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ON σ-LIMIT AND $s\sigma$-LIMIT IN BANACH SPACES

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Abstract. For bounded sequences in a normed linear space X, we introduce a notion of limit, called the $s\sigma$-limit, and discuss some interesting properties related to σ-limit and $s\sigma$-limit. It is shown that the space $X_{s\sigma}$ (resp. X_σ) of all $s\sigma$-convergent (resp. σ-convergent) sequences in X is a Banach space, and the space $\mathbb{C}_{s\sigma}$ is a unital Banach subalgebra of ℓ^∞ such that every Banach limit restricted to $\mathbb{C}_{s\sigma}$ is a multiplicative linear functional. We also use $s\sigma$-limit to characterize continuity of functions and prove two versions of the dominated convergence theorem in terms of σ-limit and $s\sigma$-limit.

1. Introduction

A Banach limit ϕ on ℓ^∞, the space of all bounded sequences in the complex field $\mathbb{C}$ with the sup-norm, is a positive linear functional on ℓ^∞ such that

$$\phi(\{a_{n+k}\}) = \phi(\{a_n\})$$

for all $k = 1, 2, \ldots$, and such that $\phi(\{a_n\}) = \lim_{n\to\infty} a_n$ whenever the limit exists. Let π_σ denote the set of all Banach limits on ℓ^∞. It is known that π_σ is a weakly*-compact set in $(\ell^\infty)^*$.

In 1948, Lorentz[6] defined the σ-limit of a sequence $\{a_n\} \in \ell^\infty$ to be a number a such that $\phi(\{a_n\}) = a$ for all $\phi \in \pi_\sigma$. It is unique if it exists. Some related results and their applications can be found in [1, 7-10]. The definition of σ-limit can be generalized to $\ell^\infty(X)$, the space of all bounded sequences in a general normed linear space X. A sequence $\{x_n\} \in \ell^\infty(X)$ is said to have x as a *σ-limit* if $\sigma\text{-}\lim\langle x_n, x^*\rangle = \langle x, x^*\rangle$ (i.e., $\phi(\{\langle x_n, x^*\rangle\}) = \langle x, x^*\rangle$ for all $\phi \in \pi_\sigma$)

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for all $x^* \in X^*$; in this case, we write $\sigma\text{-}\lim x_n = x$. $\{x_n\}$ is said to be *weakly almost-convergent to* x if for every $x^* \in X^*$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \langle x_{k+m}, x^* \rangle = \langle x, x^* \rangle$$

uniformly for $m \geq 0$. A bounded sequence $\{x_n\}$ in X has the σ-limit if and only if it is weakly almost-convergent. See [5] for these and further generalization.

We define two new notions of convergence for bounded sequences in $\ell^\infty(X)$ as follows:

x is said to be a *$s\sigma$-limit* of $\{x_n\}$ if $\sigma\text{-}\lim ||x_n - x|| = 0$; in this case, we write $s\sigma\text{-}\lim x_n = x$. $\{x_n\}$ is said to be *strongly almost-convergent to* x if

$$s\text{-} \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_{k+m} = x$$

uniformly for $m \geq 0$.

Clearly, the $s\sigma$-limit is weaker than the strong limit but stronger than the σ-limit, and the strong-almost-convergence implies the weak-almost-convergence. When X is a Euclidean space, the latter two kinds of convergence are equivalent.

The purpose of this paper is to study some properties of σ-limit and $s\sigma$-limit and their applications to description of continuity and convergence of vector-valued functions.

In Section 2, we examine basic properties of $s\sigma$-limit. It is seen that a bounded sequence in a Banach space having the $s\sigma$-limit x must be strongly almost-convergent to x (Proposition 2.4). We also show that the space X_σ of all σ-convergent sequences and the space $X_{s\sigma}$ of all $s\sigma$-convergent sequences in X are Banach spaces (see Theorems 2.6 and 2.7). In particular, the space $\mathbb{C}_{s\sigma}$ is a unital Banach subalgebra of ℓ^∞ (see Corollary 2.9). This also implies that every Banach limit restricted to $\mathbb{C}_{s\sigma}$ is a multiplicative linear functional.

In Section 3, we shall study how $s\sigma$-limit is related to or implies some properties of functions, such as continuity (see Proposition 3.1), uniform continuity (see Proposition 3.2) and measurability (see Lemma 3.3) of limit function, and dominated convergence theorem (Theorem 3.5).

2. Basic Properties of σ-limit and $s\sigma$-limit

Some basic properties of σ-limit on Banach spaces can be found in [5]. We state a characterization theorem for σ-limits in the following.

Theorem 2.1. *[5, Theorem 3.2(d)] Let $\{x_n\}$ be a bounded sequence in a normed linear space X. Then $\{x_n\}$ is weakly almost-convergent to x if and only if $\sigma\text{-}\lim x_n = x$.*

Lemma 2.2. *Let $\{a_n\}$ be a bounded sequence of nonnegative numbers such that σ-$\lim a_n = 0$. If $0 \leq b_n \leq a_n$ for all $n \geq 1$, then $s\sigma$-$\lim b_n = 0$.*

Proof. For every Banach limit ϕ, we have $0 \leq \phi(\{b_n\}) \leq \phi(\{a_n\}) = \sigma$-$\lim a_n = 0$. Therefore σ-$\lim b_n = 0$. Then we have σ-$\lim |b_n| = \sigma$-$\lim b_n = 0$. Hence $s\sigma$-$\lim b_n = 0$.

Lemma 2.3. *Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in a normed linear space X and let $x \in X$. The following statements hold.*

(i) *If $s\sigma$-$\lim x_n = x$, then σ-$\lim x_n = x$.*

(ii) *$s\sigma$-$\lim$ is linear, i.e., if $s\sigma$-$\lim x_n = x$ and $s\sigma$-$\lim y_n = y$, then $s\sigma$-$\lim(cx_n + y_n) = cx + y$ for every scalar c.*

(iii) *Define, for every $\varepsilon > 0$, $E_\varepsilon := \{n \in \mathbb{N}; ||x_n - x|| \geq \varepsilon\}$. Then the following are equivalent:*

(a) $s\sigma$-$\lim x_n = x$;

(b) σ-$\lim ||x_n - x|| = 0$;

(c) σ-$\lim I_{E_\varepsilon}(n) = 0$ *for all $\varepsilon > 0$, where I_{E_ε} is the indicator function of the set E_ε.*

Proof. (i) Suppose $s\sigma$-$\lim x_n = x$. For every Banach limit ϕ there is a $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $\lambda\phi(\{\langle x_n - x, x^*\rangle\}) = |\phi(\{\langle x_n - x, x^*\rangle\})|$. Since Φ is a positive linear functional, we have for every $x^* \in X^*$

$$\begin{aligned}|\phi(\{\langle x_n - x, x^*\rangle\})| &= \phi(\lambda\{\langle x_n - x, x^*\rangle\}) \\ &= \phi(\mathrm{Re}(\lambda\{\langle x_n - x, x^*\rangle\})) + \imath\phi(\mathrm{Im}(\lambda\{\langle x_n - x, x^*\rangle\})) \\ &= \phi(\mathrm{Re}(\lambda\{\langle x_n - x, x^*\rangle\})) \\ &\leq \phi(\{|\langle x_n - x, x^*\rangle|\}) \leq \phi(\{||x_n - x||\})||x^*|| = 0.\end{aligned}$$

Therefore σ-$\lim x_n = x$.

(ii) Since the σ-limit is linear and $||cx_n + y_n - (cx + y)|| \leq |c| \cdot ||x_n - x|| + ||y_n - y||$ for all $n \geq 1$, it follows from the hypothesis and Lemma 2.2 that $s\sigma$-$(cx_n + y_n) = cx + y$.

(iii) The equivalence of (a) and (b) follows from the definition of $s\sigma$-limit.

(b) $\Rightarrow$ (c). Let $\phi \in \pi_\sigma$ and $\varepsilon > 0$ be arbitrary. Then we have

$$0 = \phi(\{||x_n - x||\}) = \phi(\{||x_n - x||I_{E_\varepsilon}(n)\}) + \phi(\{||x_n - x||I_{E_\varepsilon^c}(n)\}),$$

where $E_\varepsilon^c := \mathbb{N} \setminus E_\varepsilon$. Since ϕ is positive, we must have

$$0 \leq \varepsilon\phi(\{I_{E_\varepsilon(n)}\}) \leq \phi(\{||x_n - x||I_{E_\varepsilon}(n)\}) = 0.$$

Therefore $\phi(\{I_{E_\varepsilon}(n)\}) = 0$ for all $\phi \in \pi_\sigma$ and hence (c) holds.

(c) $\Rightarrow$ (b). Suppose $\phi \in \pi_\sigma$. Then for any $0 < \varepsilon < 1$

$$\begin{aligned} 0 &\le \phi(\{||x_n - x||\}) \\ &= \phi(\{||x_n - x||I_{E_\varepsilon}(n)\}) + \phi(\{||x_n - x||I_{E_\varepsilon^c}(n)\}) \\ &\le \sup_{n\ge1} ||x_n - x||\phi(\{I_{E_\varepsilon}(n)\}) + \varepsilon\phi(\{I_{E_\varepsilon^c}(n)\}) \\ &\le 0 + \varepsilon. \end{aligned}$$

This shows that $\phi(\{||x_n - x||\}) = 0$ for all $\phi \in \pi_\sigma$ and hence (b) holds.

Proposition 2.4. *Let $\{x_n\}$ be a bounded sequence in a normed linear space X and let $x \in X$.*

(i) *If $s\sigma$-$\lim x_n = x$, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that s-$\lim x_{n_k} = x$.*

(ii) *If $s\sigma$-$\lim x_n = x$, then $\{x_n\}$ is strongly almost-convergent to x.*

Proof. (i) Let E_ε be as defined in Lemma 2.3. If $s\sigma$-$\lim x_n = x$, it follows from Lemma 2.3(iii) that $\phi(\{I_{E_\varepsilon}(n)\}) = 0$ for all $\phi \in \pi_\sigma$. Therefore $\phi(\{I_{E_\varepsilon^c}(n)\}) = 1$ for $\phi \in \pi_\sigma$ and hence E_ε^c is an infinite set for all $\varepsilon > 0$. So there is a subsequence $\{n_k\}$ of $\{n\}$ such that $n_k \in E_{1/k}^c$. Then $||x_{n_k} - x|| < \frac{1}{k}$ for $k \ge 1$. Thus $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ converging to x.

(ii) Since σ-$\lim ||x_n - x|| = 0$ and since for every positive integer n and nonnegative integer m

$$0 \le ||\frac{1}{n+1}\sum_{k=0}^{n} x_{k+m} - x|| \le \frac{1}{n+1}\sum_{k=0}^{n} ||x_{k+m} - x||,$$

it follows by applying Theorem 2.1 to $\{||x_n - x||\}$ that $\{x_n\}$ is strongly almost-convergent to x.

Remarks. (i) In general, σ-$\lim x_n = x$ does not imply $s\sigma$-$\lim x_n = x$. For example, if $X = \mathbb{C}$ and $x_n = (-1)^n$, $n \ge 1$, then $\{(-1)^n\}$ is strongly almost-convergent to 0 and hence σ-$\lim x_n = 0$, by Theorem 2.1. But $s\sigma$-$\lim x_n$ does not exist. In fact, if $s\sigma$-$\lim x_n = x$, then $x = \sigma$-$\lim x_n = 0$, by Proposition 2.3(i). Thus we have $s\sigma$-$\lim x_n = 0$, which contradicts the fact that $|x_n - 0| = |(-1)^n - 0| = 1$.

(ii) Clearly, strong convergence implies $s\sigma$-convergence. But the converse is not true. For example, consider the sequence

$$a_n = \begin{cases} 1 & \text{if } n = 2^k \text{ for some } k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

$\{a_n\}$ does not converge to 0, while $s\sigma\text{-}\lim a_n = 0$. Indeed, we have for every $k \in \mathbb{N}$ and $2^k \leq n < 2^{k+1}$

$$0 \leq \frac{1}{n+1} \sum_{j=0}^{n} a_j \leq \frac{k+1}{2^k} \to 0 \text{ as } n \to \infty$$

and

$$\begin{aligned}
\frac{1}{n+1} \sum_{j=0}^{n} a_{j+m} &= (n+1)^{-1}(\#\{k \in \mathbb{N}; 0 \leq 2^k \leq n+m\} - \#\{k \in \mathbb{N}; 0 \leq 2^k < m\}) \\
&\leq \frac{\log_2(n+m) - \log_2(m) + 1}{n+1} \\
&= \frac{1}{n+1}(\log_2(\frac{n}{m} + 1) + 1) \\
&\leq \frac{1}{n+1}(\log_2(n+1) + 1) \to 0 \text{ as } n \to \infty
\end{aligned}$$

uniformly for $m \geq 1$. Therefore $\sigma\text{-}\lim a_n = 0$ by Theorem 2.1. It follows from Lemma 2.2 that $s\sigma\text{-}\lim a_n = 0$.

Theorem 2.5. *Let $\{a_n\} \in \ell^\infty$ and let $\{x_n\}$ be a bounded sequence in a normed linear space X. Suppose $\sigma\text{-}\lim a_n = a$ and $\sigma\text{-}\lim x_n = x$.*

(a) *If $s\sigma\text{-}\lim a_n = a$ or $s\sigma\text{-}\lim x_n = x$, then $\sigma\text{-}\lim a_n x_n = ax$.*

(b) *If $s\sigma\text{-}\lim a_n = a$ and $s\sigma\text{-}\lim x_n = x$, then $s\sigma\text{-}\lim a_n x_n = ax$.*

Proof. Let $M := \sup_{n \geq 1}(||x_n|| + |a_n|)$. First, we assume $s\sigma\text{-}\lim a_n = a$. Since $||(a_n - a)x_n|| \leq M|a_n - a|$ for all $n \geq 1$, it follows from Lemma 2.2 and Lemma 2.3(ii) that $s\sigma\text{-}\lim(a_n - a)x_n = 0$. Therefore we obtain

$$\sigma\text{-}\lim(a_n x_n) = \sigma\text{-}\lim(a_n - a)x_n + \sigma\text{-}\lim(a x_n) = 0 + ax.$$

The proof of (a) for the case $s\sigma\text{-}\lim x_n = x$ is similar. Next, we assume $s\sigma\text{-}\lim a_n = a$ and $s\sigma\text{-}\lim x_n = x$. Since for every $n \geq 1$

$$||a_n x_n - ax|| \leq ||(a_n - a)x_n|| + ||a(x_n - x)|| \leq M|a_n - a| + M||x_n - x||,$$

it follows from Lemma 2.2 that $s\sigma\text{-}\lim(a_n x_n) = ax$. This proves (b).

Remark. The $s\sigma$-lim in (b) of Theorem 2.5 can not be replaced by σ-limit. To see this, we put $a_n = (-1)^n$ and $x_n = (-1)^n$ then $\{a_n x_n\}$=1 for all $n \in \mathbb{N}$. So,

$$\sigma\text{-}\lim a_n x_n = 1 \neq 0 = (\sigma\text{-}\lim a_n)(\sigma\text{-}\lim x_n).$$

We consider the following spaces of sequences in a Banach space X:

$\ell^\infty(X) :=$ the space of all bounded sequences in X equipped with the sup-norm.
$X_\sigma := \{\{x_n\} \in \ell^\infty(X); \sigma\text{-}\lim x_n = x \text{ for some } x \in X\}$.
$X_{s\sigma} := \{\{x_n\} \in \ell^\infty(X); s\sigma\text{-}\lim x_n = x \text{ for some } x \in X\}$.

Since σ-lim and $s\sigma$-lim are linear, the spaces X_σ and $X_{s\sigma}$ are linear subspaces of $\ell^\infty(X)$. By Lemma 2.3(i), we have $X_{s\sigma} \subset X_\sigma \subset \ell^\infty(X)$. When $X = \mathbb{C}$, we obtain from Theorem 2.5(b) that $\mathbb{C}_{s\sigma}$ is a unital algebra.

Theorem 2.6. *Suppose X is a Banach space. Then X_σ is a closed linear subspace of $\ell^\infty(X)$.*

Proof. Suppose $\{\mathbf{w}^{(m)}\}$ is a sequence in X_σ. Then for every $m \geq 1$ there exists $y_m \in X$ such that

$$\sigma\text{-}\lim\langle x_n^{(m)}, x^*\rangle = \langle y_m, x^*\rangle \text{ for all } x^* \in X^*, \tag{2.1}$$

where $\mathbf{w}^{(m)} = \{x_n^{(m)}\}$ for $m = 1, 2, \ldots$. If $\{\mathbf{w}^{(m)}\}$ converges to a point $\{x_n\}$ in $\ell^\infty(X)$, then we have

$$\sup_{n\geq 1} ||x_n^{(m)} - x_n|| \to 0 \text{ as } m \to \infty, \tag{2.2}$$

so that for any $\varepsilon > 0$ there is an integer $m_0 \geq 1$ such that

$$\sup_{n\geq 1} ||x_n^{(m_1)} - x_n^{(m_2)}|| < \varepsilon \text{ for all } m_1, m_2 \geq m_0. \tag{2.3}$$

First, we show that $\{y_m\}$ strongly converges to a point $y \in X$. It follows from (2.1) and (2.3) that for every $m_1, m_2 \geq m_0$ and $\phi \in \pi_\sigma$

$$\begin{aligned}
&|\langle y_{m_1} - y_{m_2}, x^*\rangle| \\
&= |\phi(\{\langle x_n^{(m_1)} - x_n^{(m_2)}, x^*\rangle\})| \\
&\leq \sup_{n\geq 1} ||x_n^{(m_1)} - x_n^{(m_2)}|| \cdot ||x^*|| \\
&\leq \varepsilon \cdot ||x^*||.
\end{aligned}$$

Therefore we have

$$||y_{m_1} - y_{m_2}|| \leq \varepsilon \text{ for all } m_1, m_2 \geq m_0.$$

This proves that $\{y_m\}$ is a Cauchy sequence in X and hence $\{y_m\}$ strongly converges to a point $y \in X$.

Next, we show that $\{x_n\} \in X_\sigma$. For arbitrary $\phi \in \pi_\sigma$ and $x^* \in X^*$, using (2.1), (2.2) and the fact that $y_m \to y$ strongly as $m \to \infty$ we have for every $x^* \in X^*$

$$
\begin{aligned}
&|\phi(\{\langle x_n - y, x^*\rangle\})| \\
&\leq |\phi(\{\langle x_n - x_n^{(m)}, x^*\rangle\})| + |\phi(\{\langle x_n^{(m)} - y_m, x^*\rangle\})| + |< y_m - y, x^* >| \\
&\leq \sup_{n\geq 1} ||x_n - x_n^{(m)}|| \cdot ||x^*|| + 0 + ||y_m - y|| \cdot ||x^*|| \\
&\to 0 + 0 + 0 \text{ as } m \to \infty.
\end{aligned}
$$

This shows that $\phi(\{\langle x_n - y, x^*\rangle\}) = 0$ for all $\phi \in \pi_\sigma$. Therefore σ-$\lim x_n = y$ and hence $\{x_n\} \in X_\sigma$. This proves that X_σ is closed in $\ell^\infty(X)$.

Theorem 2.7. *Suppose X is a Banach space. Then $X_{s\sigma}$ is a closed linear subspace of $\ell^\infty(X)$.*

Proof. Suppose $\{\mathbf{w}^{(m)}\}$ is a sequence in $X_{s\sigma}$ converging to a point $\{x_n\}$ in $\ell^\infty(X)$, i.e., (2.2) holds. Then we have for every $m \geq 1$

$$s\sigma\text{-}\lim x_n^{(m)} = y_m \text{ for some } y_m \in X, \tag{2.4}$$

where $\mathbf{w}^{(m)} = \{x_n^{(m)}\}$ for $m = 1, 2, \ldots$. We need to show that $\{x_n\} \in X_{s\sigma}$. Since (2.4) implies σ-$\lim x_n^{(m)} = y_m$, by the proof of Theorem 2.6, we have that $\{y_m\}$ strongly converges to a point $y \in X$.

Since

$$||x_n - y|| \leq ||x_n - x_n^{(m)}|| + ||x_n^{(m)} - y_m|| + ||y_m - y|| \text{ for every } n, m \geq 1,$$

we obtain from assumptions (2.2) and (2.4) that for every $\phi \in \pi_\sigma$

$$
\begin{aligned}
&\phi(\{||x_n - y||\}) \\
&\leq \phi(\{||x_n - x_n^{(m)}||\}) + \phi(\{||x_n^{(m)} - y_m||\}) + ||y_m - y|| \\
&\leq \sup_{n\geq 1} ||x_n - x_n^{(m)}|| + 0 + ||y_m - y|| \\
&\to 0 \text{ as } m \to \infty.
\end{aligned}
$$

Therefore $s\sigma$-$\lim x_n = y$ and hence $\{x_n\} \in X_{s\sigma}$. Therefore $X_{s\sigma}$ is closed in $\ell^\infty(X)$. This completes the proof.

Corollary 2.8. *Suppose X is a Banach space. Let $\{\mathbf{w}^{(m)}\}$ be a sequence sequence in X_σ converging to a point $\{x_n\}$ in $\ell^\infty(X)$. Suppose for every $m \geq 1$ there is some $y_m \in X$ such that*

$$s\sigma\text{-}\lim_{n\to\infty} \langle x_n^{(m)}, x^*\rangle = \langle y_m, x^*\rangle \textit{ for all } x^* \in X^*, \tag{2.5}$$

where $\mathbf{w}^{(m)} = \{x_n^{(m)}\}$ for $m = 1, 2, \ldots$. Then

(i) $\{y_m\}$ *converges strongly to a point* $y \in X$;

(ii) $s\sigma$-$\lim\langle x_n, x^*\rangle = \langle y, x^*\rangle$ *for all* $x^* \in X^*$.

Proof. Since (2.5) implies (2.1), it follows from the proof of Theorem 2.6 that (i) holds. (ii) follows from Theorem 2.7 for the case $X \equiv \mathbb{C}$.

The following corollary is deduced from Theorem 2.5(b) and Theorem 2.7.

Corollary 2.9. $\mathbb{C}_{s\sigma}$ *is a unital Banach algebra.*

Remark. In view of Theorem 2.5(a) and Corollary 2.9, we see that every Banach limit on $\mathbb{C}_{s\sigma}$ is a multiplicative linear functional on $\mathbb{C}_{s\sigma}$. Now, suppose $\{a_n\}$ and $\{b_n\}$ are two bounded sequences of complex numbers such that $s\sigma$-$\lim a_n = a \neq 0$ with $\inf\limits_{n\geq 0} |a_n| > 0$ and $s\sigma$-$\lim b_n = b$. It follows from Proposition 3.1 by taking $f(x) := \frac{1}{x}$ that $s\sigma$-$\lim b_n/a_n = b/a$.

3. Description of Continuity and Convergence of Functions in Terms of σ-limit and $s\sigma$-limit.

In this section, we describe continuity and convergence of functions in terms of σ-limit and $s\sigma$-limit.

Proposition 3.1. *Let* X *and* Y *be two normed linear spaces and let* $f : \Omega \to Y$ *be a function, where* Ω *is a nonempty subset of* X. *If* $x \in \Omega$ *and* f *is locally bounded on an open ball* $B_\Omega(x; r)$ *with center at* x, *then the following are equivalent*:

(a) f *is continuous at* x.

(b) $s\sigma$-$\lim f(x_n) = f(x)$ *for all those bounded sequence* $\{x_n\}$ *in* $B_\Omega(x; r)$ *which satisfy* $s\sigma$-$\lim x_n = x$.

Proof. $(a) \Longrightarrow (b)$: Suppose that f is continuous at x. Let $\{x_n\}$ be an arbitrary bounded sequence in $B_\Omega(x; r)$ such that $s\sigma$-$\lim x_n = x$. It follows from Lemma 2.3(iii) that σ-$\lim I_{E_\delta}(n) = 0$ for all $\delta > 0$, where E_δ is defined as in Lemma 2.3(iii). Since f is continuous at x, we have for every $\varepsilon > 0$ there is a $\delta > 0$ such that $||f(y) - f(x)|| < \varepsilon$ whenever $y \in \Omega$ and $||x - y|| < \delta$. Thus we have

$$K_\varepsilon := \{n; ||f(x_n) - f(x)|| \geq \varepsilon\} \subset E_\delta.$$

Since $\{f(x_n)\}$ is bounded and σ-$\lim I_{E_\delta}(n) = 0$, it follows from Lemma 2.2 that σ-$\lim I_{K_\varepsilon}(n) = 0$. By Lemma 2.3(iii) again, we obtain that $s\sigma$-$\lim f(x_n) = f(x)$. This proves (b).

$(b) \Rightarrow (a)$: Suppose f is not continuous at x. Then there is a positive number $\varepsilon > 0$ and a sequence $\{x_n\}$ in $B_\Omega(x;r)$ such that $\lim_{n\to\infty} x_n = x$ but $||f(x_n) - f(x)|| \geq \varepsilon$ for all $n \geq 1$. By the assumption (b), we have $s\sigma\text{-}\lim f(x_n) = f(x)$. By Proposition 2.4, $\{f(x_n)\}$ has a subsequence converging strongly to $f(x)$. This contradicts the choice of $\{x_n\}$. Therefore f must be continuous at x.

In Proposition 3.1, the assertion $s\sigma\text{-}\lim x_n = x$ can not be replaced by $\sigma\text{-}\lim x_n = x$. To see this, we take X=ℓ_2, and $f(x) := ||x||_2$, $x \in X$. Then f is continuous. Let e_n, $n = 1, 2, 3\ldots$, be the standard coordinate unit vectors of ℓ_2. Then $e_n \to 0$ weakly, so $\sigma\text{-}\lim e_n = 0$. But $f(e_n) = 1$ for all $n \geq 1$, so $\sigma\text{-}\lim f(e_n) = 1 \neq f(0)$.

Since the function $f(x) := \ln(1 + |x|)$, $x \geq 0$, is continuous on $[0, \infty)$, it follows from Proposition 3.1 that, for a nonnegative bounded sequence $\{a_n\}$ in $\mathbb{R}$, if $\sigma\text{-}\lim a_n = 0$, then $\sigma\text{-}\lim f(a_n) = 0$.

Similarly, since the function $f(x) := x^r$, where $r \in \mathbb{R}$ (the real field) is a constant and $x \in \mathbb{R}$ such that x^r is well-defined, is continuous on its natural domain, if $\{a_n\}$ is a sequence in the domain of f such that $s\sigma\text{-}\lim a_n = a \in$ the domain of f, then, by Proposition 3.1, $s\sigma\text{-}\lim a_n^r = a^r$.

Proposition 3.2. *Suppose Ω is a metric space and a sequence of functions $f_n : \Omega \to X$, $n = 1, 2, \ldots$, is equicontinuous, that is, for every $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$||f_n(x) - f_n(y)|| < \varepsilon$$

for all $n = 1, 2, \ldots$, whenever $x, y \in \Omega$ with $d(x, y) < \delta$. If $\sigma\text{-}\lim f_n(\omega)$ exists for all $\omega \in D$, where D is a dense subset of Ω, then $\sigma\text{-}\lim f_n(\omega)$ exists for every $w \in \Omega$ and the limit function

$$g(\omega) := \sigma\text{-}\lim f_n(\omega),\ \omega \in \Omega,$$

is uniformly continuous.

Proof. We first show that g is uniformly continuous on $= D$. Then g can be extended to Ω as a uniformly continuous function. Let $\varepsilon > 0$ be arbitrary. Fix an $\phi \in \pi_\sigma$. Since $\{f_n\}$ is equicontinuous, there exists a $\delta > 0$ such that $||f_n(x) - f_n(y)|| < \varepsilon$ for every $x, y \in \Omega$ with $d(x, y) < \delta$ and for all $n = 1, 2, \ldots$. Hence for every $x^* \in X^*$ and $x, y \in D$ with $d(x, y) < \delta$,

$$\begin{aligned}|\langle g(x) - g(y), x^*\rangle| &= |\phi(\{\langle f_n(x) - f_n(y), x^*\rangle\})| \\ &\leq \phi(\{|\langle f_n(x) - f_n(y), x^*\rangle|\}) \leq \phi(\epsilon||x^*|| \cdot \mathbf{1}) = \varepsilon,\end{aligned}$$

where $\mathbf{1}$ is the unit in ℓ^∞. This shows that $||g(x) - g(y)|| \leq \varepsilon$. Therefore g is uniformly continuous on D and hence g can be extended to Ω as a uniformly continuous function such that $||g(x) - g(y)|| \leq \varepsilon$ whenever $d(x, y) < \delta$. Now, let $w_0 \in \Omega$, $\phi \in \pi_\sigma$ be arbitrary. Then there is a $w \in D$ such that $d(w, w_0) < \delta$, so $||g(w) - g(w_0)|| \leq \varepsilon$. Therefore we have for every $x^* \in X^*$, and $m = 1, 2, \ldots$,

$$\begin{aligned} &|\phi(\{\langle f_n(w_0), x^*\rangle\}) - \langle g(w_0), x^*\rangle| \\ &\leq |\phi(\{\langle f_n(w_0) - f_n(w), x^*\rangle\})| + |\langle g(w) - g(w_0), x^*\rangle| \\ &\leq \phi(\{||f_n(w_0) - f_n(w)|| \cdot ||x^*||\}) + ||g(w) - g(w_0)|| \cdot ||x^*|| \\ &\leq \varepsilon||x^*|| + \varepsilon||x^*|| = 2\varepsilon||x^*||. \end{aligned}$$

Since $\phi \in \pi_\sigma$, $\varepsilon > 0$ are arbitrary, this proves that $\sigma\text{-}\lim f_n(w_0)$ exists and is equal to $g(w_0)$. The proof is complete.

Lemma 3.3. *Suppose* (Ω, Σ, μ) *is a measure space and* $f_n : \Omega \to \mathbb{C}, n = 1, 2, \ldots,$ *are Lebesgue measurable functions such that*

$$\sigma\text{-}\lim f_n = f \ a.e.[\mu].$$

Then f *is measurable.*

Proof. Define $f(\omega) := \sigma\text{-}\lim f_n(\omega)$. It follows from Theorem 2.1 that for every $m = 0, 1, 2, \ldots$

$$f(\omega) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f_{k+m}(\omega) \ \text{ a.e.}[\mu].$$

This proves that f is measurable.

Theorem 3.4. *Suppose* (Ω, Σ, μ) *is a measure space and let* $\{E_n\}$ *be a sequence of* μ*-measurable subsets of* Ω *such that* $s\sigma\text{-}\lim \mu(E_n) = 0$*. If* $f : \Omega \to X$ *is a Bochner integrable function, where* X *is a Banach space, then* $s\sigma\text{-}\lim \int_{E_n} f d\mu = 0$.

Proof. First, we suppose $X = \mathbb{R}$. Let $\{E_n\}$ be an arbitrary sequence of measurable subsets of Ω such that $\sigma\text{-}\lim \mu(E_n) = 0$. Let $D := \{f \in L^1(\mu); s\sigma\text{-}\lim \int_{E_n} f d\mu = 0\}$. Then D is a linear space. If A is a measurable subset of Ω, then $0 \leq \int_{E_n} I_A d\mu = \mu(A \cap E_n) \leq \mu(E_n)$ for all $n \geq 0$. By Lemma 2.2, we have $s\sigma\text{-}\lim \int_{E_n} I_A d\mu = 0$. Therefore $I_A \in D$ if $\mu(A) < \infty$, and hence D contains all simple functions in $L^1(\mu)$. Therefore D is dense in $L^1(\mu)$.

If $f \in L^1(\mu)$, then for every $\varepsilon > 0$ there is a simple function $h \in L^1(\mu)$ such that $||f - h||_1 < \varepsilon$. Since h is simple, we have for every $n \geq 0$

$$0 \leq \int_{E_n} |f| d\mu \leq \int_{E_n} |f - h| d\mu + \int_{E_n} |h| d\mu \leq ||f - h||_1 + \int_{E_n} |h| d\mu.$$

This implies that for every $\phi \in \pi_\sigma$

$$0 \le \phi(\{\int_{E_n} |f| d\mu\}) \le \varepsilon + \phi(\{\int_{E_n} |h| d\mu\}) = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\sigma\text{-lim} \int_{E_n} |f| d\mu = 0$.

Finally, if X is an arbitrary Banach space and $f : \Omega \to X$ is Bochner integrable, then

$$|| \int_{E_n} f d\mu|| \le \int_{E_n} ||f(\omega)||\mu(d\omega) \text{ for all } n \ge 1.$$

Since $\sigma\text{-lim} \int_{E_n} ||f(\omega)||\mu(d\omega) = 0$ by the above argument, we obtain from Lemma 2.2 that $s\sigma\text{-lim} \int_{E_n} f d\mu = 0$. This completes the proof.

In general, if $\{f_n\}$ is a sequence in $L^1(\mu)$ such that $\sup_{n\ge1} \int_\Omega |f_n|\, d\mu < \infty$, the equality

$$\sigma\text{-}\lim \int_\Omega f_n\, d\mu = \int_\Omega \sigma\text{-}\lim f_n(\omega)\mu(d\omega) \tag{3.1}$$

may be false. For instance, if $f_n := n\chi_{[0,\frac{1}{n}]}$, $n = 1, 2, \ldots$, then $\int_0^1 f_n(x)dx = 1$ for all $n \ge 1$ and $f_n \to 0$ a.e. and so $\sigma\text{-lim} f_n(\omega) = 0$ a.e. Thus $\int_0^1 \sigma\text{-}\lim f_n(\omega)d\omega = 0$, but $\sigma\text{-}\lim \int_0^1 f_n(\omega)d\omega = 1$. The following theorem presents two versions of dominated convergence theorem with respect to σ-limit and $s\sigma$-limit.

Theorem 3.5. *Suppose (Ω, Σ, μ) is a complete measure space and X is a complex Banach space. Then the following hold*:

(a) *If $f, f_n : \Omega \to \mathbb{C}$, $n = 1, 2, \ldots$, are measurable functions such that $\sigma\text{-}\lim f_n = f$ a.e. $[\mu]$, and if $|f_n| \le g$ a.e. $[\mu]$ for some $g \in L^1(\mu)$ and all $n = 1, 2, 3, \ldots$, then $\sigma\text{-lim} \int_\Omega f_n\mu = \int_\Omega f d\mu$.*

(b) *If $f, f_n : \Omega \to X$, $n = 1, 2, \ldots$, are strongly measurable functions such that $s\sigma\text{-lim} f_n = f$ a.e. $[\mu]$, and if $||f_n(w)|| \le ||g(w)||$ a.e. $[\mu]$ for some $g \in L^1(\mu)$ and all $n = 1, 2, \ldots$, then $s\sigma\text{-lim} \int_\Omega f_n\mu = \int_\Omega f d\mu$.*

Proof. (a) Suppose $\sigma\text{-lim} f_n = f$ a.e. $[\mu]$. By Lemma 3.3, f is measurable. If we define $h_n(w) := \sup_{k\ge0} |\frac{1}{n} \sum_{i=1}^{n} f_{i+k}(w) - f(w)|$ for $w \in \Omega$ and $n \ge 1$, then $h_n \to 0$ a.e. $[\mu]$ by Theorem 2.1 (with $X = \mathbb{C}$), so that $f(\omega) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n f_k(\omega)$ a.e.$[\mu]$. Since $|f_n| \le g$ a.e $[\mu]$ implies $|h_n| \le 2g$ a.e $[\mu]$ for all $n \ge 1$, by the dominated convergence theorem we have $\int_\Omega h_n d\mu \to 0$ as $n \to \infty$. Therefore

$$\sup_{k\ge0} |\frac{1}{n} \sum_{i=1}^{n} \int_\Omega f_{i+k} d\mu - \int_\Omega f d\mu| \le \int_\Omega h_n d\mu \to 0 \text{ as } n \to \infty.$$

This shows that $\{\int_\Omega f_n d\mu\}$ is strongly almost-convergent to $\int_\Omega f d\mu$ and hence weakly almost convergent. Hence we must have that σ-lim$\int_\Omega f_n d\mu = \int_\Omega f d\mu$ by Theorem 2.1.

(b) Since $s\sigma$-lim $f_n = f$ a.e. $[\mu]$, we have σ-lim $||f_n(w) - f(w)|| = 0$ a.e. $[\mu]$. By the assumption of (b) and Proposition 2.4(i), we have $||f_n(w) - f(w)|| \le 2g(w)$ a.e. $[\mu]$ for all $n \ge 1$. It follows from (a) that σ-lim$\int_\Omega ||f_n(w) - f(w)||\mu(dw) = 0$. Since for every $n \ge 1$

$$|| \int_\Omega f_n d\mu - \int_\Omega f d\mu|| \le \int_\Omega ||f_n(w) - f(w)||\mu(dw),$$

it follows from Lemma 2.2 that $s\sigma$-lim$\int_\Omega f_n d\mu = \int_\Omega f d\mu$. This proves (b).

The ordinary dominated convergence theorem can be found in [2] for the scalar-valued version and in [4, pg. 27] for the vector-valued version. As an application of Proposition 3.5, we end this paper with the following analog of the Riemann-Lebesgue lemma (cf. [3, p. 22])

Corollary 3.6. *Suppose* (Ω, Σ, μ) *is a complete measure space and* X *is a complex Banach space. Let* $f \in L^1(\Omega, \Sigma, \mu)$. *If* $h : \Omega \to \mathbb{R}$ *is a measurable function such that*

$$\mu\{\omega \in \Omega; (2\pi)^{-1}h(\omega) \text{ is an integer}\} = 0,$$

then

$$\sigma\text{- lim} \int_\Omega f \sin(nh) d\mu = \sigma\text{- lim} \int_\Omega f \cos(nh) d\mu = 0.$$

Proof. If $\theta \in \mathbb{R}$, then $\phi(\{e^{\imath n\theta}\}) = \phi(\{e^{\imath(n+1)\theta}\}) = e^{\imath\theta}\phi(\{e^{\imath n\theta}\})$ for every $\phi \in \pi_\sigma$. Therefore, for every $\theta \in \mathbb{R} \setminus (2\pi\mathbb{Z})$, we have $\phi(\{e^{\imath n\theta}\}) = 0$ for all $\phi \in \pi_\sigma$ and hence σ-lim $e^{\imath n\theta} = 0$, where $\mathbb{Z}$ is the set of all integers. This also implies that σ- lim $\cos(n\theta) = \sigma$- lim $\sin(n\theta) = 0$ for every $\theta \in \mathbb{R} \setminus (2\pi\mathbb{Z})$.

By the hypothesis, we have σ- lim $\cos(nh) = \sigma$- lim $\sin(nh) = 0$ a.e. $[\mu]$. It follows from Theorem 3.5(a) that

$$\sigma\text{- lim} \int_\Omega f \sin(nh) d\mu = \sigma\text{- lim} \int_\Omega f \cos(nh) d\mu = 0.$$

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