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GENERATION OF LOCAL C-SEMIGROUPS AND SOLVABILITY OF THE ABSTRACT CAUCHY PROBLEMS

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Abstract. For a bounded linear injection C on a Banach space X and a (not necessarily densely defined) closed linear operator A which commutes with C, we present various conditions for A to generate a local C-semigroup. A Hille-Yosida type generation theorem in terms of the asymptotic C-resolvent of A is proved, and various characterizations of a generator by means of existence of unique strong solutions of the associated abstract Cauchy problems are obtained.

1. Introduction

Let X be a Banach space with norm $\|\cdot\|$, and let $B(X)$ be the set of all bounded linear operators from X into itself. Consider the abstract Cauchy

$$ACP(A; f, x) \qquad \begin{cases} \dfrac{d}{dt}u(t) = Au(t) + f(t), \ 0 \le t < T, \\ u(0) = x, \end{cases}$$

where $A : D(A) \subset X \to X$ is a closed linear operator and f is an X-valued function on $[0, T)$, where T may be finite or infinite. A function $u : [0, T) \to X$ is called a strong solution of ACP($A; f, x$) if u is continuously differentiable, $u(t) \in D(A)$ for $0 \le t < T$, and satisfies ACP($A; f, x$).

The ACP is closely related to the theory of operator semigroup. It is known that ACP($A; 0, x$) has a unique strong solution for every $x \in D(A)$ if and only if the part

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A_1 of A in the Banach space X_1 (the space $D(A)$ equipped with the graph norm $\|x\|_1 = \|x\| + \|Ax\|$) generates a (C_0)-semigroup on X_1 (see [10, A-II, Theorem 1.1]). Moreover, when A has nonempty resolvent set $\rho(A)$, these two conditions are also equivalent to that A generates a (C_0)-semigroup on X [10, A-II, Corollary 1.2].

Let $C \in B(X)$ be an injection, and $T < \infty$ (resp. $T = \infty$). A family $\{S(t); 0 \leq t < T\}$ in $B(X)$ is called a *local C-semigroup* (resp. *C-semigroup*) if

(a) $S(\cdot)x : [0, T) \to X$ is continuous for each $x \in X$; and

(b) $S(s+t)C = S(s)S(t)$ for all $0 \leq t, s, t+s < T$ and $S(0) = C$.

The *generator* A of $S(\cdot)$ is defined as

(c) $$\begin{cases} D(A) = \{x \in X; \lim\limits_{h \to 0^+} (S(h)x - Cx)/h \in R(C)\} \\ Ax = C^{-1} \lim\limits_{h \to 0^+} (S(h)x - Cx)/h \text{ for } x \in D(A). \end{cases}$$

C-semigroups and their connections with the ACP have been studied in [2], [3], [7], [8], [14, 15], and other papers. If A is the generator of a C-semigroup, then A commutes with C and ACP$(A; 0, x)$ has a unique strong solution for each initial value x in $C(D(A))$ [3, Theorem 4.1]. The converse of the last statement is also true when $\rho(A) \neq \emptyset$ [15, Corollary 2.2]. These results extend the above cited result in [10, A-II, Corollary 1.2] to C-semigroups.

The concept of a local C-semigroup was first introduced and studied by Tanaka and Okazawa in [17]. Clearly, every C-semigroup can be viewed as a local C-semigroup on $[0, T)$ for any $0 < T < \infty$. In general, a local C-semigroup on $[0, T)$ for some $T < \infty$ is not necessarily extendable to the whole half line $[0, \infty)$ except when $C = I$, the identity operator. For instance, the family $\{S(t); 0 \leq t \leq 1\}$ of operators on c_0, defined by $S(t)x := (e^{-n}e^{nt}x_n)$ for $x = (x_n) \in c_0$ and $t \in [0, 1]$, is a local C-semigroup (with $C : x \to (e^{-n}x_n)$) which is strongly continuous on $[0, 1]$ and satisfies $\|S(t)\| = e^{t-1} \to \|S(1)\| = 1$ as $t \to 1^-$. But it cannot be continuously extended beyond the point $t = 1$. Results concerning extension of local C-semigroups can be found in [4] and [18].

Under the assumption that C has a dense range $R(C)$ (which implies that the generator A has a dense domain $D(A)$, but not the converse), Tanaka and Okazawa proved a Hille-Yosida type generation theorem [17, Theorem 2.1] for $S(\cdot)$ in terms of the *asymptotic C-resolvent* of its *complete infinitesimal generator* $\overline{G}$. In [4, Theorem 2.4], Gao proved a version of the generation theorem for densely defined generators (though without the assumption of denseness of $R(C)$).

In this paper, for local C-semigroups with generators not necessarily densely defined, we prove generalizations and improvements of the aforementioned theorems in [10] and [14], and we characterize generators in terms of unique existence of solutions of the associated Cauchy problems and integral equations.

In Section 2, we first prepare some preliminary results about basic properties of local C-semigroups, and then we prove a generation theorem (Theorem 2.5) in terms of the asymptotic C-resolvent of its generator A (instead of its complete infinitesimal generator $\overline{G}$). Our formulation and assumption are simpler than those in [17], and our proof is based on a different approach, namely, the Widder-Arendt theorem about Laplace transform (cf. [1], [7, Theorem 2.2]).

In Section 3, we show that, under the assumption that A commutes with C, the problem ACP($A; 0, Cx$) has a unique strong solution for every $x \in D(A)$ if and only if A_1 is the generator of a local C_1-semigroup on X_1, where C_1 is the restriction of C to X_1 (Theorem 3.2). It is also shown that A generates a local C-semigroup on X if and only if $C^{-1}AC = A$ and the problem ACP($A; Cx+\int_0^t Cg(s)ds, 0$) has a unique solution for every $g \in L^1_{loc}([0,T), X)$ and $x \in X$, if and only if $C^{-1}AC = A$ and the problem ACP($A; Cx, 0$) has a unique strong solution $u(\cdot; Cx, 0)$ for every $x \in X$ (Theorem 3.4). In case that A has a dense domain, these conditions are also equivalent to the condition that $C^{-1}AC = A$ and for each $x \in D(A)$, the ACP($A; 0, Cx$) has a unique strong solution $u(\cdot; 0, Cx)$ which depends continuously on x (Corollary 3.6). An illustrating example will be given at the end of the paper. Finally, for further discussion on strong and weak solutions of the abstract Cauchy problems associated with local C-semigroups and perturbation of local C-semigroups, the readers are referred to [6, 9, 13].

2. Some Basic Properties and a Generation Theorem

In this section, we prove some basic properties and a generation theorem for local C-semigroups.

Lemma 2.1. *Let $C \in B(X)$ be an injection and $\{S(t); 0 \le t < T\}$ be a local C-semigroup with generator A. The following are true:*

(2.1) $S(s)S(t) = S(t)S(s)$ *for all* $0 \le s,t < T$;

(2.2) *If* $x \in D(A)$, *then* $S(t)x \in D(A)$, $AS(t)x = S(t)Ax$ *and* $\int_0^t S(s)Axds = S(t)x - Cx$ *for* $0 \le t < T$;

(2.3) $\int_0^t S(s)xds \in D(A)$ *and* $A\int_0^t S(s)xds = S(t)x - Cx$ *for* $x \in X$ *and* $0 \le t < T$;

(2.4) *A is closed and satisfies* $C^{-1}AC = A$;

(2.5) $R(C) \subset \overline{D(A)}$.

Proof. First, taking $s = 0$ in (b) gives $S(t)C = CS(t)$ for all $0 \le t < T$. To show (2.1), we fix an arbitrary $t \in [0,T)$. It is seen from (b) that the identity in (2.1) holds for all $s \in [0, T-t)$. We need to extend s to the whole interval $[0,T)$.

Take a $\theta > 1$ such that $t < \theta t < T$. For any $s \in [0, T)$, we write $s = n(T - \theta t) + r$ for some $n \in \mathbb{N}$ and $0 \le r < T - \theta t$. Then, by (b), we have

$$\begin{aligned} C^n S(s)S(t) &= C^n S(n(T-\theta t)+r)S(t) = (S(T-\theta t))^n S(r)S(t) \\ &= (S(T-\theta t))^n S(t)S(r) = S(t)(S(T-\theta t))^n S(r) \\ &= S(t)C^n S(s) = C^n S(t)S(s), \end{aligned}$$

so that $S(s)S(t) = S(t)S(s)$, by the injectivity of C.

To show that (2.2) holds, let $x \in D(A)$ and $t \in [0, T)$. Then for all $s \in [0, T-t)$ we have

$$\begin{aligned} &s^{-1}[S(s)\,S(t)x - C\,S(t)x] = S(t)[s^{-1}(S(s)x - Cx)] \\ &\to S(t)CAx = CS(t)Ax \in R(C) \end{aligned}$$

as $s \to 0$. This means that $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$. It also shows $\frac{d}{dt}CS(t)x = CS(t)Ax$, so that $C\int_0^t S(s)Axds = CS(t)x - C^2x$, hence (2.2) is proved. We next prove (2.3). Using (a) and (b), we have, for all $x \in X$

$$\begin{aligned} &s^{-1}[S(s)\int_0^t S(\tau)xd\tau - C\int_0^t S(\tau)xd\tau] \\ &= s^{-1}[\int_0^t S(s+\tau)Cxd\tau - \int_0^t S(\tau)Cxd\tau] \\ &= s^{-1}[\int_t^{t+s} S(\tau)Cxd\tau - \int_0^s S(\tau)Cxd\tau] \\ &\to S(t)Cx - C^2x = C(S(t)x - Cx) \text{ as } s \to 0. \end{aligned}$$

Thus $\int_0^t S(\tau)xd\tau \in D(A)$ and $A\int_0^t S(\tau)xd\tau = S(t)x - Cx$ for $x \in X$ and $0 \le t < T$.

To show that A is closed, let $x_n \in D(A)$, $x_n \to x$, and $Ax_n \to y$. Then by (2.2) we have $S(t)x_n - Cx_n = \int_0^t S(s)Ax_nds$, from which it follows that, as $n \to \infty$, $S(t)x - Cx = \int_0^t S(s)yds$ for all $t \in [0, T)$ and

$$t^{-1}(S(t)x - Cx) = t^{-1}\int_0^t S(s)yds \to Cy \text{ as } t \to 0.$$

This shows that $x \in D(A)$ and $Ax = y$, and so A is a closed linear operator in X. Finally we show that $C^{-1}AC = A$. The relation $A \subset C^{-1}AC$ follows immediately from (2.2) with $t = 0$. To show the converse, let $x \in D(C^{-1}AC)$, i.e., $Cx \in D(A)$ and $ACx \in R(C)$. Then by (2.2), we have

$$C(S(t)x - Cx) = S(t)Cx - C^2x = \int_0^t S(\tau)ACxd\tau = C\int_0^t S(\tau)C^{-1}ACxd\tau$$

from which it follows that

$$t^{-1}(S(t)x - Cx) = t^{-1}\int_0^t S(\tau)C^{-1}ACxd\tau \to ACx \in R(C)$$

as $t \to 0$. This means that $x \in D(A)$ and $Ax = C^{-1}ACx$. Finally, the fact that $\int_0^t S(s)xds \in D(A)$ for all $x \in X$ and $t^{-1}\int_0^t S(s)xds \to Cx$ as $t \to 0$ implies (2.5).

Remark. It is easy to see from (2.2) that the definition of generators as given in (c) is equivalent to

$$x \in D(A) \text{ and } Ax = y \Longleftrightarrow \int_0^t S(s)yds = S(t)x - Cx \text{ for all } t \in [0, T).$$

Proposition 2.2. *Let $C \in B(X)$ be an injection and $\{S(t); 0 \leq t < T\}$ be a strongly continuous family of bounded linear operators on X. If A is a closed operator such that*

$$\text{(2.6)} \qquad R(\int_0^t S(s)ds) \subset D(A) \text{ and } \int_0^t S(s)Ads \subset A\int_0^t S(s)ds = S(t) - C$$

for all $0 \leq t < \tau$, then $S(\cdot)$ is a local C-semigroup with generator $C^{-1}AC$.

Proof. For any fixed $t \in (0, T)$, all $r \in (0, t)$, and all $x \in X$,

$$\begin{aligned}\frac{d}{dr}[S(t-r)\int_0^r S(u)xdu] &= -S(t-r)A\int_0^r S(u)xdu + S(t-r)S(r)x \\ &= -S(t-r)[S(r)x - Cx] + S(t-r)S(r)x \\ &= S(t-r)Cx,\end{aligned}$$

so that , by integration with respect to r on $[0, s]$, we have for $s \in [0, t]$

$$S(t-s)\int_0^s S(u)xdu = \int_0^s S(t-r)Cxdr.$$

Since A is closed, it follows from (2.6) that $S(t)Ax = AS(t)x$ for all $x \in D(A)$. Using these facts we can write

$$\begin{aligned}S(t-s)S(s)x &= S(t-s)[A\int_0^s S(u)xdu + Cx] \\ &= A(S(t-s)\int_0^s S(u)xdu) + S(t-s)Cx \\ &= A\int_0^s S(t-r)Cxdr + S(t-s)Cx \\ &= A\int_{t-s}^t S(\tau)Cxd\tau + S(t-s)Cx\end{aligned}$$

$$= (S(t) - C)Cx - (S(t-s) - C)Cx + S(t-s)Cx$$
$$= S(t)Cx$$

for all $x \in X$ and $0 \leq s \leq t < T$. Since $S(0) = C$ by (2.6), we have shown that $S(\cdot)$ is a local C-semigroup.

Let B be the generator of $S(\cdot)$. It follows immediately from (2.6) and the definition of generator that $A \subset B$. As B is a generator, (2.4) implies $B = C^{-1}BC \supset C^{-1}AC$. To show the converse, let $x \in D(B)$. Then (2.6) and (2.2) (applied to B) imply that

$$\int_0^t S(s)Bxds = S(t)x - Cx = A \int_0^t S(s)xds.$$

Differentiating both sides and using the closedness of A, we obtain $S(t)x \in D(A)$ and $AS(t)x = S(t)Bx$ for all $t \in [0, T)$. In particular, $Cx \in D(A)$ and $ACx = CBx$, i.e., $x \in D(C^{-1}AC)$ and $C^{-1}ACx = Bx$. The proof is complete. Now

Lemma 2.1 and Proposition 2.2 yield the next characterization of a generator.

Corollary 2.3. *Let $C \in B(X)$ be an injection and $\{S(t); 0 \leq t < T\}$ be a strongly continuous family of bounded linear operators on X. Then $S(\cdot)$ is a local C-semigroup with generator A if and only if* (2.4) *and* (2.6) *hold.*

In general, the generator A is not necessarily densely defined; it is so when the range $R(C)$ of C is dense in X, by (2.5). In [17], Tanaka and Okazawa considered local C-semigroups for the case that C has a dense range. They considered the so-called *complete infinitesimal generator*, which is the closure $\overline{G}$ of the closable operator G defined by

$$(2.7) \qquad \begin{cases} D(G) = \{x \in R(C); \lim_{h \to 0^+} (C^{-1}S(h)x - x)/h \text{ exists}\} \\ Gx = \lim_{h \to 0^+} (C^{-1}S(h)x - x)/h \text{ for } x \in D(G). \end{cases}$$

The operator $\overline{G}$ also satisfies (2.2) and (2.3) with A replaced by $\overline{G}$ (see [17]). From the identity $\int_0^t S(s)\overline{G}xds = S(t)x - Cx$ ($x \in D(\overline{G})$) it follows that $\overline{G} \subset A$, and hence $C^{-1}GC \subset C^{-1}\overline{G}C \subset C^{-1}AC = A$. Conversely, if $x \in D(A)$, then $Cx \in D(G)$ and $GCx = CAx$, by (2.7). Thus the following relation holds:

$$(2.8) \qquad G \subset \overline{G} \subset C^{-1}\overline{G}C = C^{-1}GC = A.$$

Proofs of this fact for the case $T = \infty$ can be found in [3] and [7]. Moreover, if $\rho(\overline{G}) \neq \emptyset$, then $\overline{G} = C^{-1}\overline{G}C$ (see [15, Proposition 1.4]) and so $\overline{G} = A$, by (2.8).

The *local Laplace transform* of $\{S(t); 0 \le t < T\}$ is the family $\{L_\tau(\lambda); \tau \in (0,T), \lambda \in \mathbb{R}\}$ of operators defined by

$$L_\tau(\lambda)x = \int_0^\tau e^{-\lambda t} S(t)x dt \ (x \in X). \tag{2.9}$$

This is clearly a commutative family which also commutes with C because $S(\cdot)$ is so, by (2.1).

Lemma 2.4. *Let $C \in B(X)$ be an injection and $\{S(t); 0 \le t < T\}$ be a local C-semigroup with generator A. The following hold:*

(2.10) *$R(L_\tau(\lambda)) \subset D(A)$ and $L_\tau(\lambda)(\lambda - A) \subset (\lambda - A)L_\tau(\lambda) = C - e^{-\tau\lambda}S(\tau)$ for all $\tau \in [0,T)$ and $\lambda \in \mathbb{R}$;*

(2.11) *for $x \in X$, $L_\tau(\lambda)x$ is infinitely differentiable in λ and*

$$\left\| \frac{\lambda^n}{(n-1)!}(d/d\lambda)^{n-1} L_\tau(\lambda) \right\| \le M_\tau := \sup_{t\in[0,\tau]} \|S(t)\|$$

for $\lambda > 0$ and $n \in \mathbb{N}$.

Proof. Using integration by parts and the closedness of A, we have for all $x \in X$

$$\begin{aligned}(\lambda - A)L_\tau(\lambda)x &= \lambda L_\tau(\lambda)x - A[e^{\lambda\tau}\int_0^\tau S(s)xds + \lambda\int_0^\tau e^{-\lambda t}\int_0^t S(s)xdsdt] \\ &= \lambda L_\tau(\lambda)x - e^{-\lambda\tau}(S(\tau)x - Cx) - \lambda\int_0^\tau e^{-\lambda t}(S(t)x - Cx)dt \\ &= -e^{-\lambda\tau}S(\tau)x + e^{-\lambda\tau}Cx + \lambda\int_0^\tau e^{-\lambda t}Cxdt \\ &= Cx - e^{-\lambda\tau}S(\tau)x.\end{aligned}$$

That $L_\tau(\lambda)A \subset AL_\tau(\lambda)$ follows from (2.2) and the closedness of A. Thus (2.10) holds.(2.11) is proved in [17, Proposition 1.2].

For a closed linear operator A in X and $\tau > 0$, a family $\{L_\tau(\lambda); 0 < \tau < T, \lambda > 0\}$ in $B(X)$ will be called an *asymptotic C-resolvent* of A if, for each $\tau \in (0,T)$, the subfamily $\{L_\tau(\lambda); \lambda > 0\}$ is commutative and satisfies the following condition:

(2.12) $R(L_\tau(\lambda)) \subset D(A)$ and $L_\tau(\lambda)(\lambda - A) \subset (\lambda - A)L_\tau(\lambda) = C + V_\tau(\lambda)$ for all $\lambda > 0$, where for $x \in X$, both $L_\tau(\lambda)x$ and $V_\tau(\lambda)x$ are infinitely differentiable for $\lambda > 0$ and there exists a constant $M_\tau > 0$, depending on τ, such that

$$\|(d/d\lambda)^{n-1}V_\tau(\lambda)x\| \le M_\tau \tau^{n-1} e^{-\tau\lambda}\|x\|$$

for $x \in X$, $\lambda > 0$ and $n \in \mathbb{N}$.

It is easy to see that Lemmas 2.1 and 2.4 verify part (i) of the following generation theorem.

Theorem 2.5. *Let $C \in B(X)$ be an injection.*

(i) *If an operator A is the generator of a local C-semigroup $\{S(t); 0 \le t < T\}$ on X, then there exists an asymptotic C-resolvent $\{L_\tau(\lambda); 0 < \tau < T, \lambda > 0\}$ of A such that*

$$\left\| \frac{\lambda^n}{(n-1)!}(d/d\lambda)^{n-1} L_\tau(\lambda) \right\| \le M_\tau \tag{2.13}$$

for all $\tau \in (0, T)$, $\lambda > 0$ and $n \in \mathbb{N}$, and (2.14) *A is closed and satisfies $C^{-1}AC = A$.*

(ii) *If a closed linear operator A has an asymptotic C-resolvent $\{L_\tau(\lambda); 0 < \tau < T, \lambda > 0\}$ satisfying* (2.13)*, then $X_0 := \overline{D(A)}$ is invariant under C, and the operator $C_0^{-1} A_0 C_0$ generates a local C_0-semigroup $\{S(t); 0 \le t < T\}$ on X_0, with $C_0 := C|X_0$ and A_0 the part of A in X_0. If, in addition,* (2.14) *holds, then A_0 is the generator of $S(\cdot)$.*

To prove (ii), we need the following lemmas. The following key lemma is due to Y.-C. Li.

Lemma 2.6. *Let $V_\tau(\lambda)$ be as in (2.12), and let $H_n(\lambda) := \frac{\lambda^{n+1}}{n!}\frac{d^n}{d\lambda^n}[\lambda^{-2}V_\tau(\lambda)]$ for $\lambda > a$ and $n \in \mathbb{N}$. Then $\lim_{n\to\infty} H_n(\frac{n}{t}) = 0$ for $t \in (0, \tau)$.*

Proof. We have for every $n = 0, 1, 2, \ldots$ and $\lambda > 0$,

$$\begin{aligned} H_n(\lambda) &= \frac{\lambda^{n+1}}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{d\lambda^k}(\lambda^{-2})V_\tau^{(n-k)}(\lambda) \\ &= \sum_{k=0}^{n}\frac{k+1}{(n-k)!}(-1)^k\lambda^{n-k-1}V_\tau^{(n-k)}(\lambda) \\ &= \sum_{k=0}^{n}\frac{n-k+1}{k!}(-1)^{n-k}\lambda^{k-1}V_\tau^{(k)}(\lambda). \end{aligned}$$

By assumption, we have for every $0 < t < \tau$,

$$\begin{aligned} ||H_n(\frac{n}{t})|| &\le \sum_{k=0}^{n}\frac{n-k+1}{k!}(\frac{n}{t})^{k-1}||V_\tau^{(k)}(\frac{n}{t})|| \\ &\le \sum_{k=0}^{n}\frac{n-k+1}{k!}(\frac{n}{t})^{k-1}M\tau^k e^{-\tau\frac{n}{t}} \\ &= \frac{Mt}{n}\sum_{k=0}^{n}\frac{n-k+1}{k!}(\frac{n}{t})^k\tau^k e^{-\tau\frac{n}{t}} \end{aligned}$$

$$
\begin{aligned}
&= Mt\{\frac{n+1}{n}\sum_{k=0}^{n}\frac{1}{k!}(\frac{n\tau}{t})^k e^{-\tau\frac{n}{t}} - \frac{\tau}{t}\sum_{k=0}^{n-1}\frac{1}{k!}(\frac{n\tau}{t})^k e^{-\tau\frac{n}{t}}\} \\
&\le M\tau\frac{1}{n!}(\frac{n\tau}{t})^n e^{-\tau\frac{n}{t}} \text{ for } \frac{n+1}{n} < \frac{\tau}{t}.
\end{aligned}
$$

Using the Stirling formula: $\lim_{t\to\infty}\frac{\Gamma(t+1)}{(t/e)^t\sqrt{2\pi t}} = 1$ (cf. [11, p.194]) and the fact that $w - 1 - \ln(w) > 0$ for $w > 1$, we have for every $w := \frac{\tau}{t} > 1$,

$$
\begin{aligned}
\lim_{n\to\infty}\frac{1}{n!}(nw)^n e^{-nw} &= \lim_{n\to\infty}\frac{1}{\sqrt{2\pi}n^{n+1/2}}e^n(nw)^n e^{-nw} \\
&\le \frac{1}{\sqrt{2\pi}}\lim_{n\to\infty} w^n e^{n-nw} \\
&= \frac{1}{\sqrt{2\pi}}\lim_{n\to\infty} e^{n(\ln(w)+1-w)}.
\end{aligned}
$$

Since $\ln(w) + 1 - w < 0$, the last limit equals to 0. Therefore $\lim_{n\to\infty} H_n(\frac{n}{t}) = 0$ for $t \in (0, \tau)$.

Lemma 2.7. *If* $C^{-1}AC = A$, *then* $C_0^{-1}A_0C_0 = A_0$.

Proof. First, $A \subset C^{-1}AC$ implies $CD(A) \subset D(A)$ and so $R(C_0) = C(\overline{D(A)}) \subset \overline{D(A)} = X_0$. Furthermore, to deduce $A_0 \subset C_0^{-1}A_0C_0$ from $A \subset C^{-1}AC$, let $x \in D(A_0)$. Then $x \in D(A)$ and $Ax \in X_0$, which imply $C_0x = Cx \in D(A)$ and $AC_0x = ACx = CAx = C_0A_0x \in R(C_0) \subset X_0$. This means that $C_0x \in D(A_0)$ and $A_0C_0x = AC_0x = C_0A_0x \in R(C_0)$ so that $x \in D(C_0^{-1}A_0C_0)$ and $C_0^{-1}A_0C_0x = A_0x$ for $x \in D(A_0)$, i.e., $A_0 \subset C_0^{-1}A_0C_0$.

Next, to show $D(C_0^{-1}A_0C_0) \subset D(A_0)$, let $x \in D(C_0^{-1}A_0C_0)$. Then $C_0x \in D(A_0)$ and $ACx = A_0C_0x \in R(C_0) \subset R(C)$. So, $Ax = C^{-1}ACx = C^{-1}A_0C_0x = C_0^{-1}A_0C_0x \in X_0$. Hence $x \in D(A_0)$.

Proof of (ii) of Theorem 2.5.

First, we fix a $\tau \in (0, T)$. We obtain from the Widder-Arendt Theorem (cf. [1] or [7, Theorem 2.2]) and (2.13) that there is a strongly continuous function $W_\tau : [0, \infty) \to B(X)$ such that $W_\tau(0) = 0$, $||W_\tau(t + h) - W_\tau(t)|| \le M_\tau h$ for all $t, h \ge 0$, and

$$
L_\tau(\lambda)x = \lambda\int_0^\infty e^{-\lambda t}W_\tau(t)xdt = \lambda^2\int_0^\infty e^{-\lambda t}(1 * W_\tau)(t)xdt \tag{2.15}
$$

for all $\lambda > 0$ and $x \in X$. From (2.12) we have that

$$
\int_0^\infty e^{-\lambda t}W_\tau(t)Adt \subset A\int_0^\infty e^{-\lambda t}W_\tau(t)dt \tag{2.16}
$$

and

$$(2.17)\qquad \begin{aligned} &A\int_0^\infty e^{-\lambda t}(1*W_\tau)(t)xdt\\ &= \int_0^\infty e^{-\lambda t}W_\tau(t)xdt - \int_0^\infty e^{-\lambda t}tCxdt - \lambda^{-2}V_\tau(\lambda)x \end{aligned}$$

for every $x \in X$

Next, we need to employ the Post inverse formula (cf. [19, Chapter 7] or [12, p. 250]), which states that if $L(g,\lambda) := \int_0^\infty e^{-\lambda t}g(t)dt$ for $\lambda > 0$, where $g : [0,\infty) \to X$ is an exponentially bounded continuous function, then $\lim\limits_{n\to\infty} \frac{(-1)^n(\frac{n}{t})^{n+1}}{n!}D^nL(g,\frac{n}{t})$ = $g(t)$ uniformly on bounded subsets of $(0,\infty)$. Since $W_\tau(\cdot)$ is Lipschitz continuous, we can apply the Post inverse formula to the functions $g(t) = (1*W_\tau)(t)x,\ W_\tau(t)x,\ tCx.$ By applying it to the equation (2.17) and using Lemma 2.6 and the closedness of A, we obtain, for every $0 < t < \tau$ and $x \in X$, that $(1*W_\tau)(t)x \in D(A)$ and

$$(2.18)\qquad A(1*W_\tau)(t)x = W_\tau(t)x - tCx.$$

Since $W_\tau(0) = 0$, (2.17) also holds for $t = 0$. Similarly, from (2.16) we obtain $W_\tau(t)A \subset AW_\tau(t)$, $0 \le t < \tau$. In particular, if $x \in D(A)$, then

$$W_\tau(t)x = A\int_0^t W_\tau(s)xds + tCx = \int_0^t W_\tau(s)Axds + tCx,$$

so that $W_\tau(\cdot)x : [0,\tau) \to X$ is continuously differentiable and

$$(2.19)\qquad W'_\tau(t)x = AW_\tau(t)x + Cx = W_\tau(t)Ax + Cx$$

for all $x \in D(A)$. Since $W_\tau(\cdot)$ is Lipschitz continuous on $[0,\tau)$, the set

$$\{x \in X; W_\tau(\cdot)x \text{ is continuously differentiable on } [0,\tau)\}$$

is a closed linear subspace of X, which contains $D(A)$. Hence $W_\tau(\cdot)x$ is continuously differentiable for all $x \in X_0 = \overline{D(A)}$.

We set $S_\tau(t) := W'_\tau(t)|_{X_0}$ for $0 \le t < \tau$. Then $S_\tau(\cdot)x$ is continuous on $[0,\tau)$ for every $x \in X_0$. From

$$W_\tau(t)X_0 = W_\tau(t)\overline{D(A)} \subset \overline{W_\tau(t)(D(A))} \subset \overline{D(A)} = X_0$$

and the Banach-Steinhaus theorem, we can see that the operator $S_\tau(t) : X_0 \to X_0$ is bounded for each $0 \le t < \tau$. Thus $\{S_\tau(t); 0 \le t < \tau\}$ is a strongly continuous family in $B(X_0)$. It follows from (2.19) that

$$(2.20)\qquad \int_0^t S_\tau(s)A_0xds = W_\tau(t)Ax = W'_\tau(t)x - Cx = S_\tau(t)x - C_0x$$

and

$$A\int_0^t S_\tau(s)xds = AW_\tau(t)x = W'_\tau(t)x - Cx = S_\tau(t)x - C_0x \in X_0$$

for all $x \in D(A_0)$ and $0 \le t < \tau$. Since A is closed, the latter equation actually holds for all $x \in X_0$. Thus

$$R(\int_0^t S_\tau(s)ds) \subset D(A_0) \text{ and } A_0\int_0^t S_\tau(s)ds = S_\tau(t) - C_0 \tag{2.21}$$

for all $0 \le t < \tau$. In view of (2.20) and (2.21), it follows from Proposition 2.2 that $\{S_\tau(t); 0 \le t < \tau\}$ is a local C_0-semigroup on X_0 with $C_0 = C|X_0$ and generator $B := C_0^{-1}A_0C_0$.

The family $\{S(t); 0 \le t < T\}$, defined on X_0 by $S(t)x = S_\tau(t)x$ for $t \in [0,\tau)$, $\tau \in (0,T)$ and $x \in X_0$, is a well-defined local C_0-semigroup on X_0 with $C_0 = C|X_0$. Indeed, for $x \in X_0$, $0 < \tau_1 < \tau_2 < T$, and all $t \in [0,\tau_1)$

$$\begin{aligned}
&\frac{d}{dr}[S_{\tau_2}(t-r)\int_0^r S_{\tau_1}(u)xdu]\\
&= -S_{\tau_2}(t-r)B\int_0^r S_{\tau_1}(u)xdu + S_{\tau_2}(t-r)S_{\tau_1}(r)x\\
&= S_{\tau_2}(t-r)Cx,
\end{aligned}$$

and so $C\int_0^t S_{\tau_1}(u)xdu = \int_0^t S_{\tau_2}(t-r)Cx = C\int_0^t S_{\tau_2}(u)xdu$ for all $t \in [0,\tau_1)$. Since C is injective, we have $S_{\tau_1}(t) = S_{\tau_2}(t)$ for all $t \in [0,\tau_1)$. Finally, the last assertion in (ii) follows from Lemma 2.7.

From Theorem 2.5 and (2.5) of Lemma 2.1 follows immediately the next corollary.

Corollary 2.8. *Let $C \in B(X)$ be an injection and A be a densely defined linear operator. Then A is the generator of a local C-semigroup $\{S(t); 0 \le t < T\}$ on X if and only if it is closed, satisfying $C^{-1}AC = A$, and has an asymptotic C-resolvent $\{L_\tau(\lambda); 0 < \tau < T, \lambda > 0\}$ which satisfies* (2.13).

Remarks. (i) Theorem 2.5 is analogous to generation theorems (see e.g. [7, Theorem 6.2], [14]) for exponentially bounded C-semigroups. (ii) Since $R(C) \subset \overline{D(A)}$, Corollary 2.8 characterizes the generator A of a local C-semigroup $S(\cdot)$ for the case that C has dense range. Compared to Tanaka and Okazawa's generation theorem [17, Theorem 2.1] which characterizes the complete infinitesimal generator $\overline{G}$ of $S(\cdot)$, Corollary 2.8 is subject to simpler condition

3. Solutions of Abstract Cauchy Problems

This section is concerned with connections between a generator A and strong solutions of the associated abstract Cauchy problems. In the next lemma, X_1 denotes the Banach space which is $D(A)$ equipped with the graph norm $\|x\|_1 = \|x\|+\|Ax\|$.

Lemma 3.1. *Let $C \in B(X)$ be an injection and $\{S(t); 0 \le t < T\}$ be a local C-semigroup on X with generator A. Then $S_1(t) := S(t)|_{X_1}, 0 \le t < T$, form a local C_1-semigroup on X_1 with $C_1 := C|_{X_1}$ and generator A_1, the part of A in X_1.*

Proof. It is easy to see that $S_1(\cdot)$ is a local C_1-semigroup on X_1, with $C_1 = C|X_1$. To show that its generator B is equal to A_1, first let $x \in D(A_1) = D(A^2)$. Then we have

$$t^{-1}(S_1(t)x - C_1x) = t^{-1}(S(t)x - Cx) \to CAx = C_1A_1x,$$

$$At^{-1}(S_1(t)x - C_1x) = t^{-1}(S(t)Ax - CAx) \to CA^2x = AC_1A_1x,$$

which show that $t^{-1}(S_1(t)x - C_1x) \to C_1A_1x$ in $\|\cdot\|_1$, i.e., $x \in D(B)$ and $Bx = A_1x$. Hence $A_1 \subset B$. Conversely, if $x \in D(B)$, then

$$t^{-1}(S(t)x - Cx) = t^{-1}(S_1(t)x - C_1x) \to C_1Bx = CBx,$$

so that $x \in D(A)$ and $Ax = Bx \in X_1 = D(A)$. Hence $D(B) \subset D(A^2) = D(A_1)$.

Theorem 3.2. *Let $C \in B(X)$ be an injection on X and A be a closed linear operator satisfying*

(3.1) $$Cx \in D(A) \text{ and } ACx = CAx \text{ for } x \in D(A).$$

Then the following statements are equivalent.

- (i) *A_1 is the generator of a local C_1-semigroup $S_1(\cdot)$ on X_1, where C_1 is the restriction of C to X_1.*
- (ii) *There exists a unique strong solution $u(\cdot; 0, Cx)$ of* ACP$(A; 0, Cx)$ *for every $x \in D(A)$.*

In this case, $u(\cdot; 0, Cx)$ is given by $u(t; 0, Cx) = S_1(t)x$.

Proof. (i) ⇒ (ii): Assume that A_1 is the generator of a local C_1-semigroup $\{S_1(t); 0 \le t < T\}$ on X_1, let $x \in D(A)$ and set $u(t) = S_1(t)x$ for $0 \le t < T$. Then $u \in C([0, T), X_1)$ so that both u and Au are continuous functions. Since A is closed we have $\int_0^t u(s)ds \in D(A)$ and $A\int_0^t u(s)ds = \int_0^t Au(s)ds$ for $0 \le t < T$. Moreover, (i) implies that $\int_0^t S_1(s)xds \in D(A_1)$ and

$$A\int_0^t u(s)ds = A_1\int_0^t S_1(s)xds = S_1(t)x - C_1x = u(t) - Cx$$

for $0 \le t < T$. Consequently, $u(t) = Cx + \int_0^t Au(s)ds$ for $0 \le t < T$. Hence $u \in C^1([0,T),X)$ and $u' = Au$. Thus u is a solution of ACP$(A;0,Cx)$. In order to show the uniqueness, assume that u is a solution of ACP$(A;0,0)$, we have to show that $u \equiv 0$. Let $v(t) = \int_0^t u(s)ds$ for $0 \le t < T$. Then the closedness of A implies that $v(t) \in D(A)$ and $Av(t) = \int_0^t Au(s)ds = \int_0^t u'(s)ds = u(t) \in D(A)$. Consequently, $v(t) \in D(A^2) = D(A_1)$ for all $0 \le t < T$. Moreover, $v' = u = Av$ and $Av' = Au = u'$ are continuous on $[0,T)$. Thus $v \in C^1([0,T),X_1)$ and $v' = A_1v$. Since $v(0) = 0$, it follows that

$$\begin{aligned} C_1v(t) &= \int_0^t (d/ds)(S_1(t-s)v(s))ds \\ &= \int_0^t [S_1(t-s)A_1v(s) - S_1(t-s)A_1v(s)]ds = 0 \end{aligned}$$

for all $0 \le t < T$. Thus $u = v \equiv 0$.

(ii) $\Rightarrow$ (i): Assume that (ii) holds, i.e., for every $x \in D(A)$ there exists a unique solution $u(\cdot;0,Cx) \in C^1([0,T),X)$ of ACP$(A;0,Cx)$. For each $0 \le t < T$, we define a mapping $S_1 : X_1 \to X_1$ by $S_1(t)x = u(t;0,Cx)$ for $x \in X_1$. By the uniqueness of solution one can easily see that $S_1(t)$ is a linear operator on X_1 satisfying $S_1(0) = C_1$ and $S_1(t+s)C_1 = S_1(t)S_1(s)$ for $0 \le t,s,t+s < T$. In particular, this implies that $S_1(\cdot)$ commutes with C_1. Moreover, since $u(\cdot;0,Cx)$ is continuously differentiable on $[0,T)$ and has values in $D(A)$, both $u(\cdot;0,Cx)$ and $Au(\cdot;0,Cx) = u'(\cdot;0,Cx)$ are continuous, so that $t \to S_1(t)x$ is continuous from $[0,T)$ into X_1, i.e., $S_1(\cdot)x \in C([0,T),X_1)$.

We next show that $S_1(t)$ is a bounded linear operator on X_1 for all $0 \le t < T$. Let $0 < t < T$. Consider the linear map $\eta_t : X_1 \to C([0,t],X_1)$ given by $\eta_t(x) = S_1(\cdot)x = u(\cdot;0,Cx)$. We claim that η_t is a closed operator. In fact, let $x_n \to x$ in X_1 and $\eta_t(x_n) = u(\cdot;0,Cx_n) \to v$ in $C([0,t],X_1)$. Then $u(s;0,Cx_n) = Cx_n + \int_0^s Au(r;0,Cx_n)dr$. Letting $n \to \infty$ we obtain $v(s) = Cx + \int_0^s Av(r)dr$ for $0 \le s \le t$. Let $\tilde{v}(s) = Cv(s)$ for $0 \le s \le t$ and $\tilde{v}(s) = S_1(s-t)v(t) = u(s-t;0,Cv(t))$ for $t < s < T$. Then, by (3.1) one can easily check that $\tilde{v}$ is a solution of ACP$(A;0,C^2x)$. The uniqueness of solution implies that $\tilde{v} = u(\cdot;0,C^2x) = S_1(\cdot)Cx = CS_1(\cdot)x$. In particular, for $0 \le s \le t$ we have $Cv(s) = \tilde{v}(s) = C\eta_t(x)(s)$, and so $v = \eta_t(x)$ on $[0,t]$, by the injectivity of C. We have shown that η_t is closed. By the closed graph theorem, η_t is a continuous linear operator from X_1 to $C([0,t],X_1)$. This shows in particular that $S_1(s) \in B(X_1)$ for each $s \in [0,t]$ and $S_1(\cdot)x = \eta_t(x)$ is continuous on $[0,t]$ for all $x \in X_1$. Since t is arbitrary in $[0,T)$, $\{S_1(t); 0 \le t < T\}$ is a strongly continuous family of operators on X_1. Hence $\{S_1(t); 0 \le t < T\}$ is a local C_1-semigroup on X_1.

We now prove that A_1 is its generator. First, to show that

(3.2) $S_1(t)x \in D(A_1)$ and $A_1S_1(t)x = S_1(t)A_1x,\ x \in D(A_1),\ 0 \le t < T.$

In fact, for $x \in D(A_1) = D(A^2)$ let $w(t) = Cx + \int_0^t u(s; 0, CAx)ds$. Then by the closedness of A, the continuity of the function $Au(\cdot; 0, CAx)$ and by (3.1), we have

$$\begin{aligned}\frac{dw}{dt} = u(t; 0, CAx) &= CAx + \int_0^t Au(s; 0, CAx)ds \\ &= A(Cx + \int_0^t u(s; 0, CAx)ds) \\ &= Aw(t).\end{aligned}$$

Since $w(0) = Cx$, it follows from (ii) that $w(\cdot) \equiv u(\cdot; 0, Cx)$. Hence we have

$$AS_1(t)x = Au(t; 0, Cx) = Aw(t) = \frac{d}{dt}w(t) = u(t; 0, CAx) = S_1(t)A_1x \in X_1,$$

which implies (3.2). In particular, $C_1x \in D(A_1)$ and $A_1C_1x = C_1A_1x$ for $x \in D(A_1)$. Now denote by B the generator of $\{S_1(t); 0 \le t < T\}$. For $x \in D(A)$ we have, by ACP$(A; 0, Cx)$, $\lim_{h\to 0^+}(S_1(h)x - Cx)/h = u'(0; 0, Cx) = Au(0; 0, Cx) = ACx = CAx$. Furthermore, if $x \in D(A_1) = D(A^2)$, then, by (3.2),

$$\lim_{h\to 0^+} A(S_1(h)x - C_1x)/h = \lim_{h\to 0^+}(S_1(h)A_1x - CA_1x)/h = CA_1A_1x = AC_1A_1x$$

in the norm of X. Hence $\lim_{h\to 0^+}(S_1(h)x - C_1x)/h = C_1A_1x$ in the norm of X_1 for $x \in D(A_1)$. This shows that $A_1 \subset B$. In order to show the converse, let $x \in D(B)$. Then $\lim_{h\to 0^+}(S_1(h)x - C_1x)/h = C_1Bx$ in the norm of X_1. On the other hand, since $D(B) \subset D(A)$, as shown above, we have $\lim_{h\to 0^+}(S_1(h)x - C_1x)/h = CAx$. Hence $CAx = CBx$ and so $Ax = Bx \in X_1 = D(A)$. Thus $x \in D(A^2) = D(A_1)$ and $A_1x = Bx$. This shows $B \subset A_1$.

It follows from Lemma 3.1 and Theorem 3.2 that $u := S(\cdot)x$ is the unique strong solution of ACP$(A; 0, Cx)$ for each $x \in D(A)$. Moreover, we have the following proposition for the nonhomogeneous Cauchy problem ACP$(A; Cf, Cx)$. A proof of it is given in [9].

Proposition 3.3. *Let $C \in B(X)$ be an injection and A be the generator of a local C-semigroup $S(\cdot)$ on X. If either* (i) $f \in C^1([0, \tau), X)$, *or* (ii) $f \in C([0, \tau), D(A))$ *and* $Af \in C([0, \tau), X)$, *then for each* $x \in D(A)$ ACP$(A; Cf, Cx)$ *has the unique strong solution* $u(t) := S(t)x + (S * f)(t)$, $0 \le t < T$.

Remarks. Note that Theorem 1.1 in [10, A-II] is the special case $C = I$ of our Theorem 3.2. Versions of Proposition 3.3 for (C_0)-semigroups and global C-semigroups can be found in [5, Theorem II.1.3] and [7, Corollary 7.5], respectively. See also [3, Theorem 4.1] for the case $f = 0$. In general, the condition that $u := S(\cdot)x$ is the unique strong solution of ACP$(A; 0, Cx)$ for every $x \in D(A)$ is not sufficient for A to generate a local C-semigroup (even if $C^{-1}AC = A$) except when A has dense domain (see Corollary 3.6).

Theorem 3.4. *Let $C \in B(X)$ be an injection on X and A be a closed linear operator satisfying* (3.1).

(i) *If A is the generator of a local C-semigroup $S(\cdot)$ on X, then $C^{-1}AC = A$ and the problem* ACP$(A; Cx + \int_0^t Cg(s)ds, 0)$ *has a unique strong solution* $u(t; Cx + \int_0^t Cg(s)ds, 0) = \int_0^t S(s)xds + \int_0^t \int_0^s S(s-r)g(r)drds$ *for every* $g \in L^1_{loc}([0,T), X)$ *and* $x \in X$.

(ii) *If the problem* ACP$(A; Cx, 0)$ *has a unique strong solution $u(\cdot; Cx, 0)$ for every $x \in X$, then the family $\{S(t); 0 \le t < T\}$, defined by $S(t)x := u'(t; Cx, 0)$, $x \in X$, is a local C-semigroup with generator $C^{-1}AC$.*

Proof. (i) Let $u(t) = \int_0^t S(s)xds + \int_0^t \int_0^s S(s-r)g(r)drds$. Using (2.3) and the closedness of A we have

$$\begin{aligned} Au(t) &= A\int_0^t S(s)xds + A\int_0^t\int_0^s S(s-r)g(r)drds \\ &= S(t)x - Cx + A\int_0^t \Big(\int_r^t S(s-r)g(r)ds\Big)dr \\ &= S(t)x - Cx + \int_0^t A\Big(\int_0^{t-r} S(s)g(r)ds\Big)dr \\ &= S(t)x - Cx + \int_0^t \big[S(t-r)g(r) - Cg(r)\big]dr \\ &= u'(t) - Cx - \int_0^t Cg(r)dr. \end{aligned}$$

Hence u satisfies ACP$(A; Cx + \int_0^t Cg(s)ds, 0)$. The uniqueness of solution of the problem ACP$(A; Cx + \int_0^t Cg(s)ds, 0)$ follows from the fact that $u \equiv 0$ is the unique solution of ACP$(A; 0, 0)$ (see Proposition 3.3).

(ii) Assume that for every $x \in X$ there exists a unique solution $u(\cdot; Cx, 0)$ of ACP$(A; Cx, 0)$, and let $S(t)x := u'(t; Cx, 0)$ for $x \in X$, $0 \le t < T$. $S(\cdot)x$ is strongly continuous and $S(0)x = u'(0; Cx, 0) = Au(0; Cx, 0) + Cx = Cx$ for $x \in X$. By the uniqueness of solution one can see that $S(t)$ is a linear operator and $u(\cdot; C^2x, 0) = Cu(\cdot; Cx, 0)$.

Next, we claim that $u(\cdot;CAx,0)=Au(\cdot;Cx,0)$ and $u'(\cdot;CAx,0)=Au'(\cdot;Cx,0)$ for $x\in D(A)$. Indeed, let $w(t):=\int_0^t u(s;CAx,0)ds+tCx$ for $x\in D(A)$ and $0\le t<T$. Since $u'(s;CAx,0)=Au(s,CAx,0)+CAx=A(u(s;CAx,0)+Cx)$, taking integration we obtain that $u(s;CAx,0)=Aw(s)$ for $0\le s<T$, so that $w(t)=\int_0^t Aw(s)ds+tCx$ and $w'(t)=Aw(t)+Cx$ for $0\le t<T$. The uniqueness of solution shows $w(t)=u(t;Cx,0)$. Thus $\int_0^t u(s;CAx,0)ds+tCx=u(t;Cx,0)$ and, after differentiation, $u(t;CAx,0)+Cx=u'(t;Cx,0)=Au(t;Cx,0)+Cx$. We have shown $u(t;CAx,0)=Au(t;Cx,0)$ for $x\in D(A)$ and $0\le t<T$. By the closedness of A we also have $u'(\cdot;CAx,0)=Au'(\cdot;Cx,0)$.

We next show that $\{S(t);0\le t<T\}$ is a strongly continuous family of bounded linear operators on X. Let $0<t<T$. Consider the linear map $\eta_t:X\to C([0,t],X)$ given by $\eta_t(x)=S(\cdot)x=u'(\cdot;Cx,0)$. We show that η_t is a closed operator. In fact, let $x_n\to x$ in X and $\eta_t(x_n)=u'(\cdot;Cx_n,0)\to v$ in $C([0,t],X)$. Then, for all $s\in[0,t]$, $u(s;Cx_n,0)\to\int_0^s v(r)dr$ and $Au(s;Cx_n,0)=u'(s;Cx_n,0)-Cx_n\to v(s)-Cx$. The closedness of A implies that $\int_0^s v(r)dr\in D(A)$ and $A\int_0^s v(r)dr=v(s)-Cx$ for all $0\le s\le t$. Let $\tilde v(s):=C\int_0^s v(r)dr$ for $0\le s\le t$ and $\tilde v(s):=u(s-t;Cv(t),0)+\tilde v(t)$ for $t\le s<T$. Then $\tilde v\in C([0,T),X)$. Furthermore, by (3.1) we have

$$\tilde v'(s)=Cv(s)=C(A\int_0^s v(r)dr+Cx)=A\tilde v(s)+C^2x$$

for $0\le s\le t$, and

$$\begin{aligned}\tilde v'(s)&=u'(s-t;Cv(t),0)=Au(s-t;Cv(t),0)+Cv(t)\\&=A\tilde v(s)-A\tilde v(t)+(A\tilde v(t)+C^2x)\\&=A\tilde v(s)+C^2x\end{aligned}$$

for $t\le s<T$. Hence $\tilde v$ is a solution of ACP$(A;C^2x,0)$ on $[0,T)$. By the uniqueness of solution, we have $\tilde v(s)=u(s;C^2x,0)=Cu(s;Cx,0)$ and hence $v(s)=u'(s;Cx,0)$ for all $0\le s\le t$, i.e., $v=\eta_t(x)$. Therefore η_t is closed. By the closed graph theorem, η_t is a continuous linear operator from X to $C([0,t],X)$. Since t is arbitrary, this implies the continuity of $S(t)$ on X for all $0\le t<T$ and the strong continuity of $S(\cdot)$ on $[0,T)$.

For $x\in X$ we have

$$\int_0^t S(s)xds=\int_0^t u'(s;Cx,0)ds=u(t;Cx,0)\in D(A)$$

and so,

$$A\int_0^t S(s)xds=Au(t;Cx,0)=u'(t;Cx,0)-Cx=S(t)x-Cx.$$

Since $S(t)Ax = u'(t; CAx, 0) = Au'(t; Cx, 0) = AS(t)x$ for $x \in D(A)$, it follows from Proposition 2.2 that $C^{-1}AC$ is the generator of $S(\cdot)$.

The fact that $S(\cdot)$ satisfies $S(t+s)C = S(t)S(s)$ for $0 \leq t, s, t+s < T$ can also be seen directly in the following way. For any fixed $x \in X$ and $0 \leq s < T$, define $w \in C([0, T-s), X)$ by $w(t) := u(t+s; C^2x, 0) - u(s; C^2x, 0)$, $0 \leq t < T-s$. Then $w(0) = 0$ and

$$\begin{aligned} w'(t) &= u'(t+s; C^2x, 0) = Au(t+s; C^2x, 0) + C^2x \\ &= Aw(t) + Au(s; C^2x, 0) + C^2x = Aw(t) + C(Au(s; Cx, 0) + Cx) \\ &= Aw(t) + Cu'(s; Cx, 0) \end{aligned}$$

for all $t \in [0, T-s)$. That is, w satisfies ACP$(A; Cu'(s; Cx, 0), 0)$ on $[0, T-s)$. Since $u(\cdot; Cu'(s; Cx, 0), 0)$ is the unique solution of ACP$(A; Cu'(s; Cx, 0), 0)$ on $[0, T)$, it must coincide with w on $[0, T-s)$. Hence we have $w(t) = u(t; Cu'(s; Cx, 0), 0)$, so that

$$\begin{aligned} S(t+s)Cx &= u'(t+s; C^2x, 0) = w'(t) = u'(t; Cu'(s; Cx, 0), 0) \\ &= S(t)u'(s; Cx, 0) = S(t)S(s)x \end{aligned}$$

for all $t \in [0, T-s)$. The proof is complete.

Corollary 3.5. *Let $C \in B(X)$ be an injection. If A is a densely defined closed operator and if for each $x \in D(A)$ ACP$(A; 0, Cx)$ has a unique strong solution $u(\cdot; 0, Cx)$ which depends continuously on x (i.e., if $\{x_n\}$ is a Cauchy sequence in $D(A)$, then $\{u(\cdot; 0, Cx_n)\}$ is uniformly Cauchy on compact subsets of $[0, T)$), then $C^{-1}AC$ generates a local C-semigroup on $[0, T)$.*

Proof. In view of Theorem 3.4(ii), we need only to show that ACP$(A; Cx, 0)$ has a unique strong solution $u(\cdot; Cx, 0)$ for every $x \in X$. For any $x \in X$ let $\{x_n\}$ be a sequence in $D(A)$ such that $x_n \to x$. Let $u(\cdot; 0, Cx_n)$ be the unique strong solution of ACP$(A; 0, Cx_n)$, and let $v_n(t) = \int_0^t u(s; 0, Cx_n)ds$. Then there is a continuous function u such that $u(t; 0, Cx_n) \to u(t)$ and $v_n(t) \to v(t) = \int_0^t u(s)ds$ uniformly on compact subsets of $[0, T)$. Since A is closed and $Au(\cdot; 0, Cx_n) = u'(\cdot; 0, Cx_n)$ is continuous, we have

$$\begin{aligned} Av_n(t) &= A\int_0^t u(s; 0, Cx_n)ds = \int_0^t Au(s; 0, Cx_n)ds \\ &= u(t; 0, Cx_n) - Cx_n, \end{aligned}$$

which converges to $u(t) - Cx$. It follows that $v(t) \in D(A)$ and $Av(t) = u(t) - Cx = v'(t) - Cx$. Hence v is a strong solution of ACP$(A; Cx, 0)$. That this function v is the unique strong solution of ACP($A; Cx, 0$) follows from the unique existence of the strong solution of ACP($A; 0, 0$).

Corollary 3.6. *Let $C \in B(X)$ be an injection and A be a closed linear operator satisfying $C^{-1}AC = A$. Then the following statements are equivalent.*

(i) *A is the generator of a local C-semigroup $S(\cdot)$ on X.*

(ii) *The problem* ACP($A; Cx + \int_0^t Cg(s)ds, 0$) *has a unique strong solution $u(t; Cx + \int_0^t Cg(s)ds, 0)$ for every $g \in L^1_{loc}([0, T), X)$ and $x \in X$.*

(ii') *The integral equation*

$$v(t) = A\int_0^t v(s)ds + Cx + \int_0^t Cg(s)ds \tag{3.3}$$

has a unique strong solution $v \in C([0, T); X)$ for every $g \in L^1_{loc}([0, T), X)$ and $x \in X$.

(iii) *The problem* ACP($A; Cx, 0$) *has a unique strong solution $u(\cdot; Cx, 0)$ for every $x \in X$.*

Moreover, we have $v(t) = u'(t; Cx + \int_0^t Cg(s)ds, 0) = S(t)x + \int_0^t S(t-s)g(s)ds$ and $u(t; Cx + \int_0^t Cg(s)ds, 0) = \int_0^t v(s)ds$ for $0 \le t < T$. If, in addition, A has dense domain, then each of the above conditions is also equivalent to

(iv) *The problem* ACP($A; 0, Cx$) *has a unique strong solution for every $x \in D(A)$, and the solution depends continuously on x.*

Proof. By setting $v(t) = u'(t; Cx + \int_0^t Cg(s)ds, 0)$, one easily sees that statement (ii) is equivalent to (ii'). "(ii) $\Rightarrow$ (iii)" is obvious. "(i) $\Rightarrow$ (ii)" and "(iii) $\Rightarrow$ (i)" follow from Theorem 3.4. "(i) $\Rightarrow$ (iv)" is contained in Proposition 3.3, and "(iv) $\Rightarrow$ (i)" follows from Corollary 3.5 in the case that A is densely defined.

Applying Theorem 3.2 and Corollary 3.6 we prove the following result.

Corollary 3.7. *Let $C \in B(X)$ be an injection. The following statements have the relations:* (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv).

(i) *A is the generator of a local C-semigroup $S(\cdot)$ on X.*

(ii) *$C^{-1}AC = A$, and the problem* ACP($A; Cx, 0$) *has a unique strong solution $u(\cdot; Cx, 0)$ for every $x \in X$.*

(iii) *A is a closed linear operator satisfying* (3.1), *and A_1 is the generator of a local C_1-semigroup $S_1(\cdot)$ on X_1.*

(iv) *A is a closed linear operator satisfying* (3.1), *and* ACP($A; 0, Cx$) *has a unique strong solution $u(\cdot; 0, Cx)$ for every $x \in D(A)$.*

In case A has nonempty resolvent set, the above statements are equivalent. Moreover, $S_1(\cdot)$ is the restriction of $S(\cdot)$ to X_1, and $u(\cdot; 0, Cx) = S(t)x$ for $x \in D(A)$.

Proof. "(i) $\Leftrightarrow$ (ii)" follows from Corollary 3.6, "(i) $\Rightarrow$ (iii)" follows from Lemma 3.1, and "(iii) $\Leftrightarrow$ (iv)" follows from Theorem 3.2.

It remains to show "(iii) $\Rightarrow$ (i)" under the assumption $\rho(A) \neq \emptyset$. Suppose A_1 is the generator of a local C_1-semigroup $S_1(\cdot)$ on X_1. Let $\lambda \in \rho(A)$, and define $S(\cdot) = (\lambda - A)S_1(\cdot)(\lambda - A)^{-1}$. Since $(\lambda - A)^{-1}$ is a topological linear isomorphism from X onto X_1 and since $CA \subset AC$ on $D(A)$, it is obvious that $S(\cdot)$ is a local C-semigroup on X with generator $(\lambda - A)A_1(\lambda - A)^{-1} = A$.

Remarks. In the case $C = I$, this corollary reduces to Corollary 1.2 of [10, A-II]. The equivalence of (i) and (ii) for the case $T = \infty$ was proved in [16, Corollary 2.4]. That (i) implies (iv), for the case $T = \infty$, was proved by deLaubenfels [3, Theorem 4.1]. The equivalence of (i) and (iv) in case $\rho(A) \neq \emptyset$ and $T = \infty$, was proved by Tanaka and Miyadera [15, Corollary 2.2] in different way.

Next, we include the following simple example for illustration. Consider the following initial value problems in c_0:

$$(3.4)\qquad \begin{cases} u_n'(t) = nu_n(t) + e^{-n}f_n(t),\ 0 < t < 1; \\ u_n(0) = e^{-n}q_n, \end{cases} \quad n \geq 1,$$

$$(3.5)\qquad \begin{cases} v_n'(t) = nv_n(t) + e^{-n}q_n + \int_0^t e^{-n}g_n(s)ds,\ 0 < t < 1; \\ v_n(0) = 0, \end{cases} \quad n \geq 1.$$

The family $\{S(t); 0 \leq t < 1\}$, defined by $S(t)x := (e^{-n}e^{nt}x_n),\ x = (x_n) \in c_0$, is a local C-semigroup with $C := \bigoplus_{n=1}^{\infty} e^{-n} \in B(c_0)$ and with generator $A := \bigoplus_{n=1}^{\infty} n$.

If, for instance, $(nf_n(t)) \in c_0$ for all $t \in [0, 1)$ and the functions $\{nf_n\}$ are uniformly continuous on $[0, 1)$, then $Af \in C([0, 1); c_0)$. Now it follows from Proposition 3.3 that, for any $q \in c_0$ with $\lim_{n\to\infty} nq_n = 0$, (3.4) has a unique solution $u \in C([0, 1); c_0)$, which is given by

$$\begin{aligned} u(t) &= S(t)q + \int_0^t S(t-s)f(s)ds \\ &= \left(e^{-n}\left[e^{nt}q_n + \int_0^t e^{n(t-s)}f_n(s)ds\right]\right),\ 0 \leq t < 1. \end{aligned}$$

Next we consider (3.5). If $g = (g_n(t)) \in c_0$ for all $t \in [0, 1)$ and the functions $\{g_n\}$ are uniformly continuous on $[0, 1)$, then $g \in C([0, 1); c_0)$. It follows from

Theorem 3.4(i) that, for any $q \in c_0$, (3.5) has a unique solution $v \in C([0,1);c_0)$, which is given by

$$\begin{aligned} v(t) &= \int_0^t S(s)q\,ds + \int_0^t \int_0^s S(s-r)g(r)\,dr\,ds \\ &= \left(e^{-n} \left[\frac{1}{n+}(e^{nt}-1)q_n + \int_0^t \int_0^s e^{n(s-r)} g_n(r)\,dr\,ds \right] \right) \end{aligned}$$

for $0 \le t < 1$. Finally, we remark that Corollary 3.6 can be used to prove the following bounded perturbation theorem [13] for local C-semigroups.

Theorem 3.8. *Let $C \in B(X)$ be an injection and A be the generator of a local C-semigroup $S(\cdot)$ on X. If $B \in B(X)$ satisfies $R(B) \subset R(C)$ and $BCx = CBx$ for $x \in D(A)$, then $A + B$ is the generator of a local C-semigroup $T(\cdot)$ on X, which satisfies*

$$(1) \qquad T(t)x = S(t)x + \int_0^t S(t-s)C^{-1}BT(s)x\,ds, \; x \in X, \, 0 \le t < T.$$

Moreover, in the case $T = \infty$, if $\|S(t)\| \le Me^{wt}$ for some $M, w > 0$ and all $t \ge 0$, then $\|T(t)\| \le Me^{(w+M\|C^{-1}B\|)t}$ for $t \ge 0$.

We also remark that perturbation by unbounded operators has been discussed in [9].

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