$. In a similar fashion, by using (6) and (26) we get $\mathcal{A}_g\tau(\mathcal{B}_1,\mathcal{B}_h) = 0$. Hence $\omega(\mathcal{B}_1,\mathcal{B}_g)\tau(\mathcal{B}_1,\mathcal{B}_h) \subseteq \mathcal{A}_g\tau(\mathcal{B}_1,\mathcal{B}_h) = 0$, as desired. Since $\omega(\mathcal{B}_g,\mathcal{B}_1) \subseteq \mathcal{Z}_\epsilon(\mathcal{A})$ it follows that $\omega(\mathcal{B}_1,\mathcal{B}_g)\mathcal{A}\tau(\mathcal{B}_1,\mathcal{B}_h) = 0$. The primeness of $\mathcal{A}$ yields $\omega(\mathcal{B}_1,\mathcal{B}_g) = 0$ or $\tau(\mathcal{B}_1,\mathcal{B}_h) = 0$, a contradiction.

Suppose now that $\mathcal{A}_- = 0$. Hence $\mathcal{A} = \mathcal{A}_+$. We proved (35) for every $h \in G_+$ such that $\tau(\mathcal{B}_1,\mathcal{B}_h) \neq 0$. But this relation is also true when $\tau(\mathcal{B}_1,\mathcal{B}_h) = 0$. Hence $\omega(\mathcal{B}_1,\mathcal{B}_g)\tau(\mathcal{B}_1,\mathcal{B}_+) = 0$. Making use of (6) it follows that $\omega(\mathcal{B}_1,\mathcal{B}_g)\varphi(x)\tau(y,z) = 0$ for all $x \in \mathcal{H}(\mathcal{B}_+)$ and $y,z \in \mathcal{B}_1$. Thus $\omega(\mathcal{B}_1,\mathcal{B}_g)\mathcal{A}_+\tau(\mathcal{B}_1,\mathcal{B}_1) = 0$ which yields $\omega(\mathcal{B}_1,\mathcal{B}_g) = 0$ or $\tau(\mathcal{B}_1,\mathcal{B}_1) = 0$, a contradiction.

We proved that $\tau(\mathcal{B}_1,\mathcal{B}_+) = 0$ or $\omega(\mathcal{B}_1,\mathcal{B}_+) = 0$. Using Lemma 3.2 and Lemma 3.3 the result follows. ■

Example 3.5. Let $\mathcal{A}$ be a graded prime algebra with unity such that $[\mathcal{A}_+,\mathcal{A}]_\epsilon = 0$ and $\mathcal{A}_- \circ_\epsilon \mathcal{A}_- = 0$ (cf. [2, Proposition 2.7] where such algebras are described in greater detail). Suppose there is an invertible $a \in \mathcal{A}_1$ such that $a^2 \neq \pm 1$. Now define $\varphi : \mathcal{A} \to \mathcal{A}$ by $\varphi(x+y) = x + ay$ for all $x \in \mathcal{H}(\mathcal{A}_+)$ and $y \in \mathcal{H}(\mathcal{A}_-)$. Then φ is a Jordan ϵ-automorphism which is neither an ϵ-automorphism nor an ϵ-antiautomorphism. Note that this example is actually an extension of Example 4 in [1].

4. Jordan ϵ-Derivations

Let us fix the notation. Throughout, $\mathcal{A}$ will be an associative algebra graded by G, and ϵ will be a bicharacter. We fix $k \in G$ and let D_k be a Jordan ϵ-derivation of degree k on $\mathcal{A}$. We set

$$\delta_k(x,y) \;\; = \;\; D_k(xy) - D_k(x)y - \epsilon(k,x)xD_k(y)$$

for all elements $x, y \in \mathcal{H}(\mathcal{A})$. In the case when x lies in $\mathcal{A}_1$ it follows that $\delta_k(x,y) = D_k(xy) - D_k(x)y - xD_k(y)$. Of course δ_k is an ϵ-derivation of degree k if and only if $\delta_k(x,y) = 0$ for all $x, y \in \mathcal{H}(\mathcal{A})$.

A straightforward calculation shows us that

$$\delta_k(x,y) = -\epsilon(x,y)\delta_k(y,x), \tag{36}$$

$$\delta_k(xy,z) + \delta_k(x,y)z = \delta_k(x,yz) + \epsilon(k,x)x\delta_k(y,z) \tag{37}$$

for all $x, y, z \in \mathcal{H}(\mathcal{A})$. Note that (36) implies that $\delta_k(\mathcal{A}_g, \mathcal{A}_h) = 0$ if and only if $\delta_k(\mathcal{A}_h, \mathcal{A}_g) = 0$, $g, h \in G$. Using (2) it follows immediately from the definition of a Jordan ϵ-derivation that

$$\begin{aligned} D_k([[x,y]_\epsilon, z]_\epsilon) &= [[D_k(x), y]_\epsilon, z]_\epsilon + \epsilon(k,x)[[x, D_k(y)]_\epsilon, z]_\epsilon \\ &\quad + \epsilon(k,xy)[[x,y]_\epsilon, D_k(z)]_\epsilon. \end{aligned} \tag{38}$$

Lemma 4.1. *Let $p, q, r \in G$. If $\delta_k(\mathcal{A}_{pq}, \mathcal{A}_r) = 0$, then $[\delta_k(\mathcal{A}_p, \mathcal{A}_q), \mathcal{A}_r]_\epsilon = 0$.*

Proof. Let $x \in \mathcal{A}_p$, $y \in \mathcal{A}_q$ and $z \in \mathcal{A}_r$. Since $[x,y]_\epsilon \in \mathcal{A}_{pq}$ it follows that

$$\begin{aligned} D_k([[x,y]_\epsilon, z]_\epsilon) &= D_k([x,y]_\epsilon z) - \epsilon(xy,z)D_k(z[x,y]_\epsilon) \\ &= D_k([x,y]_\epsilon)z + \epsilon(k,xy)[x,y]_\epsilon D_k(z) \\ &\quad -\epsilon(xy,z)D_k(z)[x,y]_\epsilon - \epsilon(kxy,z)zD_k([x,y]_\epsilon) \\ &= [D_k([x,y]_\epsilon), z]_\epsilon + \epsilon(k,xy)[[x,y]_\epsilon, D_k(z)]_\epsilon. \end{aligned}$$

Comparing this relation with (38) and using that $D_k(x \circ_\epsilon y) = D_k(x) \circ_\epsilon y + \epsilon(k,x)x \circ_\epsilon D_k(y)$ we arrive at $[\delta_k(x,y), z]_\epsilon = 0$ for all $x \in \mathcal{A}_p$, $y \in \mathcal{A}_q$ and $z \in \mathcal{A}_r$. ∎

From now on $\mathcal{A}$ will be graded prime and $[\mathcal{A}_1, \mathcal{A}_1] \neq 0$.

Lemma 4.2. *If $\delta_k(\mathcal{A}_1, \mathcal{A}_+) = 0$, then D_k is an ϵ-derivation.*

Proof. Note that $\delta_k(\mathcal{A}_+, \mathcal{A}_1) = 0$ as well. Therefore Lemma 4.1 implies

$$[\delta_k(\mathcal{A}_+, \mathcal{A}_+), \mathcal{A}_1]_\epsilon = 0. \tag{39}$$

By (37) we get $[\delta_k(x, y)z, \mathcal{A}_1] = 0$ for all $x, y \in \mathcal{H}(\mathcal{A}_+)$ and $z \in \mathcal{A}_1$. Using (1) and (39) we arrive at $\delta_k(x, y)[z, \mathcal{A}_1] = 0$. Thus

$$\delta_k(\mathcal{A}_+, \mathcal{A}_+)[\mathcal{A}_1, \mathcal{A}_1] = [\mathcal{A}_1, \mathcal{A}_1]\delta_k(\mathcal{A}_+, \mathcal{A}_+) = 0. \tag{40}$$

Let $x, y, z \in \mathcal{H}(\mathcal{A}_+)$. Multiply (37) on the right by $[\mathcal{A}_1, \mathcal{A}_1]$. Using (40) it follows that $\delta_k(x, y)z[\mathcal{A}_1, \mathcal{A}_1] = 0$. Similarly, by multiplying (37) on the left by $[\mathcal{A}_1, \mathcal{A}_1]$ we get $[\mathcal{A}_1, \mathcal{A}_1]x\delta_k(y, z) = 0$, which yields

$$\delta_k(\mathcal{A}_+, \mathcal{A}_+)\mathcal{A}_+[\mathcal{A}_1, \mathcal{A}_1] = [\mathcal{A}_1, \mathcal{A}_1]\mathcal{A}_+\delta_k(\mathcal{A}_+, \mathcal{A}_+) = 0. \tag{41}$$

Again using our assumption that $\delta_k(\mathcal{A}_+, \mathcal{A}_1) = 0$ it follows by Lemma 4.1 that

$$[\delta_k(\mathcal{A}_-, \mathcal{A}_-), \mathcal{A}_1]_\epsilon = 0. \tag{42}$$

Therefore (37) implies that $[x\delta_k(y, z), z]_\epsilon = 0$ for all $x, y \in \mathcal{H}(\mathcal{A}_-)$ and $z \in \mathcal{A}_1$. Hence by (1) it follows that

$$0 = [wx\delta_k(y, z), z]_\epsilon - w[x\delta_k(y, z), z]_\epsilon = [w, z]_\epsilon x\delta_k(y, z)$$

for all $w \in \mathcal{H}(\mathcal{A}_+)$, $x, y \in \mathcal{H}(\mathcal{A}_-)$ and $z \in \mathcal{A}_1$. Thus we have

$$[\mathcal{A}_+, z]\mathcal{A}_-\delta_k(\mathcal{A}_-, z) = 0 \tag{43}$$

for all $z \in \mathcal{A}_1$. Similarly, by (37) and (42) we have $[\delta_k(x, y)z, x]_\epsilon = 0$ for all $x \in \mathcal{A}_1$, $y, z \in \mathcal{H}(\mathcal{A}_-)$, and hence $\delta_k(x, y)z[w, x]_\epsilon = 0$ for all $w \in \mathcal{H}(\mathcal{A}_+)$. This implies

$$\delta_k(\mathcal{A}_-, x)\mathcal{A}_-[\mathcal{A}_+, x] = 0 \tag{44}$$

for all $x \in \mathcal{A}_1$. We now divide the proof into two parts: when $k \in G_+$ and when $k \in G_-$.

Case 1. Suppose first that $k \in G_+$. Now compare the identities (43) and (44) and note that Lemma 2.6 (iii) can be used. Therefore for each $x \in \mathcal{A}_1$ either

$\delta_k(\mathcal{A}_-, x) = 0$ or $[\mathcal{A}_+, x] = 0$. Using the fact that a group cannot be the union of its proper subgroups and the assumption $[\mathcal{A}_1, \mathcal{A}_1] \neq 0$ it follows that

$$\delta_k(\mathcal{A}_-, \mathcal{A}_1) = 0.$$

Using Lemma 4.1 we arrive at

$$[\delta_k(\mathcal{A}_-, \mathcal{A}_+), \mathcal{A}_1]_\epsilon = 0. \tag{45}$$

Hence from (37) we get $[\delta_k(x, y)z, \mathcal{A}_1] = 0$ for all $x \in \mathcal{H}(\mathcal{A}_-)$, $y \in \mathcal{H}(\mathcal{A}_+)$ and $z \in \mathcal{A}_1$. Consequently, from (1) and (45) we get $\delta_k(x, y)[z, \mathcal{A}_1] = 0$. Thus

$$\delta_k(\mathcal{A}_-, \mathcal{A}_+)[\mathcal{A}_1, \mathcal{A}_1] = [\mathcal{A}_1, \mathcal{A}_1]\delta_k(\mathcal{A}_-, \mathcal{A}_+) = 0. \tag{46}$$

Let $x \in \mathcal{H}(\mathcal{A}_-)$ and $y, z \in \mathcal{H}(\mathcal{A}_+)$. Multiply (37) on the right by $[\mathcal{A}_1, \mathcal{A}_1]$. Using (40) and (46) we arrive at $\delta_k(x, y)z[\mathcal{A}_1, \mathcal{A}_1] = 0$ which yields

$$\delta_k(\mathcal{A}_-, \mathcal{A}_+)\mathcal{A}_+[\mathcal{A}_1, \mathcal{A}_1] = 0. \tag{47}$$

Let $x, y \in \mathcal{H}(\mathcal{A}_+)$ and $z \in \mathcal{H}(\mathcal{A}_-)$. Again multiply (37) on the left by $[\mathcal{A}_1, \mathcal{A}_1]$ to get $[\mathcal{A}_1, \mathcal{A}_1]x\delta_k(y, z) = 0$ by (40) and (46). Hence

$$[\mathcal{A}_1, \mathcal{A}_1]\mathcal{A}_+\delta_k(\mathcal{A}_-, \mathcal{A}_+) = 0 \tag{48}$$

by (36). Therefore (47) and (48) together with Lemma 2.6 (iv) imply

$$\delta_k(\mathcal{A}_-, \mathcal{A}_+) = 0. \tag{49}$$

Using (37) it follows that $x\delta_k(y, z) = 0$ for all $x \in \mathcal{H}(\mathcal{A}_-$ and $y, z \in \mathcal{H}(\mathcal{A}_+)$. That is, $\mathcal{A}_-\ \delta_k(\mathcal{A}_+, \mathcal{A}_+) = 0$. Hence Lemma 2.6 (ii) implies $\mathcal{A}_- = 0$ or $\delta_k(\mathcal{A}_+, \mathcal{A}_+) = 0$. Suppose that

$$\delta_k(\mathcal{A}_+, \mathcal{A}_+) = 0. \tag{50}$$

Using Lemma 4.1 we get

$$[\delta_k(\mathcal{A}_-, \mathcal{A}_-), \mathcal{A}_+]_\epsilon = 0. \tag{51}$$

Hence by (37), (49) and (50) we arrive at $[\delta_k(x, y)z, \mathcal{A}_+]_\epsilon = 0$ for all $x, y \in \mathcal{H}(\mathcal{A}_-)$ and $z \in \mathcal{H}(\mathcal{A}_+)$. Therefore $\delta_k(x, y)[z, \mathcal{A}_+]_\epsilon = 0$ by (1) and (51). Thus

$$\delta_k(\mathcal{A}_-, \mathcal{A}_-)[\mathcal{A}_+, \mathcal{A}_+]_\epsilon = [\mathcal{A}_+, \mathcal{A}_+]_\epsilon\delta_k(\mathcal{A}_-, \mathcal{A}_-) = 0. \tag{52}$$

Using (51) we also have

$$\delta_k(\mathcal{A}_-, \mathcal{A}_-)\mathcal{A}_+[\mathcal{A}_+, \mathcal{A}_+]_\epsilon = 0. \tag{53}$$

Now pick any $x, y, z \in \mathcal{H}(\mathcal{A}_-)$ and multiply (37) on the right by $[\mathcal{A}_+, \mathcal{A}_+]_\epsilon$. Using (49) and (52) it follows that $\delta_k(x, y)z[\mathcal{A}_+, \mathcal{A}_+]_\epsilon = 0$. Therefore $\delta_k(\mathcal{A}_-, \mathcal{A}_-)\mathcal{A}_-[\mathcal{A}_+, \mathcal{A}_+]_\epsilon = 0$ which together with (53) yields

$$\delta_k(\mathcal{A}_-, \mathcal{A}_-)\mathcal{A}[\mathcal{A}_+, \mathcal{A}_+]_\epsilon = 0.$$

Since $\mathcal{A}$ is prime and $[\mathcal{A}_1, \mathcal{A}_1] \neq 0$ we get $\delta_k(\mathcal{A}_-, \mathcal{A}_-) = 0$, and the result follows.

Assume that $\mathcal{A}_- = 0$. This yields $\delta_k(\mathcal{A}_-, \mathcal{A}_-) = 0$. By (41) we arrive at $\delta_k(\mathcal{A}_+, \mathcal{A}_+) = 0$ since $\mathcal{A} = \mathcal{A}_+$ is a graded prime algebra. Consequently, in the case when $k \in G_+$ we proved that δ_k is an ϵ-derivation on $\mathcal{A}$.

Case 2. Suppose now that $k \in G_-$. From (41) and Lemma 2.6 (iv) it follows that

$$\delta_k(\mathcal{A}_+, \mathcal{A}_+) = 0 \tag{54}$$

since $\delta_k(\mathcal{A}_+, \mathcal{A}_+) \subseteq \mathcal{A}_-$. Hence

$$[\delta_k(\mathcal{A}_-, \mathcal{A}_-), \mathcal{A}_+]_\epsilon = 0 \tag{55}$$

by Lemma 4.1. Using (43), (44) and Lemma 2.6 (v) we have for each $x \in \mathcal{A}_1$ either $[\mathcal{A}_+, x]_\epsilon = 0$ or $\delta_k(\mathcal{A}_-, x)\mathcal{A}_-\delta_k(\mathcal{A}_-, x) = 0$. Thus the union of the sets $\mathcal{B} = \{x \in \mathcal{A}_1 \mid [\mathcal{A}_+, x]_\epsilon = 0\}$ and $\mathcal{C} = \{x \in \mathcal{A}_1 \mid \delta_k(\mathcal{A}_-, x)\mathcal{A}_-\delta_k(\mathcal{A}_-, x) = 0\}$ is, by what we proved, equal to $\mathcal{A}_1$. Using the assumption that $[\mathcal{A}_1, \mathcal{A}_1] \neq 0$, we have $\mathcal{B} \neq \mathcal{A}_1$. Therefore there exists $x \notin \mathcal{B}$. Let us show that $\mathcal{C} = \mathcal{A}_\infty$. Suppose that there exists $y \in \mathcal{A}_1$ such that $y \notin \mathcal{C}$. Hence $x + y, x - y \notin \mathcal{B}$ which yields $x + y, x - y \in \mathcal{C}$. Therefore it is easy to see that this implies $y \in \mathcal{C}$, a contradiction. Consequently, we have

$$\delta_k(\mathcal{A}_-, x)\mathcal{A}_-\delta_k(\mathcal{A}_-, x) = 0 \tag{56}$$

for all $x \in \mathcal{A}_1$. From (37) and (55) we get

$$[\delta_k(x, y)z, w]_\epsilon = [\epsilon(k, x)x\delta_k(y, z), w]_\epsilon$$

for all $x, z \in \mathcal{H}(\mathcal{A}_-)$, $y \in \mathcal{A}_1$ and $w \in \mathcal{H}(\mathcal{A}_+)$. Multiply this relation on the right by $\mathcal{A}_-\delta_k(\mathcal{A}_-, y)$. Using (56) we get

$$[\delta_k(x, y)z, w]_\epsilon\mathcal{A}_-\delta_k(\mathcal{A}_-, y) = 0.$$

Again using (56) we also have

$$[\delta_k(x, y)z, w]_\epsilon\mathcal{A}_+\delta_k(\mathcal{A}_-, y) = 0.$$

Hence $[\delta_k(x,y)z,w]_\epsilon \mathcal{A}\delta_k(\mathcal{A}_-,y) = 0$ for all $x,z \in \mathcal{H}(\mathcal{A}_-)$, $y \in \mathcal{A}_1$ and $w \in \mathcal{H}(\mathcal{A}_+)$. Thus

$$[\delta_k(\mathcal{A}_-,y)\mathcal{A}_-,\mathcal{A}_+]_\epsilon \mathcal{A}\delta_k(\mathcal{A}_-,y) = 0$$

for all $y \in \mathcal{A}_1$. By the primeness of $\mathcal{A}$ and the fact that a group cannot be the union of its proper subgroups we get $[\delta_k(\mathcal{A}_-,\mathcal{A}_1)\mathcal{A}_-,\mathcal{A}_+]_\epsilon = 0$ or $\delta_k(\mathcal{A}_-,\mathcal{A}_1) = 0$. In any case the first relation holds true. In particular,

$$[\delta_k(\mathcal{A}_-,\mathcal{A}_1)\mathcal{A}_-,\delta_k(\mathcal{A}_-,\mathcal{A}_1)\mathcal{A}_+]_\epsilon = 0$$

since $\delta_k(\mathcal{A}_-,\mathcal{A}_1)\mathcal{A}_+ \subseteq \mathcal{A}_+$. Using (56) we arrive at

$$\delta_k(\mathcal{A}_-,x)\mathcal{A}_+\delta_k(\mathcal{A}_-,x)\mathcal{A}_- = 0$$

for all $x \in \mathcal{A}_1$, which yields $\mathcal{A}_- = 0$ or $\delta_k(\mathcal{A}_-,x)\mathcal{A}_+\delta_k(\mathcal{A}_-,x) = 0$ by Lemma 2.6 (ii). In both cases we have

$$\delta_k(\mathcal{A}_-,\mathcal{A}_1) = 0.$$

Namely, if the second relation holds true the result follows by Lemma 2.6 (i). Consequently

$$[\delta_k(\mathcal{A}_-,\mathcal{A}_+),\mathcal{A}_1]_\epsilon = 0 \tag{57}$$

by Lemma 4.1. From this relation together with (37) and (54) we get $[\delta_k(x,y)z,\mathcal{A}_1]_\epsilon = 0$ for all $x \in \mathcal{H}(\mathcal{A}_-)$, $y,z \in \mathcal{H}(\mathcal{A}_+)$. Hence by (1) and (57) we arrive at $\delta_k(x,y)[z,\mathcal{A}_1]_\epsilon = 0$. Therefore

$$\delta_k(x,y)w[z,v]_\epsilon = \delta_k(x,y)[wz,v]_\epsilon - \epsilon(z,v)\delta_k(x,y)[w,v]_\epsilon z = 0$$

for all $v \in \mathcal{A}_1$, $x \in \mathcal{H}(\mathcal{A}_-)$ and $w,y,z \in \mathcal{H}(\mathcal{A}_+)$ which implies

$$\delta_k(\mathcal{A}_-,\mathcal{A}_+)\mathcal{A}_+[\mathcal{A}_+,\mathcal{A}_1]_\epsilon = 0.$$

Using Lemma 2.6 (vi) and the assumption that $[\mathcal{A}_1,\mathcal{A}_1] \neq 0$ it follows that

$$\delta_k(\mathcal{A}_-,\mathcal{A}_+)\mathcal{A}_-\delta_k(\mathcal{A}_-,\mathcal{A}_+) = 0. \tag{58}$$

In the next step of the proof we shall more or less just repeat the procedure just presented. Anyway, we shall give details. From (37) and (55) we get

$$[\delta_k(x,y)z,w]_\epsilon = [\epsilon(k,x)x\delta_k(y,z),w]_\epsilon$$

for all $x,z \in \mathcal{H}(\mathcal{A}_-)$, $w,y \in \mathcal{H}(\mathcal{A}_+)$. Multiply this relation by $\mathcal{A}_-\delta_k(\mathcal{A}_-,\mathcal{A}_+)$. Using (58) we get

$$[\delta_k(x,y)z,w]_\epsilon \mathcal{A}_-\delta_k(\mathcal{A}_-,\mathcal{A}_+) = 0.$$

Again using (58) we also have

$$[\delta_k(x,y)z,w]_\epsilon \mathcal{A}_+ \delta_k(\mathcal{A}_-,\mathcal{A}_+) = 0$$

and hence

$$[\delta_k(\mathcal{A}_-,\mathcal{A}_+)\mathcal{A}_-,\mathcal{A}_+]_\epsilon \mathcal{A}\delta_k(\mathcal{A}_-,\mathcal{A}_+) = 0.$$

By the primeness of $\mathcal{A}$ we get $\delta_k(\mathcal{A}_-,\mathcal{A}_+) = 0$ or $[\delta_k(\mathcal{A}_-,\mathcal{A}_+)\mathcal{A}_-,\mathcal{A}_+]_\epsilon = 0$. In any case the second relation holds true. In particular,

$$[\delta_k(\mathcal{A}_-,\mathcal{A}_+)\mathcal{A}_-,\delta_k(\mathcal{A}_-,\mathcal{A}_+)\mathcal{A}_+]_\epsilon = 0.$$

Using (58) we get

$$\delta_k(\mathcal{A}_-,\mathcal{A}_+)\mathcal{A}_+\delta_k(\mathcal{A}_-,\mathcal{A}_+)\mathcal{A}_- = 0$$

which yields $\mathcal{A}_- = 0$ or $\delta_k(\mathcal{A}_-,\mathcal{A}_+)\mathcal{A}_+\delta_k(\mathcal{A}_-,\mathcal{A}_+) = 0$ by Lemma 2.6 (ii). In both cases we have

$$\delta_k(\mathcal{A}_-,\mathcal{A}_+) = 0. \tag{59}$$

Namely, if the latter is true than the result follows from Lemma 2.6 (i). By (37), (54), (55) and (59) we get $[\delta_k(x,y)z,\mathcal{A}_+]_\epsilon = 0$ for all $x,y \in \mathcal{H}(\mathcal{A}_-)$, $z \in \mathcal{H}(\mathcal{A}_+)$. Using (1) and the standard argument we arrive at

$$\delta_k(\mathcal{A}_-,\mathcal{A}_-)\mathcal{A}_+[\mathcal{A}_+,\mathcal{A}_+]_\epsilon = 0. \tag{60}$$

Similarly we can show that $[x\delta_k(y,z),\mathcal{A}_+]_\epsilon = 0$ for all $x \in \mathcal{H}(\mathcal{A}_+)$ and $y,z \in \mathcal{H}(\mathcal{A}_-)$, which yields

$$[\mathcal{A}_+,\mathcal{A}_+]_\epsilon \mathcal{A}_+\delta_k(\mathcal{A}_-,\mathcal{A}_-) = 0. \tag{61}$$

Using (60) and (61) we arrive at $\delta_k(\mathcal{A}_-,\mathcal{A}_-) = 0$ by Lemma 2.6 (iv). The proof is complete. ∎

Theorem 4.3 *Let D be a Jordan ϵ-derivation on a graded prime associative algebra $\mathcal{A}$ such that $[\mathcal{A}_1,\mathcal{A}_1] \neq 0$. Then D is an ϵ-derivation.*

Proof. It suffices to prove that for every $k \in G$, a Jordan ϵ-derivation D_k of degree k is an ϵ-derivation. Let $k \in G$ and let D_k be a Jordan ϵ-derivation of degree k. By (3) it follows that

$$D_k(axa) = D_k(a)xa + \epsilon(k,a)aD_k(x)a + \epsilon(k,ax)axD_k(a).$$

Linearizing this identity we arrive at

$$\begin{aligned} D_k(axb) + D_k(bxa) &= D_k(a)xb + D_k(b)xa + \epsilon(k,a)aD_k(x)b \\ &\quad + \epsilon(k,a)bD_k(x)a + \epsilon(k,ax)axD_k(b) \\ &\quad + \epsilon(k,ax)bxD_k(a) \end{aligned}$$

for all $a, b \in \mathcal{A}_g$, $g \in G_+$ and $x \in \mathcal{H}(\mathcal{A})$. Let $a, b \in \mathcal{H}(\mathcal{A}_+)$ and $x \in \mathcal{H}(\mathcal{A})$. Considering the expression $D_k(abxba + baxab)$ we get

$$\begin{aligned}
D_k(a(bxb)a + b(axa)b) &= D_k(a)bxba + \epsilon(k,a)aD_k(bxb)a + \epsilon(k,ab^2x)abxbD_k(a) \\
&\quad + D_k(b)axab + \epsilon(k,b)bD_k(axa)b + \epsilon(k,a^2bx)baxaD_k(b) \\
&= D_k(a)bxba + \epsilon(k,a)aD_k(b)xba \\
&\quad + \epsilon(k,ab)abD_k(x)ba + \epsilon(k,abx)abxD_k(b)a \\
&\quad + \epsilon(k,ab^2x)abxbD_k(a) + D_k(b)axab \\
&\quad + \epsilon(k,b)bD_k(a)xab + \epsilon(k,ab)baD_k(x)ab \\
&\quad + \epsilon(k,abx)baxD_k(a)b + \epsilon(k,a^2bx)baxaD_k(b),
\end{aligned}$$

and on the other hand

$$\begin{aligned}
D_k((ab)x(ba) + (ba)x(ab)) &= D_k(ab)xba + \epsilon(k,ab)abD_k(x)ba \\
&\quad + \epsilon(k,axb)abxD_k(ba) + D_k(ba)xab \\
&\quad + \epsilon(k,ab)baD_k(x)ab + \epsilon(k,abx)baxD_k(ab)
\end{aligned}$$

Comparing the identities so obtained and using (36) we arrive at

$$\begin{aligned}
0 &= (D_k(ab) - D_k(a)b - \epsilon(k,a)aD_k(b))xba \\
&\quad + (D_k(ba) - D_k(b)a - \epsilon(k,b)bD_k(a))xab \\
&\quad + \epsilon(k,abx)bax(D_k(ab) - D_k(a)b - \epsilon(k,a)aD_k(b)) \\
&\quad + \epsilon(k,axb)abx(D_k(ba) - D_k(b)a - \epsilon(k,b)bD_k(a)) \\
&= \delta_k(a,b)xba + \delta_k(b,a)xab + \epsilon(k,abx)bax\delta_k(a,b) \\
&\quad + \epsilon(k,abx)abx\delta_k(b,a) \\
&= \delta_k(a,b)x[b,a]_\epsilon + \epsilon(k,abx)[b,a]_\epsilon x\delta_k(a,b)
\end{aligned} \tag{62}$$

for all $a, b \in \mathcal{H}(\mathcal{A}_+)$ and $x \in \mathcal{H}(\mathcal{A})$. Write $\delta_k = \delta_k(a,b)$ and $c = [b,a]_\epsilon$. Using (62) it follows that

$$\begin{aligned}
\delta_k xcyc &= -\epsilon(k,a^2b^2xy)cxcy\delta_k \\
&= \epsilon(k,abx)cx\delta_k yc \\
&= -\delta_k xcyc
\end{aligned}$$

which yields $\delta_k xcyc = 0$ for all $x, y \in \mathcal{H}(\mathcal{A})$. Hence $\delta_k \mathcal{A} c \mathcal{A} c = 0$ which in turn implies $\delta_k = 0$ or $c = 0$ by the primeness of $\mathcal{A}$. Thus we proved that for each pair $a, b \in \mathcal{H}(\mathcal{A}_+)$ we have $\delta_k(a,b) = 0$ or $[a,b]_\epsilon = 0$. Arguing as in the proof of Theorem 3.4 we get that for each pair $g, h \in G_+$ either $\delta_k(\mathcal{A}_g, \mathcal{A}_h) = 0$ or $[\mathcal{A}_g, \mathcal{A}_h]_\epsilon = 0$. Let us show that $\delta_k(\mathcal{A}_1, \mathcal{A}_g) = 0$ for all $g \in G_+$. Suppose that $[\mathcal{A}_1, \mathcal{A}_g]_\epsilon = 0$ for some $g \in G_+$. Therefore by (38) we have $[D_k([x,y]_\epsilon) - [D_k(x), y]_\epsilon - [x, D_k(y)]_\epsilon, z]_\epsilon = 0$ for all $x \in \mathcal{A}_1$, $y \in \mathcal{A}_g$ and $z \in \mathcal{A}$. Since also $D_k(x \circ_\epsilon y) - D_k(x) \circ_\epsilon y - x \circ_\epsilon D_k(y) = 0$ we arrive at

$$[\delta_k(\mathcal{A}_1, \mathcal{A}_g), \mathcal{A}]_\epsilon = 0. \tag{63}$$

Therefore from (37) we get $[x\delta_k(y,z), z]_\epsilon = 0$ for all $x, z \in \mathcal{A}_1$ and $y \in \mathcal{A}_g$. Using (1) and (63) it follows that $[x,z]\delta_k(y,z) = 0$ which in turn implies $[x,z]\mathcal{A}\delta_k(y,z) = 0$ for all $x, z \in \mathcal{A}_1$ and $y \in \mathcal{A}_g$. Since $\mathcal{A}$ is prime it follows that for each $z \in \mathcal{A}_1$ either $[\mathcal{A}_1, z] = 0$ or $\delta_k(\mathcal{A}_g, z) = 0$. Using the fact a group $\mathcal{A}_1$ cannot be the union of its proper subgroups and the assumption $[\mathcal{A}_1, \mathcal{A}_1] \neq 0$ it follows that $\delta_k(\mathcal{A}_g, \mathcal{A}_1) = 0$, as desired. Thus we proved that $\delta_k(\mathcal{A}_+, \mathcal{A}_1) = 0$. Using Lemma 4.2 the result follows. ∎

Example 4.4. Let $\mathcal{A}$ be an algebra such as in Example 3.5 and let $k \in G_+$. Pick $a \in \mathcal{A}_k$ and define $D_k : \mathcal{A} \to \mathcal{A}$ by $D_k(x+y) = ay$ for all $x \in \mathcal{H}(\mathcal{A}_+)$ and $y \in \mathcal{H}(\mathcal{A}_-)$. Then D_k is a Jordan ϵ-derivation of degree k which is not an ϵ-derivation provided that $a \neq 0$ and $\mathcal{A}_- \neq 0$. This example is just an extension of Example 3.3 in [3].

Acknowledgement

I would like to thank Professor Matej Brešar for help and encouragement. I would also like to thank the referee for careful reading of the paper and useful suggestions.

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