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A CHARACTERIZATION OF THE FINITE SIMPLE GROUP $L_{11}(2)$ BY ITS ELEMENT ORDERS

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Abstract. In this article we characterize the projective special linear group $L_{11}(2)$, by the set of the order of its elements.

1. Introduction

For a finite group G, we denote by $\pi_e(G)$ the set of all orders of elements in G. It is clear that the set $\pi_e(G)$ is closed and partially ordered by divisibility, hence, it is uniquely determined by $\mu(G)$, the subset of its maximal elements. Also, $\pi_e(G)$ defines the *prime graph* $\Gamma(G)$ of G whose vertices are prime factors of $|G|$ and two primes p and q are adjacent if and only if $pq \in \pi_e(G)$. The number of connected components of $\Gamma(G)$ is denoted by $t(G)$, and the connected components are denoted by $\pi_i = \pi_i(G)$, $i = 1, 2, \ldots, t(G)$. If $2 \in \pi(G)$ we always assume $2 \in \pi_1$.

In [8] and [16] the authors have obtained the connected components of $\Gamma(S)$ where S is a finite simple group. As a result of these investigations we see that $t(L_n(2)) = 2$, for $n = p$ or $p+1$, where $p > 3$ is a prime. In fact, the first components are

$$\pi_1(L_p(2)) = \pi(2\prod_{i=1}^{p-1}(2^i - 1))$$

and

$$\pi_1(L_{p+1}(2)) = \pi(2(2^{p+1} - 1)\prod_{i=1}^{p-1}(2^i - 1)),$$

where as the second component in both cases is

$$\pi_2(L_p(2)) = \pi_2(L_{p+1}(2)) = \pi(2^p - 1).$$

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Also we know that the prime graph $\Gamma(L_n(2))$ with $n \neq p$ or $p+1$, where p is a prime, is connected.

If Ω is a subset of $\mathbb{N}$ then $h(\Omega)$ denotes the number of pairwise non-isomorphic groups G such that $\pi_e(G) = \Omega$. A group G is called *k-distinguishable* if $h(\pi_e(G)) = k < \infty$; otherwise G is called *non-distinguishable*. Also a 1-distinguishable group is called a *characterizable* group.

Previously, it was proved that the following simple groups are characterizable: $L_3(2) \cong L_2(7)$ [13], $L_4(2) \cong A_8$ [12], $L_5(2)$ [3], $L_6(2)$, $L_7(2)$ ([3, 14, 4], $L_8(2)$ [5]. Moreover, in [3] we have put forward the following conjecture:

Conjecture *For all positive integers $n \geq 3$, the simple groups $L_n(2)$ are characterizable.*

In the present article, we consider the simple group $L_{11}(2)$ and we prove its characterizability using the Classification Theorem of Finite Simple Groups. Therefore, we prove the following.

Main Theorem *Let G be a finite group. Then $G \cong L_{11}(2)$ if and only if $\pi_e(G) = \pi_e(L_{11}(2))$.*

All groups discussed will be assumed to be finite in this article. Given a natural number n, denote the set of all prime divisors of n by $\pi(n)$. If G is a group, we write for short $\pi(G)$ instead of $\pi(|G|)$. The other notations are standard and can be found in [1].

2. Preliminary results

To prove the Main Theorem, we need some lemmas. First, we give the set of element orders of $L_{11}(2)$.

Lemma 1. $\mu(L_{11}(2)) = \{48, 120, 248, 315, 372, 420, 504, 508, 762, 868, 889, 930, 1020, 1022, 1023, 1533, 1785, 1905, 1953, 2047\}$.

Proof. Here, we use the Green's notations (see [6]). In general, for the number $c(n,q)$ of classes of $GL(n,q)$, there is a generating function

$$\sum_{n=0}^{\infty} c(n,q)x^n = \prod_{d=1}^{\infty} p(x^d)^{w(d,q)}, \tag{1}$$

where

$$w(d,q) = \frac{1}{d} \sum_{k|d} \mu(k) q^{d/k}, \tag{2}$$

is the number of irreducible polynomials $f(t)$ of degree d over $GF(q)$. We recall that in equations (1) and (2)

$$p(x) := \frac{1}{(1-x)(1-x^2)\dots} = \sum_{n=0}^{\infty} p_n x^n, \tag{3}$$

is the partition function (in this power series the coefficient p_n is the number of partitions of n), and μ is the Möbius function, respectively.

Now, using (2) and (3) we calculate the values of $w(d,2)$ and p_d where $1 \leq d \leq 11$, and list them in Table 1.

Table 1. The number of irreducible polynomials of degree d over $\mathbb{Z}_2$, and the number of partitions of d.

d	1	2	3	4	5	6	7	8	9	10	11
$w(d,2)$	2	1	2	3	6	9	18	30	56	99	186
p_d	1	2	3	5	7	11	15	22	30	42	56

We denote by $f_1, f_2, \dots, f_d$, the irreducible polynomials over $\mathbb{Z}_2$ with degrees $1, 2, \dots, d$, respectively. Furthermore, if $w(d,2) = k$, then we denote the k irreducible polynomials of the same degree d by $f_{d_1}, f_{d_2}, \dots, f_{d_k}$.

For $(n,q) = (11,2)$, we have $c(11,2) = 1998$. In fact, using (1) we obtain

$$\sum_{n=0}^{\infty} c(n,2)x^n = \prod_{d=1}^{\infty} p(x^d)^{w(d,2)} = 1 + x + 3x^2 + 6x^3 + 14x^4 + 27x^5 + 60x^6 \\ +117x^7 + 246x^8 + 490x^9 + 1002x^{10} + 1998x^{11} + \cdots .$$

Let

$$f(t) = t^d - a_{d-1}t^{d-1} - \cdots - a_0,$$

be a polynomial over $\mathbb{Z}_2$, of degree d, and using Green's notation we let

$$U(f) = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ a_0 & a_1 & a_2 & a_3 & \dots & a_{d-1} \end{bmatrix},$$

be its companion matrix. Also we set

$$U_l(f) := \begin{bmatrix} U(f) & 1_d & 0 & 0 & \dots & 0 \\ 0 & U(f) & 1_d & 0 & \dots & 0 \\ 0 & 0 & U(f) & 1_d & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & U(f) \end{bmatrix},$$

with l diagonal blocks $U(f)$, and 1_d is the identity matrix. If $\lambda = \{l_1, l_2, \dots, l_p\}$ is a partition of a positive integer k whose p parts written in descending order are:

$$l_1 \geq l_2 \geq \dots \geq l_p > 0,$$

then

$$U_\lambda(f) := \text{diag}\{U_{l_1}(f), U_{l_2}(f), \dots, U_{l_p}(f)\}.$$

Let $A \in GL(11, 2)$ have characteristic polynomial

$$f_1^{k_1} f_2^{k_2} \dots f_{11}^{k_{11}}.$$

Evidently $\sum_{i=1}^{11} ik_i = 11$. Moreover A is conjugate with

$$\text{diag}\{U_{\nu_1}(f_1), U_{\nu_2}(f_2), \dots, U_{\nu_{11}}(f_{11})\},$$

in $GL(11, 2)$, where $\nu_1, \nu_2, \dots, \nu_{11}$ are certain partitions of $k_1, k_2, \dots, k_{11}$ respectively. We denote the conjugacy class c of A by the symbol

$$c = (f_1^{\nu_1} f_2^{\nu_2} \dots f_{11}^{\nu_{11}}).$$

Also note that $o(U(f_i))$ divides $2^i - 1$, and also there exists an irreducible polynomial f_i such that $o(U(f_i)) = 2^i - 1$. Moreover, $o(U_k(f_i))$ with $k \geq 2$ and $ki \leq 11$ are given in Table 2. Furthermore, if B is conjugate to

$$\text{diag}\{U_{k_1}(f_1), U_{k_2}(f_2), \dots, U_{k_{11}}(f_{11})\},$$

then we have

$$o(B) = \text{l.c.m}\{o(U_{k_1}(f_1)), o(U_{k_2}(f_2)), \dots, o(U_{k_{11}}(f_{11}))\}.$$

On the other hand, it is easy to see that $o(A)$ divides $o(B)$. In the last column of Table 3, m denotes the order of $B \sim \text{diag}\{U_{k_1}(f_1), U_{k_2}(f_2), \dots, U_{k_{11}}(f_{11})\}$. Also $\text{Par}(k)$ denotes the set of partitions of k, where $k \leq 11$. In fact, m among all the conjugacy classes having the same charactristic polynomial is of maximum value. Now, we can easily derive the set $\mu(G)$ from last column of Table 3, as required. ■

Table 2. The order of $A \in GL(ki, 2)$ having characteristic polynomial f_i^k, where $k > 1$ and $ki \leq 11$.

k	f_1	f_2	f_3	f_4	f_5
2	2	6	14	30	62
3	4	12	28		
4	4	12			
5	8	24			
$6 \leq k \leq 8$	8				
$9 \leq k \leq 11$	16				

Table 3. The order of elements of the simple group $L_{11}(2)$.

Type of c	Conditions	Number	m
(f_1^r)	$r \in \text{Par}(11)$	56	$16 = 2^4$
$(f_2^r f_1)$	$r \in \text{Par}(5)$	7	$24 = 2^3.3$
$(f_2^r f_1^s)$	$r \in \text{Par}(4), s \in \text{Par}(3)$	15	$12 = 2^2.3$
$(f_2^r f_1^s)$	$r \in \text{Par}(3), s \in \text{Par}(5)$	21	$24 = 2^3.3$
$(f_2^r f_1^s)$	$r \in \text{Par}(2), s \in \text{Par}(7)$	30	$24 = 2^3.3$
$(f_2 f_1^r)$	$r \in \text{Par}(9)$	30	$48 = 2^4.3$
$(f_3^r f_2)$	$r \in \text{Par}(3)$	6	$84 = 2^2.3.7$
$(f_{3_i}^r f_{3_j} f_2)$	$r \in \text{Par}(2)$, $1 \leq i \neq j \leq 2$	4	$42 = 2.3.7$
$(f_3^r f_1^s)$	$r \in \text{Par}(3)$, $s \in \text{Par}(2)$	12	$28 = 2^2.7$
$(f_{3_i}^r f_{3_j} f_1^s)$	$r, s \in \text{Par}(2)$, $1 \leq i \neq j \leq 2$	8	$14 = 2.7$
$(f_3^r f_2^s f_1)$	$r, s \in \text{Par}(2)$	8	$42 = 2.3.7$
$(f_{3_1} f_{3_2} f_2^r f_1)$	$r \in \text{Par}(2)$	2	$42 = 2.3.7$
$(f_3^r f_2 f_1^s)$	$r \in \text{Par}(2)$, $s \in \text{Par}(3)$	12	$84 = 2^2.3.7$
$(f_{3_1} f_{3_2} f_2 f_1^r)$	$r \in \text{Par}(3)$	3	$84 = 2^2.3.7$
$(f_3^r f_1^s)$	$r \in \text{Par}(2)$, $s \in \text{Par}(5)$	28	$56 = 2^3.7$
$(f_{3_1} f_{3_2} f_1^r)$	$r \in \text{Par}(5)$	7	$56 = 2^3.7$
$(f_3 f_2^r)$	$r \in \text{Par}(4)$	10	$84 = 2^2.3.7$
$(f_3 f_2^r f_1^s)$	$r \in \text{Par}(3)$, $s \in \text{Par}(2)$	12	$84 = 2^2.3.7$
$(f_3 f_2^r f_1^s)$	$r \in \text{Par}(2)$, $s \in \text{Par}(4)$	20	$84 = 2^2.3.7$
$(f_3 f_2 f_1^r)$	$r \in \text{Par}(6)$	22	$168 = 2^3.3.7$
$(f_3 f_1^r)$	$r \in \text{Par}(8)$	44	$56 = 2^3.7$
$(f_4^r f_3)$	$r \in \text{Par}(2)$	12	$210 = 2.3.5.7$
$(f_{4_i} f_{4_j} f_3)$	$1 \leq i \neq j \leq 3$	6	$105 = 3.5.7$
$(f_4^r f_2 f_1)$	$r \in \text{Par}(2)$	6	$30 = 2.3.5$
$(f_{4_i} f_{4_j} f_2 f_1)$	$1 \leq i \neq j \leq 3$	3	$30 = 2.3.5$
$(f_4^r f_1^s)$	$r \in \text{Par}(2)$, $s \in \text{Par}(3)$	18	$60 = 2^2.3.5$
$(f_{4_i} f_{4_j} f_1^r)$	$r \in \text{Par}(3)$, $1 \leq i \neq j \leq 3$	9	$60 = 2^2.3.5$
$(f_4 f_3^r f_1)$	$r \in \text{Par}(2)$	12	$210 = 2.3.5.7$
$(f_4 f_{3_1} f_{3_2} f_1)$		3	$105 = 3.5.7$

(Continuation of Table 3)

Type of c	Conditions	Number	m
$(f_4f_3f_2^r)$	$r \in$ Par(2)	12	$210 = 2.3.5.7$
$(f_4f_3f_2f_1^r)$	$r \in$ Par(2)	12	$105 = 3.5.7$
$(f_4f_3f_1^r)$	$r \in$ Par(4)	30	$420 = 2^2.3.5.7$
$(f_4f_2^rf_1)$	$r \in$ Par(3)	9	$60 = 2^2.3.5$
$(f_4f_2^rf_1^s)$	$r \in$ Par(2), $s \in$ Par(3)	18	$60 = 2^2.3.5$
$(f_4f_2f_1^r)$	$r \in$ Par(5)	21	$120 = 2^3.3.5$
$(f_4f_1^r)$	$r \in$ Par(7)	45	$120 = 2^3.3.5$
$(f_5^rf_1)$	$r \in$ Par(2)	12	$62 = 2.31$
$(f_{5_i}f_{5_j}f_1)$	$1 \leq i \neq j \leq 6$	15	31
$(f_5f_4f_2)$		18	$465 = 3.5.31$
$(f_5f_4f_1^r)$	$r \in$ Par(2)	36	$930 = 2.3.5.31$
$(f_5f_3^r)$	$r \in$ Par(2)	24	$434 = 2.7.31$
$(f_5f_{3_1}f_{3_2})$		6	$217 = 7.31$
$(f_5f_3f_2f_1)$		12	$651 = 3.7.31$
$(f_5f_3f_1^r)$	$r \in$ Par(3)	36	$868 = 2^2.7.31$
$(f_5f_2^r)$	$r \in$ Par(3)	18	$372 = 2^2.3.31$
$(f_5f_2^rf_1^s)$	$r, s \in$ Par(2)	24	$186 = 2.3.31$
$(f_5f_2f_1^r)$	$r \in$ Par(4)	30	$315 = 3^2.5.7$
$(f_5f_1^r)$	$r \in$ Par(6)	66	$248 = 2^3.31$
(f_6f_5)		54	$1953 = 3^2.7.31$
$(f_6f_4f_1)$		27	$315 = 3^2.5.7$
$(f_6f_3f_2)$		18	$63 = 3^2.7$
$(f_6f_3f_1^r)$	$r \in$ Par(2)	36	$126 = 2.3^2.7$
$(f_6f_2^rf_1)$	$r \in$ Par(2)	18	$126 = 2.3^2.7$
$(f_6f_2f_1^r)$	$r \in$ Par(3)	27	$252 = 2^2.3^2.7$
$(f_6f_1^r)$	$r \in$ Par(5)	63	$504 = 2^3.3^2.7$
(f_7f_4)		54	$1905 = 3.5.127$
$(f_7f_3f_1)$		36	$889 = 7.127$
$(f_7f_2^r)$	$r \in$ Par(2)	36	$762 = 2.3.127$
$(f_7f_2f_1^r)$	$r \in$ Par(2)	36	$762 = 2.3.127$
$(f_7f_1^r)$	$r \in$ Par(4)	90	$508 = 2^2.127$
(f_8f_3)		60	$1785 = 3.5.7.17$
$(f_8f_2f_1)$		30	$255 = 3.5.17$
$(f_8f_1^r)$	$r \in$ Par(3)	90	$1020 = 2^2.3.5.17$
(f_9f_2)		56	$1533 = 3.7.73$
$(f_9f_1^r)$	$r \in$ Par(2)	112	$1022 = 2.7.73$
$(f_{10}f_1)$		99	$1023 = 3.11.31$
(f_{11})		186	$2047 = 23.89$
Total		1998	

In the following Lemma, we show the existence of an outer automorphism of $L_{11}(2)$, of order 22, which certainly proves that $\pi_e(L_{11}(2)) \subsetneq \pi_e(\mathrm{Aut}(L_{11}(2))$.

Lemma 2. *Aut$(L_{11}(2))$ contains an element of order* 22.

Proof. Set $G = L_{11}(2)$. Let θ be an involutary graph automorphism of G. Using the notations in [2] we have G^+=Aut$(G) = G \cdot \langle\theta\rangle = G \cup \theta G$. The conjugacy classes of G^+ which lie in θG are called negative classes and by Theorem 1 in [2], G^+ has only one negative conjugacy class of involutions with representative θI and we have

$$|C_{G^+}(\theta I)| = 2^{26}.3^6.5^2.7.11.13,$$

where I is the identity matrix. Now, it is easy to see that $22 \in \pi_e(G^+)$, as required. The lemma is proved. ■

In the next lemma, we will be concerned with order structure of a simple group satisfying some prescribed conditions.

Lemma 3. *If G is a simple group of Lie type such that*

$$\{2, 23, 89\} \subset \pi(G) \subseteq \pi(L_{11}(2)) = \{2, 3, 5, 7, 11, 17, 23, 31, 73, 89, 127\},$$

then G is isomorphic to $L_{11}(2)$.

Proof. Suppose G is a finite simple group of Lie type over a finite field of order $q = p^n$, where p is a prime and n is a natural number. If $127 \notin \pi(G)$, then $\{89\} \subset \pi(G) \subseteq \pi(89!)$. Now, by Lemma 2.6 in [10], we obtain that $G \cong L_2(89)$. But then $23 \notin \pi(G)$, which is a contradiction. Therefore $127 \in \pi(G)$. On the other hand $p \in \pi(G)$, hence p may be equal to 2, 3, 5, 7, 11, 17, 23, 31, 73, 89 or 127. If $p = 2$, then it is clear that the order of 2 modulo 127 is 7, and there is no natural number m such that $2^m + 1 \equiv 0$ (mod 127). Thus if $2^k - 1$ divides $|G|$ and $127 \in \pi(2^k - 1)$, for some k, then k must be a multiple of 7. Therefore, from Table 6 in [1], the only candidate for G under our assumptions is $A_{10}(2) \cong L_{11}(2)$. If $p = 3$, then the order of 3 modulo 127 is 126 and the least natural number m for which $3^m + 1 \equiv 0$ (mod 127) is 63. Now, from Table 6 in [1] no candidate for G will arise. Similarly, for other p we do not get a group. The Lemma is proved. ■

Lemma 4. [9] *Let G be a finite group, N a normal subgroup of G, and G/N a Frobenius group with Frobenius kernel F and cyclic complement C. If $(|F|, |N|) = 1$ and F is not contained in $NC_G(N)/N$, then $p|C| \in \pi_e(G)$ for some prime divisor p of $|N|$.*

2. Proof of the Main Theorem

We need to prove only the sufficiency part. Let G be a finite group for which

$\pi_e(G) = \pi_e(L_{11}(2))$. Then the connected components of the prime graph of $\Gamma(G)$ are

$$\pi_1 = \{2, 3, 5, 7, 11, 17, 31, 73, 127\} \quad \text{and} \quad \pi_2 = \{23, 89\}.$$

We have to prove that G is isomorphic to $L_{11}(2)$. This will be done below by going through a sequence of separately stated lemmas.

Lemma 5. *G is non-soluble. Moreover, G is neither Frobenius nor 2-Frobenius and G has a normal series which contains a non-Abelian simple section.*

Proof. If G is soluble, we consider the $\{5, 11, 23\}$-Hall subgroup H of G. Since G does not have any element of order 55, 115 or 253, H is a soluble group all of whose elements are of prime power order. By [7] Theorem 1 we must have $|\pi(H)| \leq 2$, which is a contradiction. Therefore, G is non-soluble and so G is not a 2-Frobenius group (Note: 2-Frobenius groups are always soluble).

If G is a Frobenius group with kernel K and complement C, then C is non-soluble. Now, by the structure of non-soluble Frobenius complement (see Theorem 18.6 in [11]), C has a normal subgroup C^* of index ≤ 2 such that $C^* \cong SL_2(5) \times Z$, where every Sylow subgroup of Z is cyclic and $\pi(Z) \cap \pi(30) = \varnothing$. Because G does not contain any element of order 5.73 or 5.127, $\{73, 127\} \cap \pi(Z) = \varnothing$, and hence $\{73, 127\} \subset \pi(K)$. Now since K is nilpotent we get $73.127 \in \pi_e(K) \subset \pi_e(G)$, which is a contradiction. Hence G is not a Frobenius group.

Thus [16] Theorem A implies that G has a normal series

$$G \trianglerighteq G_1 \triangleright N \trianglerighteq 1,$$

where N is a nilpotent π_1-group, $\overline{G}_1 := G_1/N$ is a non-Abelian simple group and G/G_1 is a soluble π_1-group. ∎

Now we discuss the non-Abelian simple group $\overline{G}_1$ in Lemma 5 using the Classification of Finite Simple Groups. Note that $\pi_2 = \{23, 89\} \subset \pi(\overline{G}_1)$.

Lemma 6. *The non-Abelian simple group $\overline{G}_1$ in Lemma 5 is isomorphic to $L_{11}(2)$.*

Proof. According to the Classification of Finite Simple Groups, we know that the possibilities for $\overline{G}_1$ are:

(1) alternating groups A_n, $n \geq 5$;

(2) 26 sporadic finite simple groups;

(3) simple groups of Lie type.

If $\overline{G}_1$ is an alternating group $A_n, n \geq 5$, then since $89 \in \pi(\overline{G}_1)$, we deduce $n \geq 89$. But then $13 \in \pi(G)$, which is a contradiction. Also, $\overline{G}_1$ can not be a sporadic simple group, since otherwise the maximum prime in $\pi(\overline{G}_1)$ is 71, but

$89 \in \pi(\overline{G}_1)$, which is a contradiction. Now, we assume that $\overline{G}_1$ is a simple group of Lie type. In this case, by Lemma 3, we obtain $\overline{G}_1 \cong L_{11}(2)$, as claimed. ∎

Lemma 7. $N = 1$.

Proof. Assume the contrary. Without loss of generality we may assume that $N = O_r(G)$ for some prime $r \in \pi_1$. Moreover, we may assume that N is an elementary Abelian subgroup and $C_{G_1}(N) = N$. Evidently, the group $GL(10, 2)$ has an element A of order $2^{10} - 1$, the so called Singer element. Let $K = \langle A \rangle$ and define the subgroup L of $L_{11}(2)$ as follows:

$$L = \left\{ \left[\begin{array}{c|cccc} 1 & a_1 & a_2 & \dots & a_{10} \\ \hline 0 & & & X & \end{array} \right] \big| X \in K, a_i \in \mathrm{GF}(2), 1 \leq i \leq 10 \right\}.$$

Now, if we define the subgroup

$$S = \left\{ \left[\begin{array}{c|cccc} 1 & a_1 & a_2 & \dots & a_{10} \\ \hline 0 & & & I & \end{array} \right] \big| a_i \in \mathrm{GF}(2), 1 \leq i \leq 10 \right\},$$

of L, then S is isomorphic to the additive group of the vector space of dimension 10 over $\mathrm{GF}(2)$. Moreover, the subgroup

$$T = \left\{ \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & X \end{array} \right] \big| X \in K \right\},$$

of L acts on S in the usual way and fixed point freely, hence $L = S \rtimes T$ is a Frobenius group with kernel S and complement T. This group is written in the form $L = 2^{10} : (2^{10} - 1)$. Hence $\overline{G}_1$ contains a subgroup of shape $2^{10} : 2^{10} - 1$. If $r \neq 2$, then by Lemma 4, we get $r \cdot (2^{10} - 1) \in \pi_e(G)$, which contradicts Lemma 1. Thus, N is a non-trivial 2-subgroup. Yet, in this case, we have $23 : 11 < M_{23} < M_{24} < L_{11}(2)$ (see [1] and [15]), and by Lemma 4, we obtain $22 \in \pi_e(G)$, which again contradicts Lemma 1. Therefore $N = 1$. ∎

Lemma 8. $G \cong L_{11}(2)$.

Proof. By Lemma 7, $N = 1$, hence we have $1 \trianglelefteq G_1 \trianglelefteq G$. Now since $t(G) = 2$, we obtain $C_G(G_1) = 1$, and so

$$G = \frac{N_G(G_1)}{C_G(G_1)} \hookrightarrow \mathrm{Aut}(G_1).$$

Therefore $L_{11}(2) \leq G \leq \mathrm{Aut}(L_{11}(2))$. Because $|\mathrm{Out}(L_{11}(2))| = 2$, $G \cong L_{11}(2)$ or $G \cong \mathrm{Aut}(L_{11}(2))$. Since by Lemma 2, $\mathrm{Aut}(L_{11}(2))$ contains an element of order 22 and $22 \notin \pi_e(G)$, $G \cong L_{11}(2)$, as claimed. ∎

Thus the Main Theorem is proved.

The characterization of the projective special linear group $L_n(2)$ with $n \neq p$ or $p+1$, where p is a prime, is more difficult, because its prime graph is connected and Theorem A in [16] does not work. In particular, we have the following open problem (also see [14]):

Open Problem. *Can the projective special linear groups $L_9(2)$ and $L_{10}(2)$ be characterized by the set of their element orders?*

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