TAIWANESE JOURNAL OF MATHEMATICS
Vol. 9, No. 2, pp. 175-185, June 2005
This paper is available online at http://www.math.nthu.edu.tw/tjm/

OPTIMAL LOWER ESTIMATES FOR EIGENVALUE RATIOS OF SCHRÖDINGER OPERATORS AND VIBRATING STRINGS

Chung-Chuan Chen, C. K. Law and F. Y. Sing

Abstract. We obtain optimal lower estimates for the eigenvalue ratios $(\frac{\lambda_m}{\lambda_n})$ of Dirichlet and Neumann Schrödinger operators with nonpositive potentials and Dirichlet vibrating string problems with concave and positive densities. Our results supplement those of Ashbaugh-Benguria [2] and M. J. Huang [5].

1. Introduction

Consider the one-dimensional Schrödinger operator on $[0,1]$,

$$-y'' + q(x)y = \lambda y \ , \tag{1.1}$$

and vibrating string problem on $[0,1]$,

$$-y'' = \mu\rho(x)y \ , \tag{1.2}$$

subject to linear separated boundary conditions

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0 \ ,$$
$$y(1)\cos\beta + y'(1)\sin\beta = 0 \ ,$$

where $\alpha = \beta = 0$ corresponds to the Dirichlet boundary condition and $\alpha = \beta = \pi/2$ corresponds to the Neumann boundary condition. Let λ_n (μ_n) be the n^{th} eigenvalue and y_n be the n^{th} eigenfunction with $n-1$ zeros in $(0,1)$. The functions $q, \rho \in L^1(0,1)$ and are called the potential function and density function respectively. The eigenvalue gaps and eigenvalue ratios of the above systems have been the object of

Received February 5, 2005; accepted March 16, 2005.
Communicated by Sze-Bi Hsu.
2000 *Mathematics Subject Classification*: 34B24, 34L15.
Key words and phrases: Schrödinger operators, Vibrating string problems, Eigenvalue ratios, Modified Prüfer substitution.

many studies. Recently, Lavine [8] proved an optimal lower estimate of the first eigenvalue gap for Schrödinger operators with convex potentials.

Theorem 1.1. [9] *For the Schrödinger operator* (1.1) *on* $[0,1]$, *if* q *is convex, then the first Dirichlet* $(Neumann)$ *eigenvalue gap* $\lambda_2 - \lambda_1$ *satisfies*

$$\lambda_2 - \lambda_1 \geq 3\pi^2 \quad (\lambda_2 - \lambda_1 \geq \pi^2).$$

In both cases, equality holds if and only if $q = 0$.

Lavine's theorem is a special case of a conjecture that for convex potentials q defined on any bounded domain in R^n, the first Dirichlet eigenvalue gap is smallest when $n = 1$ and $q = 0$. His theorem proves the conjecture for $n = 1$. The general case is still open. His method involves a variational approach with detailed analysis on different integrals involving $y_2^2 - y_1^2$.

Later (M. J.) Huang adapted his method to study the eigenvalue ratios of vibrating strings [5]. One of the main results is the following Theorem 1.2. It may be viewed as the dual of Theorem 1.1.

Theorem 1.2. [5] *For the vibrating string equation* (1.2), *if* ρ *is concave and positive, then the first Dirichlet eigenvalue ratio* $\frac{\mu_2}{\mu_1}$ *satisfies*

$$\frac{\mu_2}{\mu_1} \geq 4.$$

Equality holds if and only if ρ *is constant.*

The main objective of this paper is to generalize the above optimal estimate for the Dirichlet eigenvalue ratio $\frac{\mu_2}{\mu_1}$ to arbitrary $\frac{\mu_m}{\mu_n}$. Observe that in [2], Ashbaugh and Benguria introduced a method involving a modified Prüfer substitution and a comparison theorem to study the upper bounds of Dirichlet eigenvalue ratios for Schrödinger operators with nonnegative potentials. The method was then simplified and generalized to study Sturm-Liouville operators [3] and some general boundary conditions [6]. The results may be summarized as follows:

Theorem 1.3. [2,6] *For the Schrödinger operator* (1.1), *if* $q \in L^1(0,1)$ *and* $q \geq 0$ *a.e., then for any* $m > n \geq 1$, *the Dirichlet eigenvalue ratios satisfy*

$$\frac{\lambda}{\lambda_n} \leq (\lceil \frac{m}{n} \rceil)^2,$$

and the Neumann eigenvalue ratios satisfy

$$\frac{\lambda_{m+1}}{\lambda_{n+1}} \leq (2\lfloor \frac{1}{2} \lceil \frac{m}{n} \rceil \rfloor + 1)^2.$$

In each case, equality holds if and only if $q = 0$ *and* m *is a multiple of* n (*and* $\frac{m}{n}$ *is odd in the Neumann case*).

In the above theorem, the floor function of s, $\lfloor s \rfloor = \max\{k \in \mathbf{Z} : k \leq s\}$. The ceiling function of s, $\lceil s \rceil = \min\{k \in \mathbf{Z} : k \geq s\}$. It is interesting to see that the counterpart of the above result is also valid.

Theorem 1.4. *For the Schrödinger operator* (1.1), *if* $q \in L^1(0,1)$, $q \leq 0$ *a.e., and the Dirichlet, Neumann eigenvalue* λ_1*'s are positive, then for any* $m > n \geq 1$,

(a) *the Dirichlet eigenvalue ratios satisfy*

$$\frac{\lambda_m}{\lambda_n} \geq (\lfloor \frac{m}{n} \rfloor)^2.$$

Equality holds if and only if $q = 0$ *and* m *is a multiple of* n.

(b) *the Neumann eigenvalue ratios satisfy*

$$\frac{\lambda_{m+1}}{\lambda_{n+1}} \geq k^2$$

where let $s = \lfloor \frac{m}{n} \rfloor$, *and*

$$k = 2\lceil \frac{s}{2} \rceil - 1 = \begin{cases} s & \text{when } s \text{ is odd,} \\ s-1 & \text{when } s \text{ is even.} \end{cases}$$

Equality holds if and only if $q = 0$ *and* m *is an odd multiple of* n.

Theorem 1.4 helps in attaining our objective concerning the vibrating strings. For if ρ is C^2, (1.2) can be transformed [4] to a Schrödinger operator with the potential function $\hat{q}$ satisfying

$$\hat{q} = \frac{4\rho''\rho - 5(\rho')^2}{16\rho^3} = -f^3 f'' \ , \tag{1.3}$$

where $f = \rho^{-1/4}$. Hence when ρ is smooth, concave and positive, $\hat{q}$ has to be nonpositive, as required in Theorem 1.4.

Theorem 1.5. *For the vibrating string equation (1.2), if* ρ *is concave and positive, then for any* $m > n \geq 1$, *the Dirichlet eigenvalue ratios satisfy*

$$\frac{\mu_m}{\mu_n} \geq (\lfloor \frac{m}{n} \rfloor)^2.$$

In particular, if ρ *is twice differentiable, then equality holds if and only if* ρ *is constant and* m *is a multiple of* n.

It is open if the optimality result is true without the smoothness assumption on ρ. Preliminaries will be given in Section 2. Theorem 1.4 and Theorem 1.5 will be proved in section 3 and section 4 respectively.

2. Preliminaries

The Prüfer substitution [4] for the Schrödinger operator involves

$$\begin{cases} y(x) = r(x)\sin\phi(x) \\ y'(x) = r(x)\cos\phi(x) \end{cases}$$

where ϕ is the phase function. For the n^{th} Dirichlet eigenfunction y_n, the phase function ϕ_n satisfies $\phi_n(0) = 0,\ \phi_n(1) = n\pi$. The modified Prüfer substitution was introduced by Ashbaugh and Benguria [2],

$$\begin{cases} y(x) = r(x)\sin\sqrt{\lambda}\theta(x) \\ y'(x) = \sqrt{\lambda}r(x)\cos\sqrt{\lambda}\theta(x) \end{cases} \tag{2.1}$$

where the modified phase θ satisfies

$$\frac{d\theta}{dx} = 1 - \frac{q(x)}{\lambda}\sin^2(\sqrt{\lambda}\theta(x)) \equiv F(x,\theta,\lambda). \tag{2.2}$$

The modified phase θ_n satisfies $\theta_n(0) = 0$, $\theta_n(1) = n\pi/\sqrt{\lambda_n}$. Our method needs to compare the modified phases (2.2) for different eigenfunctions. Here the term $\sin^2(\sqrt{\lambda}\theta(x))/\lambda$ is important. Below we give a simpler proof for the inequality [6, Theorem 3].

Lemma 2.1. *Suppose* $c \geq 1, |\Theta| \leq \lfloor c\rfloor\pi/c$, *then*

$$\sin^2(c\Theta) \leq c^2\sin^2\Theta.$$

Proof. Clearly it is sufficient to prove the case $\Theta \geq 0$. Consider $f(\Theta) = c\sin\Theta \pm \sin(c\Theta)$, where $0 \leq \Theta \leq \lfloor c\rfloor\pi/c$. For any critical value Θ_c, we have $f'(\Theta_c) = 0$. That is

$$\cos\Theta_c = \pm\cos(c\Theta_c)\ ,$$

so that $\sin\Theta_c = \pm\sin(c\Theta_c)$, which in turn implies

$$f(\Theta_c) = (c \pm 1)\sin\Theta_c \geq 0.$$

Furthermore

$$f(0) = 0,\ f(\lfloor c\rfloor\pi/c) = c\sin(\lfloor c\rfloor\pi/c) \geq 0.$$

So we conclude that for all $0 \le \Theta \le \lfloor c \rfloor \pi / c$,

$$c \sin \Theta \pm \sin(c\Theta) \ge 0.$$

Therefore

$$c^2 \sin^2 \Theta \ge \sin^2(c\Theta). \qquad \blacksquare$$

Clearly $\frac{1}{2}\pi \le \lfloor c \rfloor \pi / c \le \pi$. Hence if $c = \sqrt{\frac{\lambda_n}{\lambda_1}}$, $\Theta = \sqrt{\lambda_1}\theta$, then for $|\theta| \le \frac{\pi}{2\sqrt{\lambda_1}}$,

$$\frac{\sin^2(\sqrt{\lambda_n}\theta(x))}{\lambda_n} \le \frac{\sin^2(\sqrt{\lambda_1}\theta(x))}{\lambda_1}. \tag{2.3}$$

Lemma 2.2. Comparison Theorem (cf. [4, p. 30]) *Consider two differential equations on* $[0, 1]$,

$$\begin{aligned} \theta_1'(x) &= F(x, \theta_1(x)), \\ \theta_2'(x) &= G(x, \theta_2(x)). \end{aligned}$$

Suppose F or G is Lipschitz in θ, and $F(x,\theta) \le G(x,\theta)$, (x,θ) in $[0,1] \times I$ for some interval I. If $\theta_1(0) \le \theta_2(0)$ and $\theta_2(x)$ lies in the interval I for every $x \in (0,1)$, then $\theta_1 \le \theta_2$ on $[0,1]$. In fact, take any $x_0 \in [0,1]$, either $\theta_1(x_0) < \theta_2(x_0)$ or $\theta_1 = \theta_2$ on $[0, x_0]$.

3. Schrödinger Operators

We shall divide the proof of Theorem 1.4 into two parts (a) and (b).

Proof of Theorem 1.4(a)

In view of [8], we may assume that q is continuous on $[0,1]$. Suppose $m = nh$. Use induction on n. When $n = 1$, the modified phases θ_1 and θ_h, corresponding to the 1^{st} and h^{th} eigenfunction respectively, satisfy

$$\begin{aligned} \theta_1(0) = 0 \ , &\qquad \theta_1(1) = \frac{\pi}{\sqrt{\lambda_1}}, \\ \theta_h(0) = 0 \ , &\qquad \theta_h(1) = \frac{h\pi}{\sqrt{\lambda_h}}. \end{aligned}$$

Let

$$F_h(x, \theta) = 1 - \frac{q(x)}{\lambda_h} \sin^2(\sqrt{\lambda_h}\theta(x)).$$

By continuity, there is some $\omega \in (0,1)$ such that $\theta_1(\omega) = \frac{\pi}{2\sqrt{\lambda_1}}$. Then by (2.3),

$$F_h(x,\theta) \le F_1(x,\theta),$$

for $(x,\theta) \in [0,\omega] \times [0, \frac{\pi}{2\sqrt{\lambda_1}}]$. Thus we may apply Lemma 2.2 to see that for all $x \in [0,\omega]$, $\theta_h(x) \le \theta_1(x)$. In particular

$$\theta_h(\omega) \le \theta_1(\omega). \tag{3.1}$$

Now define

$$\hat{\theta}_1(x) = \frac{\pi}{\sqrt{\lambda_1}} - \theta_1(x)\ , \qquad \hat{\hat{\theta}}_1(x) = \hat{\theta}_1(1-x)\ ,$$
$$\hat{\theta}_h(x) = \frac{h\pi}{\sqrt{\lambda_h}} - \theta_h(x)\ , \qquad \hat{\hat{\theta}}_h(x) = \hat{\theta}_h(1-x)\ ,$$

and

$$\hat{\hat{\theta}}_1(1-\omega) = \hat{\theta}_1(\omega) = \frac{\pi}{2\sqrt{\lambda_1}}.$$

Hence both $\hat{\hat{\theta}}_h$ and $\hat{\hat{\theta}}_1$ satisfy

$$\frac{d\hat{\hat{\theta}}}{dx} = 1 - \frac{q(1-x)}{\lambda}\sin^2(\sqrt{\lambda}\hat{\hat{\theta}}(x)) = F(x,\hat{\hat{\theta}}),$$

where

$$\hat{\hat{\theta}}_1(0) = 0,\ \hat{\hat{\theta}}_1(1) = \frac{\pi}{\sqrt{\lambda_1}},$$
$$\hat{\hat{\theta}}_h(0) = 0,\ \hat{\hat{\theta}}_h(1) = \frac{h\pi}{\sqrt{\lambda_h}}.$$

By Lemma 2.1,

$$F_h(x,\hat{\hat{\theta}}) \le F_1(x,\hat{\hat{\theta}})$$

for $(x,\hat{\hat{\theta}}) \in [0,\omega] \times [0, \frac{\pi}{2\sqrt{\lambda_1}}]$. Therefore by Lemma 2.2 again,

$$\hat{\hat{\theta}}_h(x) \le \hat{\hat{\theta}}_1(x)$$

for $x \in [0, 1-\omega]$. In particular,

$$\hat{\hat{\theta}}_h(1-\omega) = \frac{h\pi}{\sqrt{\lambda_h}} - \theta_h(\omega) \le \frac{\pi}{\sqrt{\lambda_1}} - \theta_1(\omega) = \hat{\hat{\theta}}_1(1-\omega). \tag{3.2}$$

Therefore by (3.1),

$$\frac{\lambda_h}{\lambda_1} \ge h^2. \tag{3.3}$$

In general, we follow the method in [3,6]. Fix $i \in \mathbf{N}$. For each $j < i$, let $z_j(\lambda_i)$ denote the j^{th} zero for $\lambda = \lambda_i$ of (1.1) $\in (0,1)$. Let $\omega_1 = z_1(\lambda_{n+1})$ and $\omega_2 = z_h(\lambda_{(n+1)h})$. If $\omega_1 > \omega_2$, then consider the Dirichlet problem on $(0, \omega_1)$, and let $\widetilde{\lambda_h}$ be the h^{th} eigenvalue. Then by (3.3)

$$\frac{\lambda_{(n+1)h}}{\lambda_{n+1}} \geq \frac{\widetilde{\lambda_h}}{\widetilde{\lambda_1}} \geq h^2.$$

If $\omega_1 < \omega_2$, make the transformation $t = 1 - x$, and consider the problem on $(0, 1 - \omega_1)$, then by induction hypothesis,

$$\frac{\lambda_{(n+1)h}}{\lambda_{n+1}} \geq \frac{\widetilde{\lambda_{hn}}}{\widetilde{\lambda_n}} \geq h^2.$$

Hence the statement is valid for $m = nh$. In general when m is not necessarily a multiple of n, let $h = \lfloor \frac{m}{n} \rfloor$. Then

$$\frac{\lambda_m}{\lambda_n} \geq \frac{\lambda_{hn}}{\lambda_n} \geq h^2. \tag{3.4}$$

If $m = nh$ and $q = 0$, then it is straightforward that $\lambda_n = n^2$ and $\lambda_{nh} = n^2 h^2$. Hence $\frac{\lambda_{nh}}{\lambda_n} = h^2$. If there is some λ_m and λ_n such that $\frac{\lambda_m}{\lambda_n} = h^2$, where $h = \lfloor \frac{m}{n} \rfloor$, then by (3.4), $m = nh$ by the simplicity of the eigenvalues of (1.1) under separated boundary conditions. Then we use induction on n. When $n = 1$, $\frac{\lambda_h}{\lambda_1} = h^2$ implies from (3.2) that $\theta_h(\omega) \geq \theta_1(\omega)$ which when combined with (3.1) shows that $\theta_h(\omega) = \theta_1(\omega)$. So $F_h(x, \theta) = F_1(x, \theta)$. That means $q = 0$ on $(0, \omega)$. Similarly $\hat{\hat{\theta}}_h(1 - \omega) = \hat{\hat{\theta}}_1(1 - \omega)$ implies that $q = 0$ on $(\omega, 1)$, too.

We then compare the position of $\omega_1 = z_1(\lambda_{n+1})$ and $\omega_2 = z_h(\lambda_{(n+1)h})$. Without loss of generality, let $\omega_1 \geq \omega_2$. Consider the Dirichlet problem on $(0, \omega_1)$. Let $\widetilde{\lambda_h}$ be the h^{th} eigenvalue. Hence

$$h^2 = \frac{\lambda_{(n+1)h}}{\lambda_{n+1}} \geq \frac{\widetilde{\lambda_h}}{\widetilde{\lambda_1}} \geq h^2,$$

which implies $q = 0$ on $(0, \omega_1)$. Thus $\omega_1 = \omega_2$. It then follows from induction hypothesis that $q = 0$ on $(\omega_1, 1)$ too. By continuity $q = 0$ on $[0, 1]$. The proof for part(a) is complete. ■

We note that the indirect method in [2] was used in the proof of Theorem 1.4(a). The proof of Theorem 1.4(b) is simpler, in the sense that we need to compare the modified phases only once.

Proof of Theorem 1.4(b)
Suppose $m = nh$, use induction on n. Let $n = 1 \le m$. As in [6, Theorem 8(a)], we let the phase function to be centered at 0. Thus the modified phases satisfy

$$\theta_2(0) = -\frac{\pi}{2\sqrt{\lambda_2}}\,, \qquad \theta_2(1) = \frac{\pi}{2\sqrt{\lambda_2}}\,,$$
$$\theta_{m+1}(0) = -\frac{k\pi}{2\sqrt{\lambda_{m+1}}}\,, \qquad \theta_{m+1}(1) = \frac{(2m-k)\pi}{2\sqrt{\lambda_{m+1}}}\,,$$

where $k = 2\lceil \frac{m}{2} \rceil - 1 \le m$.

Suppose $\sqrt{\frac{\lambda_{m+1}}{\lambda_2}} < k$. Then $\theta_{m+1}(0) < \theta_2(0)$. And let $\theta_{m+1}(\omega) = -\frac{\pi}{2\sqrt{\lambda_2}}$ for some $\omega \in (0,1)$. Since

$$F_{m+1}(x,\theta) \le F_2(x,\theta)$$

for all $(x,\theta) \in [\omega,1] \times [-\frac{\pi}{2\sqrt{\lambda_2}}, \frac{\pi}{2\sqrt{\lambda_2}}]$. We apply Lemma 2.2 to obtain $\theta_{m+1} < \theta_2$ on $[\omega,1]$, and hence $\theta_{m+1}(1) < \theta_2(1)$. That yields

$$2m - k < \sqrt{\frac{\lambda_{m+1}}{\lambda_2}},$$

and hence

$$k \le m \le 2m - k < \sqrt{\frac{\lambda_{m+1}}{\lambda_2}}.$$

This gives a contradiction. Therefore $\frac{\lambda_{m+1}}{\lambda_2} \ge k^2$. The rest is similar. ∎

4. Vibrating String Problems

The Liouville substitution [4] for the vibrating string involves

$$t = \int_0^x \sqrt{\rho(s)}\,ds, \quad y(x) = \frac{w(t)}{\sqrt[4]{\rho(x)}},$$

which, when ρ is C^2, transforms (1.2) into a Schrödinger equation

$$-w''(t) + \hat{q}(t)w(t) = \mu w(t),$$

where $\hat{q}$ is given in (1.3). If the original system has Dirichlet boundary conditions, so does the transformed system. Note that this is not true for Neumann boundary conditions.

Proof of Theorem 1.5

If ρ is C^2, positive and concave on $[0,1]$, (1.2) can be transformed to (1.1) with $\hat{q}$, which is negative, by the Liouville substitution. Applying Theorem 1.4(a), we obtain the lower bound as below:

$$\frac{\mu_m(\rho)}{\mu_n(\rho)} = \frac{\mu_m(\hat{q})}{\mu_n(\hat{q})} \geq (\lfloor \frac{m}{n} \rfloor)^2.$$

If ρ is not C^2, then we need the following Lemma.

Lemma 4.1. *Given $\rho \in C[0,1]$, positive and concave, for $\epsilon > 0$, there exists positive C^∞ functions $\widetilde{\rho_\epsilon}$ on $[0,1]$ such that $\widetilde{\rho_\epsilon} \to \rho$ in $L^1(0,1)$. Furthermore each $\widetilde{\rho_\epsilon}$ satisfies $\widetilde{\rho_\epsilon}'' \leq 0$ except possibly at two points in $[0,1]$.*

Proof. Choose the approximate identity which is defined as

$$k(x) = \begin{cases} ce^{\frac{1}{x^2-1}} & -1 < x < 1, \text{where } c = (\int e^{\frac{1}{x^2-1}} dx)^{-1}. \\ 0 & \text{otherwise.} \end{cases}$$

Use the convolution to define ρ_ϵ:

$$\rho_\epsilon(x) = \rho * k_\epsilon(x) = \int_{-\infty}^{\infty} \rho(x-y) k_\epsilon(y) dy \;\text{ where } k_\epsilon = \frac{1}{\epsilon} k(\frac{y}{\epsilon}).$$

It is clear that ρ_ϵ is C^∞, positive and $\rho_\epsilon \to \rho$ in $L^1(0,1)$.

We show that ρ_ϵ is concave on $[\epsilon, 1-\epsilon]$. For each $x, y \in [\epsilon, 1-\epsilon]$ and $\gamma \in [0,1]$,

$$\begin{aligned} \rho_\epsilon[\gamma x + (1-\gamma)y] &= \int_{-\epsilon}^{\epsilon} \rho[\gamma x + (1-\gamma)y - z] k_\epsilon(z) dz \;, \\ &= \int_{-\epsilon}^{\epsilon} \rho[\gamma(x-z) + (1-\gamma)(y-z)] k_\epsilon(z) dz \;, \\ &\geq \int_{-\epsilon}^{\epsilon} [\gamma\rho(x-z) + (1-\gamma)\rho(y-z)] k_\epsilon(z) dz \;, \\ &= \gamma \int_{-\epsilon}^{\epsilon} \rho(x-z) k_\epsilon(z) dz + (1-\gamma) \int_{-\epsilon}^{\epsilon} \rho(y-z) k_\epsilon(z) dz \;, \\ &= \gamma\rho_\epsilon(x) + (1-\gamma)\rho_\epsilon(y) \;. \end{aligned}$$

Hence ρ_ϵ is concave on $[\epsilon, 1-\epsilon]$. Now define

$$\widetilde{\rho_\epsilon}(x) = \begin{cases} \rho_\epsilon(x) & \text{on } [\epsilon, 1-\epsilon]. \\ L_1(x) = \rho_\epsilon(0) + \dfrac{\rho_\epsilon(\epsilon) - \rho_\epsilon(0)}{\epsilon} x & \text{on } [0, \epsilon]. \\ L_2(x) = \rho_\epsilon(1) + \dfrac{\rho_\epsilon(1-\epsilon) - \rho_\epsilon(1)}{\epsilon} (1-x) & \text{on } [1-\epsilon, 1]. \end{cases}$$

Then

$$\begin{aligned}\int_0^1 |\widetilde{\rho_\epsilon}(x)-\rho(x)|dx &\le \int_0^1 |\widetilde{\rho_\epsilon}(x)-\rho_\epsilon(x)|\,dx+\int_0^1 |\rho_\epsilon(x)-\rho(x)|dx\ ,\\ &= \int_0^\epsilon |L_1(x)-\rho_\epsilon(x)|dx+\int_{1-\epsilon}^1 |L_2(x)-\rho_\epsilon(x)|dx\\ &\quad +\int_0^1 |\rho_\epsilon(x)-\rho(x)|dx\ ,\\ &\le 2\epsilon M+\int_0^1 |\rho_\epsilon(x)-\rho(x)|dx\ \to 0 \ \text{ as } \ \epsilon\to 0\ ,\end{aligned}$$

where M is a positive constant. It is also clear that $\widetilde{\rho_\epsilon}$ is C^∞ a.e., and $\widetilde{\rho_\epsilon}'' \le 0$ except possibly at two points, ϵ and $1-\epsilon$. ■

Note that $\widetilde{\rho_\epsilon}''$ as defined above is piecewise continuous and if

$$\hat{q_\epsilon} = \frac{4\widetilde{\rho_\epsilon}''\widetilde{\rho_\epsilon} - 5(\widetilde{\rho_\epsilon}')^2}{16\widetilde{\rho_\epsilon}^3}\ ,$$

then $\hat{q_\epsilon} \le 0$ a.e., while eigenvalues are conserved. Therefore where

$$\hat{q_\epsilon} = \frac{4\widetilde{\rho_\epsilon}''\widetilde{\rho_\epsilon} - 5(\widetilde{\rho_\epsilon}')^2}{16\widetilde{\rho_\epsilon}^3} \le 0 \quad a.e.$$

In addition, the eigenvalues of Sturm-Liouville problem depend continuously on ρ [8]. Hence

$$\frac{\mu_m(\widetilde{\rho_\epsilon})}{\mu_n(\widetilde{\rho_\epsilon})} \to \frac{\mu_m(\rho)}{\mu_n(\rho)} \quad \text{as} \quad \epsilon\to 0.$$

Combining the results, we obtain

$$\frac{\mu_m(\rho)}{\mu_n(\rho)} \ge (\lfloor \frac{m}{n} \rfloor)^2.$$

When ρ is twice differentiable, then equality implies that $\hat{q}=0$ and m is a multiple of n. Hence by (1.3), $f''=0$ so that f is a linear function. That is, there exist $a, b \in \mathbf{R}$ such that

$$\rho(x) = \frac{1}{(ax+b)^4} > 0\ .$$

In this case, $\rho''(x) = 20a^2(ax+b)^{-6} \ge 0$. But ρ is concave, so $a=0$ and ρ is constant. The proof is complete. ■

ACKNOWLEDGEMENT

The authors would like to thank the anonymous referee for helpful suggestions. The second author is partially supported by National Science Council, Taiwan, R.O.C.

REFERENCES

1. M. S. Ashbaugh and R. D. Benguria, Best constant for the ratio of the first two eigenvalues of one-dimensional Schrödinger operator with positive potentials, *Proc. Amer. Math. Soc.* **99** (1987), 598-599.
2. M. S. Ashbaugh and R. D. Benguria, Optimal bounds for ratios of eigenvalues of one-dimensional Schrödinger operators with Dirichlet boundary conditions and positive potentials, *Comm. Math. Phys.* **124** (1989), 403-415.
3. M. S. Ashbaugh and R. D. Benguria, Eigenvalue ratios for Sturm-Liouville operators, *J. Diff. Eqns.* **103** (1993), 205-219.
4. G. Birkhoff and G. C. Rota, *Ordinary Differential Equations*, 4th ed. (1989), Wiley, New York.
5. M. J. Huang, On the eigenvalue ratios for vibrating strings, *Proc. Amer. Math. Soc.* **127** (1999), 1805-1813.
6. Y. L. Huang and C. K. Law, Eigenvalue ratios for the regular Sturm-Liouville system, *Proc. Amer. Math. Soc.* **124** (1996), 1427-1436.
7. Q. Kong and A. Zettl, Eigenvalues of regular Sturm-Liouville problems, *J. Diff. Eqns.* **131** (1996), 1-19.
8. R. Lavine, The eigenvalue gap for one-dimensional convex potentials, *Proc. Amer. Math. Soc.* **121** (1994), 815-821.

Chung-Chuan Chen
Department of Applied Mathematics,
National Sun Yat-Sen University,
Kaohsiung, Taiwan 804, R.O.C.

C. K. Law
Department of Applied Mathematics,
National Sun Yat-Sen University,
Kaohsiung, Taiwan 804, R.O.C.,
and
National Center for Theoretical Sciences,
Mathematics Division,
Taiwan, R.O.C.

F. Y. Sing
Department of Applied Mathematics,
Hong Kong Polytechnic University,
Hung Hom, Hong Kong.