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ESTIMATES ON SOLUTIONS TO CERTAIN QUASILINEAR EQUATIONS IN DIVERGENCE FORM

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Abstract. The convergence of approximations to solutions of nonlinear elliptic equations is closely related to the structure of the equations. As examples, we examine certain quasilinear elliptic equations with quadratic growth in the gradient defined on bounded domains. L^∞ and H^1 estimates on approximating solutions are performed to deduce the convergence to a solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$. In some cases, H^1 a priori bound can be derived without referring to L^∞ estimate. Furthermore, a $W^{2,p}(\mathbf{\Omega})$ bound is also established to deduce the existence of strong solutions in $W^{2,p}(\mathbf{\Omega}) \cap W_0^{1,p}(\mathbf{\Omega})$.

1. Introduction

The convergences of approximating solutions to nonlinear elliptic equations in various function spaces are closely related to the structure of equations as well as the constraints on nonlinear terms. As examples, we examine certain quasilinear elliptic problems with quadratic growth in the gradient defined on bounded domains. Let $\mathbf{\Omega}$ be a bounded domain in R^N, $N \geqslant 3$, which is $C^{1,1}$ diffeomorphic to a ball in R^N. L_v, L, D_v, D are the elliptic operators defined by

$$\begin{aligned} L_v u &= -\sum_{i,j=1}^{N} a_{ij}(x,v)\frac{\partial^2 u}{\partial x_i \partial x_j} + c(x,v)u, \\ Lu &= L_u u, \\ D_v u &= -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} a_{ij}(x,v)\frac{\partial u}{\partial x_j} + c(x,v)u, \\ Du &= D_u u, \end{aligned}$$

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where the coefficients a_{ij}, c, and $\frac{\partial a_{ij}}{\partial x_i}$, $\frac{\partial a_{ij}}{\partial r}$ are bounded Carathéodory functions, $c \geqslant \alpha_0 > 0$ for some constant α_0, and $\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \geqslant \lambda|\xi|^2$.

Let $f(x, r, \xi)$ be a Carathéodory function. Consider the following quasilinear elliptic problems

$$Lu = f(x, u, \nabla u) \text{ in } \mathbf{\Omega},$$

$$Du = f(x, u, \nabla u) \text{ in } \mathbf{\Omega},$$

where

$$|f(x, r, \xi)| \leqslant b(|r|) + h(|r|)|\xi|^\theta,\ 0 \leqslant \theta \leqslant 2, \tag{3}$$

$b(|r|)$ and $h(|r|)$ are locally bounded functions. When $b(|r|) \equiv C$, it has been shown [2] that Equation (2) has a solution in $H_0^1(\mathbf{\Omega}) \cap L^\infty(\mathbf{\Omega})$. L^∞, H^1 estimates and then H^1 convergence to a solution are established in successive steps. In Section 1, we extend to the case $b(|r|) = o(|r|)$. We follow the similar steps to show the existence result of solutions in $H_0^1(\mathbf{\Omega}) \cap L^\infty(\mathbf{\Omega})$. In case that $|f(x, r, \xi)| \leqslant b(|r|)(1 + |\xi|^2)$, H^1 a priori bound can be derived without refering to the L^∞ estimate. If $0 \leqslant \theta < 2$, and the oscillations of $a_{ij}(x, r)$ with respect to r are sufficiently small, we shall prove in Section 2 that there exists a solution in $W^{2,p}(\mathbf{\Omega}) \cap W_0^{1,p}(\mathbf{\Omega})$, $1 \leq p < \infty$.

2. H^1 Estimate and the Existence of Solution

Let f_n be the truncation of f_n by $\pm\, n$. Consider the approximating equation

$$Du = f_n(x, u, \nabla u) \text{ in } \mathbf{\Omega}. \tag{4}$$

Notice that the map $v \in H_0^1(\mathbf{\Omega}) \to f_n(x, v, \nabla v)$ is bounded and for every $v \in H_0^1(\mathbf{\Omega})$ there exists a unique $w \in H_0^1(\mathbf{\Omega})$ satisfying $D_v w = f_n(v, \nabla v)$. The map $v \in H_0^1(\mathbf{\Omega}) \to w \in H_0^1(\mathbf{\Omega})$ satisfies the hypotheses of Schauder Fixed Point Theorem, so there exists a solution $u_n \in H_0^1(\mathbf{\Omega})$ such that

$Du_n = f(x, u_n, \nabla u_n)$. Moreover, by the weak maximum principle, $\|u_n\|_{L^\infty} \leqslant \frac{n}{\alpha_0}$.

Now we proceed to the H^1 estimate of the solutions (u_n). Set $E_n = e^{tu_n^2}$, $v_n = E_n u_n$. Then $\frac{\partial v_n}{\partial x_i} = E_n \frac{\partial u_n}{\partial x_i} + 2tE_n u_n^2 \frac{\partial u_n}{\partial x_i}$.

Theorem 1. *Assume that* $f(x, r, \xi) \leqslant b(|r|)(1 + \xi|^2)$ *and* $b(|r|) = o(|r|)$. *Then the approximating solutions* (u_n)*are* H^1*-bounded.*

Proof. Let ε be given. Since $b(|r|) = o(|r|)$, there exists C and $K > 0$ such that $b(|r|) \leqslant C$ for $|r| \leqslant K$ and $b(|r|) \leqslant \varepsilon|r|$ for $|r| \geqslant K$. Denote $\mathbf{\Omega}_{n_1} = \{x||u_n(x)| \leqslant K\}$, $\mathbf{\Omega}_{n_2} = \mathbf{\Omega}\backslash\mathbf{\Omega}_{n_1}$.

Multiply the test function v_n on both sides of Equation (4),

$$\sum\int E_n a_{ij}\frac{\partial u_n}{\partial x_i}\frac{\partial u_n}{\partial x_j}+2t\sum\int E_n u_n^2 a_{ij}\frac{\partial u_n}{\partial x_i}\frac{\partial u_n}{\partial x_j}+\int E_n c u_n^2$$
$$=\int f_n(x,u_n,\nabla u_n)E_n u_n.$$

By the ellipticity, one has

$$\begin{aligned}
&\lambda\int_{\Omega}E_n|\nabla u_n|^2+2t\lambda\int_{\Omega}E_n u_n^2|\nabla u_n|^2+\alpha_0\int_{\Omega}E_n u_n^2 \\
&\leqslant\int_{\Omega}|f_n(x,u_n,\nabla u_n)|E_n|u_n|
\end{aligned}\tag{5}$$

$$\begin{aligned}
&\leqslant\int_{\Omega}(b(|u_n|)(1+|\nabla u_n|^2)E_n|u_n| \\
&\leqslant\int_{\Omega_{n_1}}Ce^{\mathrm{t}\mathrm{K}^2}K+\varepsilon\int_{\Omega_{n_2}}E_n|u_n|^2+\int_{\Omega_{n_1}}C|\nabla u_n|^2E_n|u_n| \\
&\quad+\varepsilon\int_{\Omega_{n_2}}|\nabla u_n|^2E_n|u_n|^2.
\end{aligned}\tag{6}$$

Use Young's inequality, the right-hand side is less than

$$\begin{aligned}
&M+\varepsilon\int_{\Omega}E_n|u_n|^2+\frac{\lambda}{2}\int_{\Omega}E_n|\nabla u_n|^2+\frac{C^2}{2\lambda}\int_{\Omega}E_n|u_n|^2|\nabla u_n|^2 \\
&+\varepsilon\textstyle\int_{\Omega_{n_2}}|\nabla u_n|^2E_n|u_n|^2.
\end{aligned}\tag{7}$$

Choose $\varepsilon=\frac{\alpha_0}{2}$, $t>\frac{1}{4\lambda^2}(C^2+\alpha_0\lambda)$ and move the last four terms of (7) to (5), we obtain

$$\int|\nabla u_n|^2\leqslant\int E_n|\nabla u_n|^2\leqslant\frac{2}{\lambda}M.$$

Finally, by the Poicaré inequality, $\|u_n\|_{H^1}\leqslant M'$. ■

Remark 1. The Proof of Theorem 1 indicates that H^1 bound can be deduced independently without employing L^∞ bound.

Consider now Equation (2) with f satisfying (3) when $b(|r|)=C$, an L^∞ estimate was performed in [2] by a sort of of weak maximum principle$''$ method. Assume that $b(|r|)=o(|r|)$. Then for every ε, $\alpha_0>\varepsilon>0$, there exists a constant $C>0$ such that

$$|f(x,r,\xi)|\leqslant C+\varepsilon(|r|)+h(|r|)|\xi|^2.\tag{8}$$

Denote $z_n = u_n - \frac{C}{\alpha_0 - \varepsilon}$, $e_n = \exp(t|z_n^+|^2)$ and $v_n = e_n z_n^+$. Following the demonstration in [2], we multiply the test function v_n on both sides of Equation (2.15) in [2, page 28]. Observe that the additional integral $\varepsilon \int |z_n| v_n = \varepsilon \int e_n |z_n^+|^2$ on the right-hand side is dominated by $\int C(u)|u_n|v_n \geqslant \alpha_0 \int e_n |z_n^+|^2$ on the left. Thus, from the estimate in [2], $u_n \leqslant \frac{C}{\alpha_0 - \varepsilon}$ for all n. In a similar way, one gets $-\frac{C}{\alpha_0-\varepsilon} \leqslant u_n$ by using $z_n = -\frac{C}{\alpha_0-\varepsilon} - u_n$. Therefore, one concludes that

Lemma 1. *Let f be given in* (3) *with* $b(|r|) = o(|r|)$. *Then the approximating solutions* (u_n) *in* (4) *are* L^∞*-bounded. Moreover, if f satisfies* (8) *then* $\| u_n \|_{L^\infty} \leqslant \frac{C}{\alpha_0 - \varepsilon}$.

Once an L^∞ bound is established, H^1 bound for (u_n) can be further established.

Lemma 2. *Let f be given as in* (3) *with* $b(|r|) = o(|r|)$. *Then* (u_n) *is* H^1*-bounded.*

Proof. Notice that (u_n) is L^∞-bounded and hence $h(|u_n(x)|) \leqslant C_1$ a.e. for some constant C_1.

If one replaces (8) for (6) in the proof of Theorem 1, then

$$
\begin{aligned}
& \int_\Omega |f(x, u_n, \nabla u_n)| E_n |u_n| \\
\leqslant & \int_\Omega (b|u_n| + h(|u_n||\nabla u_n|^2)) E_n |u_n| \\
\leqslant & \int_\Omega (C + \varepsilon |u_n| + C_1 |\nabla u_n|^2) E_n |u_n| \\
\leqslant & \ M + \varepsilon \int_\Omega E_n |u_n|^2 + \frac{\lambda}{2} \int_\Omega E_n |\nabla u_n|^2 + \frac{C_1^2}{2\lambda} \int_\Omega E_n |u_n|^2 |\nabla u_n|^2.
\end{aligned}
$$

Choose $\varepsilon = \frac{\alpha_0}{2}$ and $t > \frac{C_1^2}{4\lambda^2}$, it follows from the proof of Theorem 1 that (u_n) is H^1-bounded. ■

Together with Theorem 1, one can follow the steps in [2] to show that there exists a subsequence relabeled as (u_n), such that $u_n \to u$ in $H^1(\boldsymbol{\Omega})$ and $f_n(x, u_n, \nabla u_n) \to f(x, u, \nabla u)$ in $L^1(\Omega)$. Therefore, by passing to the limit, one concludes that $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a solution to (2).

Theorem 2. *Let f be given as in* (3) *with* $b(|r|) = o(|r|)$. *Then there exists a solution* $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *to* (2).

3. Strong Solutions to Equations in Divergence Form

In this section, we examine the existence of $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ solutions to the quasilinear equations in divergence form,

$$
\begin{aligned}
(9) \qquad Du &= -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} a_{ij}(x,u) \frac{\partial u}{\partial x_j} + C(x,u)u \\
&= f(x,u,\nabla u) \text{ in the sense of distribution } \mathfrak{D}'(\Omega),
\end{aligned}
$$

where the cofficients $a_{ij} \in C^{0,1}(\bar{\Omega} \times R)$, and

$$
(10) \qquad |f(x,r,\xi)| \leqslant C_0 + h(|r|)|\xi|^{\theta}, \ 0 \leq \theta < 2
$$

For a fixed point $x \in R^N$, we denote osc $a_{ij}(x,r)$ the oscillation of $a_{ij}(x,r)$ with respect to r, that is, osc $a_{ij}(x,r) \equiv \sup\{|a_{ij}(x,r_1) - a_{ij}(x,r_2)| : r_1, r_2 \in R\}$, and osc $a_{ij}(x,r) = \max_{1 \leq i,\, j \leq N}$ osc $a_{ij}(x,r)$.

For operators L_v, we quote the following result from [5, p.191].

Lemma 3. *Let Ω be a bounded domain in R^N which is $C^{1,1}$ diffeomorphic to a ball in R^N, and the cofficients $a_{ij} \in C^{0,1}(\bar{\Omega} \times R)$, $|a_{ij}|$, $|c| \leqslant \Lambda$, where Λ is a positive constant, i, j= $1, \ \ldots, N$. Assume that osc $a_{ij}(x,r)$ is sufficiently small with respect to r and uniformly for $x \in \Omega$. Then if $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and $L_v u \in L^p(\Omega)$,$1 < p < \infty$. One has the estimate*

$$
(11) \qquad \|u\|_{W^{2,p}(\Omega)} \leqslant C(\|L_v u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}),
$$

where C is a constant (independent of v) dependent on $N, p, \lambda, \Lambda, \partial\Omega$, and Ω, the diffeomorphism and the mpduli of continuity of $a_{ij}(x,r)$ with respect to x in $\bar{\Omega}$.

Consider now Equation (1). Suppose $f(x,r,\xi)$is bounded. For $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, the Dirichlet problem $L_v u = f(x,v,\nabla u)$ has a unique solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and by Theorem 2,

$$
\|u\|_{W^{2,p}(\Omega)} \leqslant C(\|u\|_{L^p(\Omega)} + \|f(x,v,\nabla v)\|_{L^p(\Omega)})
$$

An application of the Weak Maximum Principle of A.D Aleksendrov [4, p.220] together with the Schauder Fixed Point Theorem implies that

Lemma 4. *Suppose that $f(x,r,\xi)$ is bounded. Then for each $1 \leq p < \infty$, there exists a solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.*

Proof. See[5. p118]. ∎

Let

$$\tilde{f}(x,r,\xi) = f(x,r,\xi) + \sum \frac{\partial a_{ij}}{\partial x_i}\xi_i + \sum \frac{\partial a_{ij}}{\partial r}\xi_i\xi_j,$$
$$\tilde{g}_n(x,r,\xi) = \tilde{f}_n(x,r,\xi) + \sum \frac{\partial a_{ij}}{\partial x_i}\xi_i + \sum \frac{\partial a_{ij}}{\partial r}\xi_i\xi_j,$$

In view of Lemma 4, there exists a $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ solution u_n, satisfying

$$Lu_n = \tilde{f}_n(x,u_n,\nabla u_n), \tag{12}$$

or equivalently,

$$Du_n = \tilde{g}_n(x,u_n,\nabla u_n). \tag{13}$$

Observe that for $\varepsilon > 0$, $|\tilde{g}_n(x,r,\xi) \leqslant C_0 + \varepsilon + h_1(|r|)|\xi|^2$, where $h_1(|r|)$ is a locally bounded function. By Lemma 1, $\|u_n\|_{L^\infty} \leqslant \frac{C_0}{\alpha_0}$. One can now proceed to the $W^{2,p}$ estimate and then show the existence of $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ solutions to (9).

Lemma 5. *If in addition to the assumption of Lemma 3 and* (10)*, the cofficients a_{ij} are independent of r for $|r| \leqslant \frac{C_0}{\alpha_0}$. Then there exists a $W^{2,p}$-bounded subsequence of the approximating solutions (u_n) in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ to Equation* (9).

Proof. As described in the above paragraphs, the approximating solutions (u_n) to (13) are L^∞-bounded by $\frac{C_0}{\alpha_0}$. By the assumption that $a_{ij}(x,r) = a_{ij}(x)$ and consequently $\frac{\partial a_{ij}}{\partial r} = 0$ for $|r| \leqslant \frac{C_0}{\alpha_0}$.

$$\hat{f}(x,u_n,\nabla u_n) = f(x,u_n,\nabla u_n) - \sum \frac{a_{ij}}{\partial x_i}(x,u_n,\nabla u_n)\frac{\partial u_n}{\partial x_j}.$$

For $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $|\xi|^\theta \leqslant C_\varepsilon + \varepsilon|\xi|^2$. Combine with (10),

$$|\hat{f}_n(x,u_n,\nabla u_n)| \leqslant C_1 + C_2\varepsilon|\nabla u_n|^2,$$

for some constants C_1, C_2.

In view of the estimate (11), one has

$$\begin{aligned}\|u\|_{W^{2,p}} &\leqslant C(\|u_n\|_{L^p} + \|\tilde{f}_n(x,u_n,\nabla u_n)\|_{L^p})\\ &\leqslant C_2 + \varepsilon C_2\|\nabla u_n\|^2_{L^{2p}}.\end{aligned}$$

Now since $u_n \in L^\infty(\Omega) \cap W^{2,p}(\Omega)$, from the Interpolation Theorem of Gagliardo-Nirenberg [1, p.194], we obtain

$$\begin{aligned}\|u\|_{W^{2,p}} &\leqslant C_2 + \varepsilon C_2'\|u_n\|_{W^{2,p}}\|u_n\|_{L^\infty}\\ &\leqslant C_2 + \varepsilon C_2'\frac{C_0}{\alpha_0}\|u_n\|_{W^{2,p}}.\end{aligned}$$

Hence, by choosing $\varepsilon C_2' \frac{C_0}{\alpha_0} = \frac{1}{2}$, we conclude that $\|u_n\|_{W^{2,p}} \leqslant C_3$, i.e. (u_n) is $W^{2,p}$-bounded.

The existence of strong solutions can now be concluded from above lemmas. We summerize in our main theorem.

Theorem 3. *Let Ω be a bounded $c^{1,1}$-smooth domain in R^N, $N \geqslant 3$. The coefficients a_{ij}, $\frac{\partial a_{ij}}{\partial x_i}$, $\frac{\partial a_{ij}}{\partial x_i}$, c are bounded Carathéory functions. Assume that osc $a(x,r)$ is sufficiently small with respect to r and uniformly for $x \in \Omega, a_{ij}$ are independent of r for $|r| \leqslant \frac{C_0}{\alpha_0}$, and f satisfies (8). Then for each p, $1 \leqslant p < \infty$, there exists a solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ to (8) with $\|u\|_{L^\infty} \leqslant \frac{C_0}{\alpha_0}$.*

Proof. By Lemma 5 we get the approximating solutions (u_n) which is $W^{2,p}(\Omega)$ bounded. It follows from the compact imbedding $W^{2,p}(\Omega) \to W^{1,p}(\Omega)$ that there exists a convergent subsequence in $W^{1,p}(\Omega)$, which is still denoted by (u_n), such that $u_n \to u$ a.e., $\nabla u_n \to \nabla u$ a.e. and $u_n \to u$ in $W^{1,p}(\Omega)$.

Passing to the limit and using Vitali Convergence Theorem, one can show that $Lu_n \to Lu$ in $\mathfrak{D}'(\Omega)$ and $f_n(x, u_n, \nabla u_n) \to f(x, u, \nabla u)$ in $L^1(\Omega)$. Moreover, since $\|u_n\|_{W^{2,p}} \leqslant M$ and the set

$$\{v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) | \|u_n\|_{W^{2,p}} \leqslant M\}$$

is closed in $W_0^{1,p}(\Omega)$, the limit u of (u_n) belong to $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. The existence of solutions in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ to Problem (9) is now asserted. ■

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