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MULTIPLE POSITIVE SOLUTIONS FOR SEMILINEAR ELLIPTIC EQUATIONS IN ESTEBAN-LIONS DOMAINS WITH HOLES

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Abstract. In this paper, we study a Palais-Smale condition in unbounded domains. Furthermore, we apply this result to prove that the semilinear elliptic equation in a Esteban-Lions domain with holes has multiple positive solutions.

1. Introduction

Let $N \geq 2$ and $2 < p < 2^*$, where $2^* = \frac{2N}{N-2}$ for $N \geq 3$ and $2^* = \infty$ for $N = 2$. Consider the semilinear elliptic equation

$$(1) \qquad \begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \Omega \\ u \in H_0^1(\Omega), \end{cases}$$

where Ω is a domain in $\mathbb{R}^N$ and $H_0^1(\Omega)$ is the Sobolev space in Ω with dual space $H^{-1}(\Omega)$.

Associated with Equation (1), we consider the energy functionals a, b and J in $H_0^1(\Omega)$,

$$\begin{aligned} a(u) &= \int_\Omega (|\nabla u|^2 + u^2), \\ b(u) &= \int_\Omega |u|^p, \\ J(u) &= \frac{1}{2}a(u) - \frac{1}{p}b(u). \end{aligned}$$

It is well-known that the solutions of Equation (1) in Ω and the critical points of the energy functional J in $H_0^1(\Omega)$ are the same. By the Rellich compactness theorem,

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it is easy to obtain a solution of Equation (1) in a bounded domain. For general unbounded domains Ω, because the lack of compactness, the existence of solutions of Equation (1) in Ω is very difficult and unclear. The breakthrough was made by Esteban-Lions [8]. They asserted that Equation (1) in Esteban–Lions domain does not admit any nontrivial solution, where the definition of Esteban-Lions domain is: for a proper unbounded domain Ω in $\mathbb{R}^N$, there exists a $\chi \in \mathbb{R}^N$, $||\chi|| = 1$, such that $n(z) \cdot \chi \geq 0$ and $n(z) \cdot \chi \not\equiv 0$ on $\partial\Omega$, where $n(z)$ is the unit outward normal vector to $\partial\Omega$ at the point x. Some typical examples are:
(i) upper half space $\mathbb{R}^N_+ = \left\{(x, y) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid y > 0\right\}$;
(ii) upper half strip $\mathbf{S}^+ = \left\{(x, y) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid \ |x| < r_0,\ y > 0\right\} \cup B^N(0; r_0)$.

Thus, perturbing the Esteban–Lions domain to obtain the existence of solutions for Equation (1) is applied in a great deal of research in recent years. First, we consider a perturbation of the upper half strip $\mathbf{S}^+$ that is the interior flask domain $\mathbf{F}_r = \mathbf{S}^+ \cup B^N(0; r)$. Then $\mathbf{F}_r$ is not a Esteban–Lions domain for all $r > r_0$. Moreover, Lien-Tzeng-Wang [9] and Chen-Wang [5] proved that there is a $r^* > 0$ such that for $r > r^*$ Equation (1) in $\mathbf{F}_r$ has a ground state solution. The definition of ground state solution of Equation (1) in Ω is as follows: Consider the minimax problem

$$\alpha_\Gamma(\Omega) = \inf_{\gamma \in \Gamma(\Omega)} \max_{t \in [0,1]} J(\gamma(t)), \tag{2}$$

where

$$\Gamma(\Omega) = \{\gamma \in C([0, 1], H_0^1(\Omega)) \mid \gamma(0) = 0, \gamma(1) = e\},$$

$J(e) = 0$ and $e \neq 0$. By the well-known mountain pass lemma due to Ambrosetti-Rabinowitz [1], we call the nonzero critical point $u \in H_0^1(\Omega)$ of J a ground state solution of Equation (1) in Ω if $J(u) = \alpha_\Gamma(\Omega)$. We remark that ground state solutions of Equation (1) in Ω can also be obtained by the Nehari minimization problem

$$\alpha(\Omega) = \inf_{v \in \mathbf{M}(\Omega)} J(v),$$

where $\mathbf{M}(\Omega) = \left\{u \in H_0^1(\Omega) \backslash \{0\} \mid a(u) = b(u)\right\}$. Note that $\alpha_\Gamma(\Omega) = \alpha(\Omega) > 0$ and if there exists a nonzero solution v_0 of Equation (1) such that $J(v_0) > \alpha(\Omega)$, then we called the solution v_0 is a higher energy solution. (see Willem [14] and Wang [12]).

Next, we shall consider another perturbation. Let $\tilde{r} < r_0$ is a positive number, consider the upper half strip with hole

$$\Omega(t) = \mathbf{S}^+ \backslash B((0, t); \tilde{r}).$$

Wang [12] used the Palais-Smale decomposition lemma in the infinite strip $\mathbf{A} = \left\{(x, y) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid \ |x| < r_0\right\}$ (see Lien-Tzeng-Wang [9]) and the center mass function. He assumed that the positive solution of Equation (1) in infinite strip $\mathbf{A}$ is unique, asserting that there exists a $t_0 > 0$ such that for $t > t_0$, Equation (1) in $\Omega(t)$ has a positive higher energy solution. However, the uniqueness of positive solution of Equation (1) in infinite strip $\mathbf{A}$ only has been solved in dimension $N = 2$ (see Dancer [6]). In general, this problem is still open.

The main purpose of this paper is using a new method to improve a result of Wang [12]. In particular we do this without any assumption on the uniqueness of positive solution of Equation (1) in infinite strip $\mathbf{A}$. This paper is organized as follow. In section 2, we describe various preliminaries. In section 3, we describe a Palais-Smale condition in unbounded domains. In section 4, we proved that the semilinear elliptic equation in upper half strip with holes has multiple positive solutions.

2. Preliminary

First, we define the (PS)$-$sequences, (PS)$-$values, and (PS)$-$conditions in $H_0^1(\Omega)$ for J as follows:

Definition 1. We define

(i) For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a (PS)$_\beta-$sequence in $H_0^1(\Omega)$ for J if $J(u_n) = \beta + o(1)$ and $J'(u_n) = o(1)$ strongly in $H^{-1}(\Omega)$ as $n \to \infty$;

(ii) $\beta \in \mathbb{R}$ is a (PS)$-$value in $H_0^1(\Omega)$ for J if there exists a (PS)$_\beta-$sequence in $H_0^1(\Omega)$ for J;

(iii) (iii) J satisfies the (PS)$_\beta-$condition in $H_0^1(\Omega)$ if every (PS)$_\beta-$sequence in $H_0^1(\Omega)$ for J contains a convergent subsequence.

By Willem [14], for any $\beta \in \mathbb{R}$, a (PS)$_\beta-$sequence in $X(\Omega)$ for J is bounded. Moreover, a (PS)$-$value β should be nonnegative.

Lemma 2. *Let $\beta \in \mathbb{R}$ and $\{u_n\}$ be a (PS)$_\beta-$sequence in $H_0^1(\Omega)$ for J, then there exists a $c > 0$ such that $\|u_n\|_{H^1} \leq c$ for all $n \in \mathbb{N}$. Furthermore,*

$$a(u_n) = b(u_n) + o(1) = \frac{2p}{p-2}\beta + o(1)$$

and $\beta \geq 0$.

Now, we consider the Nehari minimization problem

$$\alpha(\Omega) = \inf_{u \in \mathbf{M}(\Omega)} J(u),$$

where $\mathbf{M}(\Omega) = \left\{u \in H_0^1(\Omega) \backslash \{0\} \mid a(u) = b(u)\right\}$. Note that $\mathbf{M}(\Omega)$ contain every nonzero solution of Equation (1) in Ω and if $u_0 \in \mathbf{M}(\Omega)$ achieves $\alpha(\Omega)$ then

u_0 is a ground state solution of Equation (1) in Ω (see Wang-Wu [13] or Willem [14]. Moreover, we have the following useful lemmas, whose proof can be found in Chen-Wang [5] and Wang-Wu [13, Lemma 7].

Lemma 3. *Let $\beta > 0$ and $\{u_n\}$ in $H_0^1(\Omega)\backslash\{0\}$ be a sequence for J such that $J(u_n) = \beta + o(1)$ and $a(u_n) = b(u_n) + o(1)$. Then there is a sequence $\{s_n\} \subset \mathbb{R}^+$ such that $s_n = 1 + o(1)$, $\{s_n u_n\}$ is in $\mathbf{M}(\Omega)$ and $J(s_n u_n) = \beta + o(1)$.*

Lemma 4. *Let $\{u_n\}$ be in $H_0^1(\Omega)$. Then $\{u_n\}$ is a (PS)$_{\alpha(\Omega)}$−sequence in $H_0^1(\Omega)$ for J if and only if $J(u_n) = \alpha(\Omega) + o(1)$ and $a(u_n) = b(u_n) + o(1)$.*

Let $\Omega^1 \subsetneqq \Omega^2$, clearly $\alpha(\Omega^2) \le \alpha(\Omega^1)$. If $\alpha(\Omega^2) = \alpha(\Omega^1)$, then we have the following useful results.

Lemma 5. *Let $\Omega^1 \subsetneqq \Omega^2$ and let $J : H_0^1(\Omega^2) \to \mathbb{R}$ be the energy functional. Suppose that $\alpha(\Omega^2) = \alpha(\Omega^1)$. We have*

(i) Equation (1) in Ω^1 does not admit any solution u^1 such that $J(u^1) = \alpha(\Omega^1)$;
(ii) J does not satisfy the (PS)$_{\alpha(\Omega^2)}$-condition.

By the Rellich compact theorem, J satisfies the (PS) $_{\alpha_X(\Omega)}$−condition in $X(\Omega)$ if Ω is a bounded domain.

Lemma 6. *Let Ω be a bounded domain in $\mathbb{R}^N$. Then the (PS)$_{\alpha(\Omega)}$−condition holds in $H_0^1(\Omega)$ for J. Furthermore, Equation (1) in Ω has a positive solution u_0 such that $J(u_0) = \alpha(\Omega)$.*

Lemma 7. *Let $u \in H_0^1(\Omega)$ be a change sign solution of Equation (1) in Ω. Then $J(u) > 2\alpha(\Omega)$.*

Proof. See the proof of Theorem A in Benci-Cerami [3]. ■

3. Palais–smale Conditions

Throughout this section, denote

$$\mathbf{A}_{s,l}(x_0, r) = \{(x, y) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid |x - x_0| < r, s < y < l\}$$

is a finite strip in $\mathbb{R}^N$. Let Ω be unbounded domain in $\mathbb{R}^N$and let

$$M(x_0, r, s, l) = \left\{ u \in \mathbf{M}(\Omega) \mid \int_{[\mathbf{A}_{s,l}(x_0,r)]^c} |u|^p < \frac{p}{(p-2)}\alpha(\Omega) \right\};$$

$$N(x_0, r, s, l) = \left\{ u \in \mathbf{M}(\Omega) \mid \int_{[\mathbf{A}_{s,l}(x_0,r)]^c} |u|^p = \frac{p}{(p-2)}\alpha(\Omega) \right\},$$

be subsets of $\mathbf{M}(\Omega)$, where $\mathbf{A}_{s,l}(x_0, r) \cap \Omega \neq \phi$ for some $x_0 \in \mathbb{R}^{N-1}$, $r > 0$ and $s, l \in \mathbb{R}$. It is easy to verify that $M(x_0, r, s, l)$ is nonempty set for all $r > 0$. Define the minimization problem in $M(x_0, r, s, l)$ and $N(x_0, r, s, l)$ for J,

$$\beta(r) = \inf_{v \in M(x_0, r, s, l)} J(v)$$

and

$$\gamma(r) = \inf_{v \in N(x_0, r, s, l)} J(v).$$

Note that, if $N(x_0, r, s, l)$ is empty then we define the $\gamma(r) = \infty$.

Let $\xi \in C^\infty([0,\infty))$ such that $0 \leq \xi \leq 1$ and

$$\xi(t) = \begin{cases} 0, & \text{for } t \in [0,1] \\ 1, & \text{for } t \in [2,\infty). \end{cases}$$

Let

$$\xi_n(z) = \xi\left(\frac{2|z|}{n}\right). \tag{3}$$

Then we have the following results.

Lemma 8. *Let $\{u_n\}$ be a (PS)$_\beta$−sequence in $H_0^1(\Omega)$ for J satisfying $u_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$ and let $v_n = \xi_n u_n$. Then there exists a subsequence $\{u_n\}$ such that $\|u_n - v_n\|_{H^1} = o(1)$ as $n \to \infty$. Furthermore, we have $a(v_n) = b(v_n) + o(1)$ and $J(v_n) = \beta + o(1)$.*

Proof. Note that

$$\begin{aligned} a(u_n - v_n) &= \langle u_n - v_n, u_n - v_n \rangle_{H^1} \\ &= a(u_n) + a(v_n) - 2\langle u_n, v_n \rangle_{H^1}. \end{aligned}$$

Thus, it suffices to show that $\langle u_n, v_n \rangle_{H^1} = a(u_n) + o(1) = a(v_n) + o(1)$. Since

$$\begin{aligned} \langle u_n, v_n \rangle_{H^1} &= \int_\Omega \nabla u_n \nabla v_n + u_n v_n \\ &= \int_\Omega \xi_n \left[|\nabla u_n|^2 + u_n^2 \right] + \int_\Omega u_n \nabla u_n \nabla \xi_n. \end{aligned}$$

Note that $|\nabla \xi_n| \leq \frac{c}{n}$ and $\{u_n\}$ is a (PS)$_\beta$−sequence in $H_0^1(\Omega)$ for J, so

$$\int_\Omega \xi_n^q u_n \nabla u_n \nabla \xi_n = o(1) \text{ for } q > 0. \tag{4}$$

Hence,

$$\langle u_n, v_n \rangle_{H^1} = \int_\Omega \xi_n \left[|\nabla u_n|^2 + u_n^2 \right] + o(1). \tag{5}$$

Similarly, we have

$$a(v_n) = \int_\Omega \xi_n^2 \left[|\nabla u_n|^2 + u_n^2 \right] + o(1). \tag{6}$$

For $r \geq 1$. Since $\{\xi_n^r u_n\}$ is bounded in $H_0^1(\Omega)$, we have

$$\begin{aligned} o(1) &= \langle J'(u_n), \xi_n^r u_n \rangle \\ &= \int_\Omega (\xi_n^r |\nabla u_n|^2 + r\xi_n^{r-1} u_n \nabla \xi_n \nabla u_n + \xi_n^r u_n^2) - \int_\Omega \xi_n^r |u_n|^p. \end{aligned}$$

By (4) we conclude that

$$\int_\Omega \xi_n^r (|\nabla u_n|^2 + u_n^2) = \int_\Omega \xi_n^r |u_n|^p + o(1). \tag{7}$$

Since $u_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$, there exists a subsequence $\{u_n\}$ such that $u_n \to 0$ strongly in $L_{loc}^p(\Omega)$, or there exists a subsequence $\{u_n\}$ such that

$$\int_{Q(n)} |u_n|^p = o(1),$$

where $Q(n) = \Omega \cap B^N(0; n)$. Clearly,

$$\int_\Omega \xi_n^r |u_n|^p = \int_\Omega |u_n|^p + o(1). \tag{8}$$

By (5-8), we have

$$\langle u_n, v_n \rangle_{H^1} = a(u_n) + o(1) = a(v_n) + o(1).$$

Therefore, $\|u_n - v_n\|_{H^1} = o(1)$ as $n \to \infty$. ∎

Then we have the following Palais-Smale conditions.

Theorem 9. *Let Ω be unbounded domain in $\mathbb{R}^N$ and let $\mathbf{A}_{s,l}(x_0, r)$ be a finite strip such that $\mathbf{A}_{s,l}(x_0, r) \cap \Omega \neq \phi$ for some $x_0 \in \mathbb{R}^{N-1}$, $r > 0$ and $s, l \in \mathbb{R}$. If $\{u_n\} \subset M(x_0, r, s, l)$ is a (PS)$_{\beta(r)}$–sequence in $H_0^1(\Omega)$ for J with $0 < \beta(r) < \gamma(r)$. Then there exist a subsequence $\{u_n\}$ and $u_0 \in M(x_0, r, s, l)$ such that $u_n \to u_0$ strongly in $H_0^1(\Omega)$ and $J(u_0) = \beta(r)$. Furthermore, u_0 is a nonzero solution of Equation (1) in Ω.*

Proof. Let $\{u_n\}$ be a $(\mathrm{PS})_{\beta(r)}$–sequence in $H_0^1(\Omega)$ for J, then by Lemma 2, there exist a subsequence $\{u_n\}$ and u_0 in $H_0^1(\Omega)$ such that $u_n \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$. Moreover, u_0 is a solution of Equation (1) in Ω. If $u_0 \equiv 0$. By Lemma 8, there exists a subsequence $\{u_n\}$ such that $J(\xi_n u_n) = \beta(r) + o(1)$ and $a(\xi_n u_n) = b(\xi_n u_n) + o(1)$, where ξ_n is as in (3). Moreover, by Lemma 3 there exists a $\{s_n\}$ such that

$$\begin{aligned} a(s_n\xi_n u_n) &= b(s_n\xi_n u_n), \\ J(s_n\xi_n u_n) &= \beta(r) + o(1), \\ s_n &= 1 + o(1). \end{aligned}$$

Let $v_n = s_n\xi_n u_n$. Then, there exist a $n_0 \in \mathbb{N}$ such that for $n > 2n_0$

$$v_n = 0 \text{ in } \mathbf{A}_{s,l}(x_0, r) \cap \Omega.$$

Moreover,

$$\begin{aligned} \left(\frac{2p}{p-2}\right)\alpha(\Omega) &\le \int_\Omega |v_n|^p = \int_\Omega |\xi_n u_n|^p + o(1) \\ &= \int_{[\mathbf{A}_{s,l}(x_0,r)]^c} |u_n|^p + o(1) < \frac{p}{(p-2)}\alpha(\Omega), \end{aligned}$$

which is a contradiction. Therefore, $u_0 \not\equiv 0$. By the Fatou lemma, we have

$$\int_{[\mathbf{A}_{s,l}(x_0,r)]^c} |u_0|^p \le \liminf \int_{[\mathbf{A}_{s,l}(x_0,r)]^c} |u_n|^p \le \frac{p}{(p-2)}\alpha(\Omega)$$

and

$$\int_\Omega |u_0|^p \le \liminf \int_\Omega |u_n|^p = \left(\frac{2p}{p-2}\right)\beta(r) < \left(\frac{2p}{p-2}\right)\gamma(r). \tag{9}$$

Thus, $u_0 \in \mathbf{M}(\Omega)$ and $\int_{[\mathbf{A}_{s,l}(x_0,r)]^c} |u_0|^p \le \frac{p}{(p-2)}\alpha(\Omega)$. If $\int_{[\mathbf{A}_{s,l}(x_0,r)]^c} |u_0|^p = \frac{p}{(p-2)}\alpha(\Omega)$. Then $u_0 \in N(x_0, r, s, l)$ and from (9) that

$$\frac{2p}{p-2}\gamma(r) \le \int_\Omega |u_0|^p < \frac{2p}{(p-2)}\gamma(r),$$

which is a contradiction. Thus, $\int_{[\mathbf{A}_{s,l}(x_0,r)]^c} |u_0|^p < \frac{p}{(p-2)}\alpha(\Omega)$. This implies $u_0 \in M(x_0, r, s, l)$ and $J(u_0) = \beta(r)$. Let $p_n = u_n - u_0$, by Bahri-Lions [2], we have

$$J(p_n) = J(u_n) - J(u_0) + o(1) = o(1)$$

and

$$\langle J'(p_n), p_n \rangle = o(1).$$

By Lemma 2, we have

$$a(p_n) = \frac{2p}{p-2} J(p_n) + o(1) = o(1).$$

Thus, $u_n \to u_0$ strongly in $H_0^1(\Omega)$.

4. Multiple Positive Solutions

We need the following definition.

Definition 10. A domain Ω in $\mathbf{A}$ is large if for any $m > 0$ there exist $s < l$ such that $l - s = m$ and $\mathbf{A}_{s,t} \subset \Omega$ where $\mathbf{A}_{s,l} = \{(x, y) \in \mathbf{A} \mid s < y < l\}$.

Then we have the following results.

Lemma 11. *If Ω is a large domain in $\mathbf{A}$, then $\alpha(\Omega) = \alpha(\mathbf{A})$. Furthermore, if Ω is a proper large domain in $\mathbf{A}$, then Equation (1) in Ω does not admits any solution u_0 such that $J(u_0) = \alpha(\Omega)$.*

Proof. By Lien-Tzeng-Wang [9, Lemma 2.5] and Lemma 5. ■

Corollary 12. *If Ω is a proper large domain in $\mathbf{A}$, then $J(v) > \alpha(\Omega)$ for all $v \in \mathbf{M}(\Omega)$.*

Throughout this section, denote $\Omega(t) = \mathbf{S}^+ \backslash \left[\cup_{i=1}^m B^N((0, it); \tilde{r})\right]$ is an upper half strip with holes and $\mathbf{A}_{s,l} = \mathbf{A}_{s,l}(0, r_0)$ is a finite strip. Then $\Omega(t)$ is a proper large domain in $\mathbf{A}$ and $\alpha(\Omega(t)) = \alpha(\mathbf{A})$ for all $t > 0$. Let $M_i(t) = M(0, r_0, (i-1)t, it)$ and $N_i(t) = N(0, r_0, (i-1)t, it)$, for $i = 1, 2, \ldots, m$. Define the minimization problem in $M_i(t)$ and $N_i(t)$ for J,

$$\beta_i(t) = \inf_{v \in M_i(t)} J(v)$$

and

$$\gamma_i(t) = \inf_{v \in N_i(t)} J(v).$$

Clearly, $\beta(t), \gamma(t) \geq \alpha(\mathbf{A})$ for each $t > 0$ and $i = 1, 2, \ldots, m$. Let $\overline{M_i(t)}$ be denoting the closure of $M_i(t)$, then we have $\overline{M_i(t)} = M_i(t) \cup N_i(t)$ and $N_i(t)$ is the boundary of $\overline{M_i(t)}$. Then, we have the following results.

Lemma 13. *For each $t > 2\tilde{r}$, we have $\overline{M_i(t)} \cap \overline{M_j(t)} = \phi$ for $i \neq j$.*

Proof. Suppose on the contrary, there exist a $v_0 \in \mathbf{M}(\Omega(t))$ and $i \neq j$ such that $v_0 \in \overline{M_i(t)} \cap \overline{M_j(t)}$. Then

$$\int_{\left[\mathbf{A}_{(i-1)t,it}\right]^c} |v_0|^p \leq \frac{p}{(p-2)} \alpha(\mathbf{A})$$

and

$$\int_{\left[\mathbf{A}_{(i-1)t,it}\right]^c} |v_0|^p \leq \frac{p}{(p-2)} \alpha(\mathbf{A}).$$

Since $\mathbf{A}_{(i-1)t,it} \cap \mathbf{A}_{(j-1)t,jt} = \phi$ for all $t > 2r'$, we have

$$\begin{aligned} \int_{\Omega(t)} |v_0|^p &\leq \int_{\left[\mathbf{A}_{(i-1)t,it}\right]^c} |v_0|^p + \int_{\left[\mathbf{A}_{(j-1)t,jt}\right]^c} |v_0|^p \\ &\leq \frac{2p}{(p-2)} \alpha(\mathbf{A}). \end{aligned}$$

Therefore,

$$J(v_0) = \left(\frac{p-2}{2p}\right) \int_{\Omega(t)} |v_0|^p \leq \alpha(\mathbf{A}) = \alpha(\Omega(t)),$$

This contradicts Corollary 12.

Lemma 14. *For each $\varepsilon > 0$ there exists a $t_1 > 0$ such that*

$$\beta_i(t) < \alpha(\mathbf{A}) + \varepsilon$$

for all $i = 1, 2, \ldots, m$ and $t \geq t_1$.

Proof. By the Lien-Tzeng-Wang [9, Lemma 2.2], we have

$$\alpha\left(\mathbf{A}_{(i-1)t,it}\right) = \alpha(\mathbf{A}_{0,t}) \searrow \alpha(\mathbf{A}) \text{ as } t \nearrow \infty.$$

Thus, there exists a $t_1 > 0$ such that

$$\alpha\left(\mathbf{A}_{(i-1)t,it}\right) < \alpha(\mathbf{A}) + \varepsilon \tag{10}$$

for each $i = 1, 2, \cdots, m$. By Lemma 6, Equation (1) in $\mathbf{A}_{(i-1)t,it}$ has a positive solution v_i such that $J(v_i) = \alpha\left(\mathbf{A}_{(i-1)t,it}\right)$, $v_i \in \mathbf{M}\left(\mathbf{A}_{(i-1)t,it}\right) \subset \mathbf{M}(\Omega(t))$ and

$$\int_{\left[\mathbf{A}_{(i-1)t,it}\right]^c} |v_i|^p = 0.$$

We obtain $v_i \in M_i(t)$ and

$$\beta_i(t) \leq \alpha\left(\mathbf{A}_{(i-1)t,it}\right) \tag{11}$$

for all $i = 1, 2, \cdots, m$ and $t \geq t_1$. By (10) and (11), we can conclude that

$$\beta_i(t) < \alpha(\mathbf{A}) + \varepsilon$$

for all $i = 1, 2, \ldots, m$ and $t \geq t_1$. ■

Lemma 15. *There exist positive numbers δ, t_2 such that for each $i = 1, 2, \ldots, m$ we have*

$$\gamma_i(t) > \alpha(\mathbf{A}) + \delta \text{ for all } t \geq t_2.$$

Proof. Fix i. Assume to the contrary that there exist $t_n \to \infty$ as $n \to \infty$ and $\{u_n\} \subset N_i(t_n) \subset \mathbf{M}(\Omega(t_n))$ such that

$$J(u_n) = \alpha(\mathbf{A}) + o(1)$$

and

$$\int_{\left[\mathbf{A}_{(i-1)t_n, it_n}\right]^c} |u_n|^p = \frac{p}{(p-2)} \alpha(\mathbf{A}). \tag{12}$$

By Lemma 4, $\{u_n\}$ is a $(\mathrm{PS})_{\alpha(\mathbf{A})}$−sequence in $H_0^1(\mathbf{A})$ for J. Applying the concentration-compactness principle of Lions [10] there exist $R > 0$, $d > 0$ and $\{(0, y_n)\} \in \mathbb{R}^{N-1} \times \mathbb{R}$ such that

$$\int_{B^N((0,y_n);R)} |u_n|^p \geq d \text{ for all } n.$$

Let $v_n(z) = u_n(x, y + y_n)$, then $\{v_n\}$ is a $(\mathrm{PS})_{\alpha(\mathbf{A})}$−sequence in $H_0^1(\mathbf{A})$ for J and $\{v_n\} \subset \mathbf{M}(\mathbf{A})$. Thus, there is a $u_0 \in H_0^1(\mathbf{A})$ such that

$$v_n \to u_0 \text{ weakly in } H_0^1(\mathbf{A}) \text{ as } n \to \infty,$$
$$v_n \to u_0 \text{ a.e. in } \mathbf{A} \text{ as } n \to \infty$$

and

$$\int_{B^N(0;R)} |v_n|^p \to \int_{B^N(0;R)} |u_0|^p \geq d \text{ as } n \to \infty.$$

Moreover, u_0 is a nonzero solution of Equation (1) in $\mathbf{A}$ and $J(u_0) = \alpha(\mathbf{A})$. By Bahri-Lions [2], Lemma 7 and the maximum principle, we may assume that

$$v_n \to u_0 \text{ strongly in } H_0^1(\mathbf{A}) \text{ as } n \to \infty$$

and u_0 is a positive solution. We complete the proof by establishing the contradiction that

$$\int_{\left[\mathbf{A}_{(i-1)t_n,it_n}\right]^c}|u_n|^p=\frac{p}{(p-2)}\alpha(\mathbf{A})\ \text{ for all } n.$$

Consider the sequence $\{it_n-y_n\}$. By passing to a subsequence if necessary, we may assume that one of the following cases occurs:
(a) $\{it_n-y_n\}$ is bounded;
(b) $it_n-y_n\to\infty$ as $n\to\infty$; and
(c) $it_n-y_n\to-\infty$ as $n\to\infty$.
In case (a), we may assume $it_n-y_n\to y_0$. Since

$$v_n\in H_0^1\left(\mathbf{A}\backslash B^N\left((0,it_n-y_n),\tilde{r}\right)\right)$$

we have

$$\Omega(r_n)\to\mathbf{A}\backslash B^N\left((0,y_0),\tilde{r}\right)\ \text{ as } n\to\infty.$$

Then $u_0\in H_0^1\left(\mathbf{A}\backslash B^N\left((0,y_0),\tilde{r}\right)\right)$, and this contradicts the fact that u_0 is a positive solution of Equation (1) in $\mathbf{A}$.
In case (b), since

$$v_n\to u_0 \text{ strongly in } H_0^1(\mathbf{A})\ \text{ as } n\to\infty,$$

u_0 is a positive solution such that $J(u_0)=\alpha(\mathbf{A})$. By the compact imbedding theorem and the Vitali convergence theorem, given $\varepsilon_0=\frac{p}{(p-2)}\alpha(\mathbf{A})$ there is a $R(\varepsilon_0)>0$ such that

$$\int_{\left[\mathbf{A}_{-R(\varepsilon_0),R(\varepsilon_0)}\right]^c}|v_n|^p<\frac{p}{(p-2)}\alpha(\mathbf{A})\ \text{ for all } n. \tag{13}$$

Since $v_n\equiv 0$ in $[\Omega(t_n)]^c$,

$$v_n\to u_0 \text{ a.e. in } \mathbf{A} \text{ as } n\to\infty$$

and u_0 is a positive solution of Equation (1) in $\mathbf{A}$, we have

$$\lim_{n\to\infty}\Omega(t_n)=\mathbf{A}.$$

By hypothesis, we have $(i-1)t_n-y_n\to-\infty$ as $n\to\infty$. Then there is a n_0 such that

$$\mathbf{A}_{-R(\varepsilon_0),R(\varepsilon_0)}\subset\left[\mathbf{A}_{(i-1)t_n,it_n}-(0,y_n)\right]\ \text{ for all } n\geq n_0.$$

Thus,

$$\begin{aligned}&\int_{\left[\mathbf{A}_{(i-1)t_n,it_n}-(0,y_n)\right]^c}|v_n|^p\\ &\quad\leq\int_{\left[\mathbf{A}_{-R(\varepsilon_0),R(\varepsilon_0)}\right]^c}|v_n|^p<\frac{p}{(p-2)}\alpha(\mathbf{A})\ \text{ for all } n\geq n_0.\end{aligned} \tag{14}$$

Since $\{u_n\} \subset \mathbf{M}(\Omega(t_n))$ and $\Omega(t_n)$ is large domain in $\mathbf{A}$, this means

$$\int_{\Omega(t_n)} |u_n|^p \geq \left(\frac{2p}{p-2}\right)\alpha(\Omega(t_n)) = \left(\frac{2p}{p-2}\right)\alpha(\mathbf{A}) \text{ for all } n. \tag{15}$$

From (14) and (15), we have

$$\begin{aligned}
\int_{\left[\mathbf{A}_{(i-1)t_n,it_n}\right]^c} |u_n|^p &= \int_{\Omega(t_n)} |u_n|^p - \int_{\mathbf{A}_{(i-1)t_n,it_n}} |u_n|^p \\
&= \int_{\Omega(t_n)} |u_n|^p - \int_{\left[\mathbf{A}_{(i-1)t_n,it_n}-(0,y_n)\right]^c} |v_n|^p \\
&> \frac{p}{(p-2)}\alpha(\mathbf{A}) \text{ for all } n \geq n_0,
\end{aligned}$$

which contradicts (13).
In case (c), the proof is similar to that of case (13). Therefore, we have completed our proof. ■

Here, we will use the idea of Cao-Noussair [4] and Tarantello [11] to get the following results.

Lemma 16. *For any $u^i \in M_i(t)$, there exist $\epsilon > 0$ and differentiable function $t^i : B(0;\epsilon) \subset H_0^1(\Omega(t)) \to \mathbb{R}^+$ such that $t^i(0) = 1$, the function $z^i = t^i(w)(u^i - w) \in M_i(t)$ and*

$$\begin{aligned}
&\left\langle (t^i)'(0), v\right\rangle \\
&= \frac{2\int_{\Omega(t)} \nabla u^i \nabla v + u^i v - p\int_{\Omega(t)} |u^i|^{p-2} u^i v}{\int_{\Omega(t)} |\nabla u^i|^2 + (u^i)^2 - (p-1)\int_{\Omega(t)} |u^i|^p} \text{ for all } v \in H_0^1(\Omega(t)).
\end{aligned} \tag{16}$$

Proof. Define a function $F : \mathbb{R} \times H_0^1(\Omega(t)) \to \mathbb{R}$ given by

$$F(t,w) = t\int_{\Omega(t)} \left|\nabla(u^i - w)\right|^2 + (u^i - w)^2 - t^{p-1}\int_{\Omega(t)} |u^i - w|^p.$$

Since $u^i \in M_i(t)$, we have $F(0,1) = 0$ and

$$\frac{d}{dt}F(1,0) = \int_{\Omega(t)} \left|\nabla u^i\right|^2 + (u^i)^2 - (p-1)\int_{\Omega(t)} |u^i|^p < 0.$$

According to the implicit function theorem, there exists a continuous function $t^i : B(0;\epsilon) \subset H_0^1(\Omega(t)) \to \mathbb{R}^+$ such that $t^i(0) = 1$ and $F(t^i(w), w) = 0$ for $w \in B(0;\epsilon)$. This is equivalent to

$$\left\langle J'\left(t^i(w)(u^i - w)\right), t^i(w)(u^i - w)\right\rangle = 0.$$

Furthermore, by the continuity of the functionals b and t^i, we have

$$\int_{\left[\mathbf{A}_{(i-1)t,it}\right]^c} \left|t^i(w)\left(u^i - w\right)\right|^p < \frac{p}{(p-2)}\alpha(\mathbf{A}),$$

if ϵ is sufficiently small. ■

Proposition 17. *There exists a $t_0 > 0$ such that for each $t \geq t_0$ and $i = 1, 2, \cdots, m$. We have $\beta_i(t) < \min\{2\alpha(\Omega(t)), \gamma_i(t)\}$ and $\beta_i(t)$ has a minimizing sequence $\{u_n^i\} \subset M_i(t)$ satisfying*

$$\begin{aligned} J\left(u_n^i\right) &= \beta_i(t) + o(1), \\ J'\left(u_n^i\right) &= o(1) \text{ in } H^{-1}(\Omega(t)). \end{aligned}$$

Proof. Using Lemmas 14 and 15, we see that there exists $t_0 > 0$ such that for $r \geq t_0$

$$\beta_i(t) < \min\{2\alpha(\Omega(t)), \gamma_i(t)\}. \tag{17}$$

It follows that for $t \geq t_0$

$$\beta_i(t) = \inf_{v \in \overline{M_i(t)}} J(v). \tag{18}$$

Since $\overline{M_i(t)}$ is a closure of $M_i(t)$. By (18) and the Ekeland variational principle [7], there exists a minimizing sequence $\{u_n^i\} \subset \overline{M_i(t)}$ such that

$$J\left(u_n^i\right) < \beta_i(t) + \frac{1}{n} \tag{19}$$

and

$$J\left(u_n^i\right) < J(w) + \frac{1}{n}\left\|w - u_n^i\right\|_{H^1} \text{ for any } w \in \overline{M_i(t)}. \tag{20}$$

Using (17) we may assume that $u_n^i \in M_i(t)$ for n sufficiently large. Applying Lemma 16 with $u^i = u_n^i$ to obtain the functions $t_n^i : B(0; \epsilon_n) \to \mathbb{R}^+$ for some $\epsilon_n > 0$, such that $t_n^i(w)\left(u_n^i - w\right) \in M^i(t)$. Choose $0 < \rho < \epsilon_n$. Let $u \in H_0^1(\Omega(t))$ with $u \not\equiv 0$ and let $w_\rho = \frac{\rho u}{\|u\|_{H^1}}$. We set $z_\rho^i = t_n^i(w_\rho)\left(u_n^i - w_\rho\right)$. Since $z_\rho^i \in M_i(t)$, we deduce from (19) that

$$J\left(z_\rho^i\right) - J\left(u_n^i\right) \geq -\frac{1}{n}\left\|z_\rho^i - u_n^i\right\|_{H^1}$$

By the mean value theorem, we have

$$\left\langle J'\left(u_n^i\right), z_\rho^i - u_n^i\right\rangle + o\left(\left\|z_\rho^i - u_n^i\right\|_{H^1}\right) \geq -\frac{1}{n}\left\|z_\rho^i - u_n^i\right\|_{H^1}.$$

Thus,

$$\begin{aligned}(21)\qquad &\left\langle J'\left(u_n^i\right), -w_\rho\right\rangle + \left(t_n^i\left(w_\rho\right) - 1\right)\left\langle J'\left(u_n^i\right), \left(u_n^i - w_\rho\right)\right\rangle \\ &\geq -\frac{1}{n}\left\|z_\rho^i - u_n^i\right\|_{H^1} + o\left(\left\|z_\rho^i - u_n^i\right\|_{H^1}\right).\end{aligned}$$

From $t_n^i\left(w_\rho\right)\left(u_n^i - w_\rho\right) \in M^i\left(\Omega\left(t\right)\right)$ and (25), we have

$$\begin{aligned}&-\rho\left\langle J'\left(u_n^i\right), \frac{u}{\|u\|_{H^1}}\right\rangle + \left(t_n^i\left(w_\rho\right) - 1\right)\left\langle J'\left(u_n^i\right) - J'\left(z_\rho^i\right), \left(u_n^i - w_\rho\right)\right\rangle \\ \geq\ &-\frac{1}{n}\left\|z_\rho^i - u_n^i\right\|_{H^1} + o\left(\left\|z_\rho^i - u_n^i\right\|_{H^1}\right).\end{aligned}$$

Thus,

$$\begin{aligned}\left\langle J'\left(u_n^i\right), \frac{u}{\|u\|_{H^1}}\right\rangle \leq\ & \frac{\left\|z_\rho^i - u_n^i\right\|_{H^1}}{n\rho} + \frac{o\left(\left\|z_\rho^i - u_n^i\right\|_{H^1}\right)}{\rho} \\ &+\frac{\left(t_n^i\left(w_\rho\right) - 1\right)}{\rho}\left\langle J'\left(u_n^i\right) - J'\left(z_\rho^i\right), \left(u_n^i - w_\rho\right)\right\rangle.\end{aligned}$$

On the other hand, by (16) we can find a constant $C > 0$, independent of ρ, such that

$$\left\|z_\rho^i - u_n^i\right\|_{H^1} \leq \rho + \left|t_n^i\left(w_\rho\right) - 1\right| C$$

and

$$\lim_{\rho \to 0}\frac{\left|t_n^i\left(w_\rho\right) - 1\right|}{\rho} \leq \left\|\left(t_n^i\right)'(0)\right\| \leq C.$$

If we let $\rho \to 0$ in (20) for a fixed n and use the fact that $z_\rho^i \to u_n^i$ in $H_0^1\left(\Omega\left(t\right)\right)$, we get

$$\left\langle J'\left(u_n^i\right), \frac{u}{\|u\|_{H^1}}\right\rangle \leq \frac{C}{n}.$$

This shows that $\left\{u_n^i\right\}$ is a $(\mathrm{PS})_{\beta_i(t)}$–sequence in $H_0^1\left(\Omega\left(t\right)\right)$ for J. ■

Theorem 18. *There exists $t_0 > 0$ such that for $t \geq t_0$, Equation (1) in $\Omega\left(t\right)$ has m positive higher energy solutions $u_0^1, u_0^2, \ldots, u_0^m$ with*

$$\int_{\left[\mathbf{A}_{(i-1)t,it}\right]^c}\left|u_0^i\right|^p < \frac{p}{(p-2)}\alpha\left(\mathbf{A}\right) \text{ for each } i = 1, 2\ldots, m.$$

Proof. It follows from Proposition 17 that there exists $t_0 > 0$ such that for each $t \geq t_0$ and $i = 1, 2, \ldots, m$ we can find minimizing sequence $\{u_n^i\}$ of $\beta_i(t)$ with

$$\beta_i(t) < \min\{2\alpha(\Omega(t)), \gamma_i(t)\}.$$

By Lemma 7 and Theorem 9, there exists $u_0^i \in M_i(t)$ such that

$$u_n^i \to u_0^i \text{ strongly in } H_0^1(\Omega(t)),$$

$J(u_0^i) = \beta_i(t)$ and u_0^i is a positive solution of Equation (1) in $\Omega(t)$. By Lemma 13, that $\overline{M_i(t)} \cap \overline{M_j(t)} = \phi$ for $i \neq j$, we get $u_0^i \neq u_0^j$ for $i \neq j$. Furthermore, $\Omega(t)$ is a large domain in $\mathbf{A}$, we have

$$J(u_0^i) = \beta_i(t) > \alpha(\Omega(t)), \text{ for each } i = 1, 2 \ldots, m.$$

This implies, $u_0^1, u_0^2, \ldots, u_0^m$ are higher energy solutions of Equation (1) in $\Omega(t)$.■

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