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ON LOCAL STABLE REDUCTION OF SINGULARITY $(y^a - x^b)(y^p - x^q)$

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Abstract. We consider a local stable reduction of a family of curves with smooth fibers except a central fiber that has a singularity like $(y^a - x^b)(y^p - x^q)$.

1. Introduction

We consider one-dimensional family of curves with smooth fibers except a central fiber. By the local stable reduction theorem, after suitable blow-ups, base changes and contractions of some rational curves we obtain a family extending the original family such that the new central fiber is a stable curve. The local stable reduction process(see [1] or [2]) can be divided in two parts; one is an embedded resolution of the singularity of the central fiber and the other is to make the curve obtained in the first part reduced via base changes and following normalizations. Both are well known and not hard to work out. The singularity of type $y^p - x^q$ is called a toric singularity and the above question for a toric singularity has been studied in [3] and [4].

In this short paper we give a simple description for both parts when the central fiber C_0 has only one singularity locally given by $(y^a - x^b)(y^p - x^q)$, a singularity that is locally given as a union of two toric singularities. For the resolution part we may assume that $C_0 \subset S_0 = \mathrm{Spec}\mathbb{C}[[x, y]]$. Let $P_0 = (0, 0)$ and call X_0, Y_0 the branches of C_0 given by $y^p - x^q$ and $y^a - x^b$ respectively. If $f : S \to S_0$ is a minimal embedded resolution of C_0 and $\mathcal{E}$ the divisor of exceptional curves of f, then we have the following. For the precise definition of a minimal embedded resolution, see [3].

Proposition 1. *$\mathcal{E}$ forms a chain of exceptional curves if* $\dfrac{q}{p} \neq \dfrac{b}{a}$.

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Proposition 2. *Let E and F be distinct components of $\mathcal{E}$ that meet the proper transform of a branch $y^p - x^q$ and the proper transform of a branch $y^a - x^b$ respectively and let E_1 be the exceptional curve in $\mathcal{E}$ we get from the first blow up. Then the greatest common divisors of the multiplicities of any two adjacent components of $\mathcal{E}$ are as follows*:

(*Type A*) *suppose a subchain from E_1 through E does not contain F, then the greatest common divisor of the multiplicities of any two adjacent exceptional curves between E_1 and E is $p+a$, the greatest common divisor of the multiplicities of any two adjacent exceptional curves between E and F is (q,a), and the greatest common divisor of the multiplicities of any two adjacent exceptional curves between F and the other end is $q+b$*;

(*Type B*) *suppose a subchain from E_1 through E contains F, then the greatest common divisor of the multiplicities of any two adjacent exceptional curves between E_1 and F is $p+a$, the greatest common divisor of the multiplicities of any two adjacent exceptional curves between F and E is (p,b), and the greatest common divisor of the multiplicities of any two adjacent exceptional curves between E and the other end is $q+b$.*

Theorem. *Let $\pi : S_0 \to \Delta^*$ be a flat family of smooth projective curves of genus $g \geq 2$ over a punctured open disk Δ^* degenerating to an irreducible curve $C_0 \subset S_0$ with only one singular point P topologically equivalent to $(y^a - x^b)(y^p - x^q)$ with $2 \leq p \leq q$. Suppose that S_0 is smooth. Then this family can be extended via stable reduction theorem to a flat family $\tilde{\pi} : \tilde{S} \to \Delta$, new central fiber of which is a stable curve consisting of three components*: *the normalization C of C_0, $\bar{E}$ and $\bar{F}$ of genus*, *respectively*,

$$\begin{cases} g(\bar{E}) = \dfrac{1}{2}\{pq + aq - (p,q) - p - a - (q,a)\} + 1 \\ g(\bar{F}) = \dfrac{1}{2}\{aq + ab - (a,b) - q - b - (a,q)\} + 1 \end{cases} \quad \text{for type } A$$

$$\begin{cases} g(\bar{E}) = \dfrac{1}{2}\{pq + pb - (p,q) - q - b - (p,b)\} + 1 \\ g(\bar{F}) = \dfrac{1}{2}\{ab + bp - (a,b) - p - a - (p,b)\} + 1 \end{cases} \quad \text{for type } B$$

$$g(C) = p_a(C_0) - \delta(P_0)$$

where $g(C)$ is a genus of C, $p_a(C_0)$ is an arithmetic genus of C_0. Here, C meets $\bar{E}$ and $\bar{F}$ respectively at (p,q) and (a,b) points, and $\bar{E}$ and $\bar{F}$ meet at (q,a) points.

Using a genus formula of a connected nodal curve, one also gets

$$\delta(P_0) = \begin{cases} \frac{1}{2}\{ab - a - b + (a,b)\} + \frac{1}{2}\{pq - p - q + (p,q)\} + aq & \text{for type } A \\ \frac{1}{2}\{ab - a - b + (a,b)\} + \frac{1}{2}\{pq - p - q + (p,q)\} + bp & \text{for type } B \end{cases}$$

Note that $\delta(y^p - x^q) = \frac{1}{2}\{pq - p - q + (p,q)\}$ and that, when P_0 is a singular point of a curve D in a surface, that $\delta(P_0)$ can be computed as $\sum \frac{1}{2} m_Q(m_Q - 1)$ taken over all infinitely near singular points Q lying over P_0 including P_0 where m_Q is a multiplicity at Q of some subsequent partial normalization of D. See [4] for example.

2. Euclidean Algorithm

We introduce Euclidean algorithm of two pairs of integers p, q and a, b, respectively:

(1) $$s_{-1} = q,\ s_0 = p,\ s_{i-1} = s_i r_{i+1} + s_{i+1},\\ 0 \le s_{i+1} < s_i,\ s_{k+1} = 0 \text{ for } 0 \le i \le k;$$

(2) $$d_{-1} = b,\ d_0 = a,\ d_{i-1} = d_i c_{i+1} + d_{i+1},\\ 0 \le d_{i+1} < d_i,\ d_{h+1} = 0 \text{ for } 0 \le i \le h.$$

Here $s_k = (p,q),\ d_h = (a,b)$, where (a,b) denotes the greatest common divisor of two integers a, b. Note $c_1 = 0$ and $d_1 = b$ if $a > b$. Define four sequences $\{p_i\},\ \{q_i\},\ \{a_i\},\ \{b_i\}$ of integers as follows:

(3) $$p_{-1} = 0,\ p_0 = 1, \cdots, p_i = p_{i-2} + p_{i-1} r_i \text{ for } 1 \le i \le k+1;$$

(4) $$q_{-1} = 1,\ q_0 = 0, \cdots, q_i = q_{i-2} + q_{i-1} r_i \text{ for } 1 \le i \le k+1;$$

(5) $$a_{-1} = 0,\ a_0 = 1, \cdots, a_i = a_{i-2} + a_{i-1} c_i \text{ for } 1 \le i \le h+1;$$

(6) $$b_{-1} = 1,\ b_0 = 0, \cdots, b_i = b_{i-2} + b_{i-1} c_i \text{ for } 1 \le i \le h+1.$$

Then as in [4],

(7) $$s_i = (-1)^i (p p_i - q q_i) \text{ for } -1 \le i \le k+1$$

(8) $$d_i = (-1)^i (a a_i - b b_i) \text{ for } -1 \le i \le h+1$$

$$(p_i, p_{i+1}) = (q_i, q_{i+1}) = (a_i, a_{i+1}) = (b_i, b_{i+1}) = 1. \tag{9}$$

Define m to be the largest integer between 0 and h such that

$$c_i = r_i \qquad \text{for all} \quad i \leq m. \tag{10}$$

Then $a_i = p_i$ and $b_i = q_i$ for all $i \leq m$. Note $m = 0$ if $c_1 \neq r_1$.

We now briefly review the case of a toric singularity in [4].

Let $X_0 \subset S_0 = \mathrm{Spec}\mathbb{C}[[x, y]]$ be given by $y^p - x^q (= 0)$ and let $f_i : S_i \to S_{i-1}$ the i-th blow up from S_0 at $P_0 = (0, 0)$ where E_i an exceptional curve of f_i. Let X_i the proper transform of X_0 in S_i and $P_i = X_i \cap E_i$. Write $(i) = \sum_{l=1}^{i} r_l$. Then we have

Lemma 1. [4, Lemma 2.4] *$X_{(i)+l}$ is tangent to $E_{(i)}$ for $0 \leq l < r_{i+1}$ and on the minimal resolution $S_{(k+1)}$ of $X_0 \subset S_0$ the proper transform X of X_0 meets only $E_{(k+1)}$ at (p, q) points transversely. Moreover all exceptional curves $\{E_l | 1 \leq l \leq (k+1) = \sum_{j=1}^{k+1} r_j\}$ on $S_{(k+1)}$ form a chain with E_1 and E_{r_1+1} as its two end components. The exceptional curves $E_{(i)+l}$ where i is even and $1 \leq l \leq r_{i+1}$ lie between E_1 and $E_{(k+1)}$, while $E_{(i)+l}$ where i is odd and $1 \leq l \leq r_{i+1}$ lie between E_{r_1+1} and $E_{(k+1)}$ in increasing order of subindex. The multiplicity of each component is as follows*:

$$m_{P_j}(X_j) = s_i \quad \text{for} \quad \sum_{l=1}^{i} r_l \leq j < \sum_{l=1}^{i+1} r_l \tag{11}$$

$$\begin{aligned} m(E_{(i)+l}) &= l\{m(E_{(i)}) + s_i\} + m(E_{(i-1)}) \text{ for } 1 \leq l \leq r_{i+1} \\ &= \begin{cases} lpp_i + pp_{i-1} & \text{for even } i \\ lqq_i + qq_{i-1} & \text{for odd } i \end{cases} \end{aligned} \tag{12}$$

$$m(E_{(i)}) = \begin{cases} pp_i & \text{for odd } i \\ qq_i & \text{for even } i\ . \end{cases} \tag{13}$$

For the proof of proposition 1, we state the following lemma 2, the proof of which is clear.

Lemma 2. *Suppose that $f : X' \to X$ is a birational morphism of smooth surfaces that is a blow-up at a point P lying on a chain $\mathcal{F}$ of rational curves in X. Then $f^*(\mathcal{F})$ forms a chain if and only if either P is an intersection point of two components of $\mathcal{F}$ or P lies on the end component of $\mathcal{E}$.*

3. Proofs of Three Statements in Section 1

For the resolution part we assume that $C_0 \subset S_0 = \mathrm{Spec}\mathbb{C}[[x,y]]$ is a union of two curves $y^p - x^q$ and $y^a - x^b$. We let $P_0 = (0,0)$ and X_0, Y_0 the branches of C_0 given by $y^p - x^q$ and $y^a - x^b$ respectively. We now blow up S_0 at P_0 until we resolve one component X_0 of C_0 completely as in Lemma 1. Let $f_i : S_i \to S_{i-1}$ be the blow-up of S_{i-1} at P_{i-1}, E_i the exceptional divisor of f_i, X_i the proper transforms of X_{i-1} respectively, $P_i = X_i \cap E_i$, $\tilde{f}_i = f_i \circ f_{i-1} \circ \cdots \circ f_1$, and $\mathcal{E}_i$ the union of exceptional curves of $\tilde{f}_i$. Suppose the proper transforms of X_0 and Y_0 separate first time on S_{i_0+1} (See the proof of Proposition 1 and equation (14)). Then for $1 \le i \le i_0 + 1$, we let Y_i be the proper transform of Y_{i-1} under f_i and $C_i = X_i + Y_i$. For, $i > i_0+1$, Y_{i_0+1} remains unchanged under f_i. Thus we still call Y_{i_0+1} for the inverse image of Y_{i_0+1} under f_i for $i > i_0 + 1$. Then the exceptional curves $\mathcal{E}_{(k+1)}$ in $S_{(k+1)}$ form a chain by Lemma 1.

Remark. Consider the case that is excluded in Proposition 1: $h = k, c_i = r_i$ for all $i \le k+1$. Then $S_{(k+1)}$ is a minimal resolution for both X_0 and Y_0 meeting each of their transforms only one exceptional curve $E_{(k+1)}$. Therefore $S_{(k+1)}$ becomes a minimal resolution of C_0 if and only if $X_{(k+1)}$ and $Y_{(k+1)}$ do not meet along $E_{(k+1)}$. If this happens $(y^a - x^b)(y^p - x^q)$ is analytically (and topologically) equivalent to $y^{a+p} - x^{b+q}$ and we have

$$b + q = (a+p)r_1 + d_1 + s_1,\ d_{i-1} + s_{i-1} = (d_i + s_i)r_{i+1} + (d_{i+1} + s_{i+1}).$$

If they meet, it produces triple points which are the intersection of $X_{(k+1)}, Y_{(k+1)}$ and $E_{(k+1)}$. Therefore we need more blow-ups at these triple points to get a minimal resolution and due to Lemma 2 we cannot have a chain of exceptional curves on a minimal resolution.

Proof of Proposition 1. If $m = h+1$, then Y_0 is resolved completely on $S_{(h+1)}$. So, the minimal resolution space S of C_0 is $S_{(k+1)}$ and $\mathcal{E} = \mathcal{E}_{(k+1)}$ forms a chain. Now assume $m \le h$ and $c_{m+1} < r_{m+1}$, otherwise we exchange the roles of X_0 and Y_0. Since $r_i = c_i$ for $i \le m$ and $c_{m+1} < r_{m+1}$, X_l and Y_l are tangential to the same exceptional curve by Lemma 1 for

$$l < \sum_{i=1}^{m+1} c_i = \sum_{i=1}^{m} r_i + c_{m+1} = i_0. \tag{14}$$

On S_{i_0}, X_{i_0} and Y_{i_0} meet at P_{i_0} while X_{i_0} is tangent to $E_{(m)}$ and Y_{i_0} is tangent to E_{i_0}. Therefore, Y_{i_0+1} becomes separated from X_{i_0+1} but still passes the intersection point of E_{i_0+1} and E_{i_0}. Now the minimal embedded resolution S of C_0 is that of $Y_{i_0+1} \subset S_{(k+1)}$ and $\mathcal{E}$ is a union of (proper transforms if necessary) of E_i and F_j

where $1 \leq i \leq (k+1)$ and $[m+1]+1 \leq j \leq [h+1]$. Therefore $\mathcal{E}$ is a chain due to Lemma 1 and 2.

Proof of Proposition 2. Call F_i the exceptional curves we get from the resolution of Y_{i_0} (so, of Y_0). Note that the subindex in F_i is the number of blow-ups when we resolve the singularity of Y_0. Since $C_j = X_j + Y_j$ and $m_{P_j}(C_j) = m_{P_j}(X_j) + m_{P_j}(Y_j)$ for $1 \leq j \leq i_0 = (m) + c_{m+1}$, we have, due to Lemma 1,

$$m(E_{(i)}) = \begin{cases} pp_i + aa_i & \text{for odd } i \leq m \\ qq_i + bb_i & \text{for even } i \leq m \end{cases} \tag{15}$$

$$\begin{aligned} m(E_{(i)+l}) &= l\{m(E_{(i)}) + s_i + d_i\} + m(E_{(i-1)}) \text{ if } i \leq m \\ &\quad \text{and } 1 \leq l \leq c_{i+1} \\ m(E_{(m)+l}) &= l\{m(E_{(m)}) + s_m\} + m(E_{(m-1)}) + c_{m+1}d_m \\ &\quad + d_{m+1} \text{ if } c_{m+1} < l \leq r_{m+1}. \end{aligned} \tag{16}$$

Thus, we have

$$\begin{aligned} m(E_{(m)+1}) &= r_{m+1}\{m(E_{(m)}) + s_m\} + m(E_{(m-1)}) + c_{m+1}d_m + d_{m+1} \\ &= \begin{cases} pp_{m+1} + bq_{m+1} & \text{if } m \text{ is even} \\ qq_{m+1} + ap_{m+1} & \text{if } m \text{ is odd} \end{cases} \end{aligned}$$

from (1)-(8), (12), (15) and (16). Note that

$$i \geq m+1 \implies m(E_{(i)+l}) = l\{m(E_{(i)}) + s_i\} + m(E_{(i-1)}), \quad \text{for } 1 \leq l \leq r_{i+1}.$$

So, we have

$$m \text{ is even and } i \geq m+1 \implies m(E_{(i)}) = \begin{cases} pp_i + bq_i & \text{for odd } i \\ qq_i + bq_i & \text{for even } i \end{cases};$$

$$m \text{ is odd and } i \geq m+1 \implies m(E_{(i)}) = \begin{cases} pp_i + ap_i & \text{for odd } i \\ qq_i + ap_i & \text{for even } i \end{cases}.$$

Let $[i] = \sum_{l=1}^{i} c_l$. Then $E_{(m)+c_{m+1}} = F_{[m+1]}$. By (16) and (3)-(9),

$$m(F_{[m+1]}) = \begin{cases} aa_{m+1} + pa_{m+1} & \text{if } m \text{ is even} \\ bb_{m+1} + qb_{m+1} & \text{if } m \text{ is odd}. \end{cases}$$

Since

$$m(F_{[m+1]+1}) = m(F_{[m+1]}) + s_m + d_{m+1} + m(E_{(m)})$$

$$m(F_{[m+1]+l}) = l(m(F_{[m+1]}) + d_{m+1}) + m(E_{(m)}) + s_m \text{ for } 1 \leq l \leq c_{m+2}$$

$$m(F_{[i]+l}) = l(m(F_{[i]}) + d_i) + m(F_{[i-1]}) \text{ for } i > m+1 \text{ and } 1 \leq l \leq c_{i+1},$$

we have

$$m \text{ is even and } i \geq m+1 \implies m(F_{[i]}) = \begin{cases} aa_i + pa_i & \text{for odd } i \\ bb_i + pa_i & \text{for even } i; \end{cases}$$

$$m \text{ is odd and } i \geq m+1 \implies m(F_{[i]}) = \begin{cases} aa_i + qb_i & \text{for odd } i \\ bb_i + qb_i & \text{for even } i. \end{cases}$$

For the remaining part of proof, we assume that m is odd. Recall from the proof of Proposition 1, Y_{i_0+1} becomes separated from X_{i_0+1} but still passes the intersection point of E_{i_0+1} and E_{i_0} where $i_0 = (m)+c_{m+1}$. Since m is odd, E_{i_0} and E_{i_0+1} lies between E_{r_1+1} and $E_{(k+1)}$. Therefore all exceptional curves E_j between E_1 and $E_{(k+1)}$ will be remained untouched when we resolve Y_{i_0+1} and $F_{[h+1]}$ which meets the normalization of Y_0 at (a, b) distinct points will lie between E_{i_0} and E_{i_0+1}. So, if m is odd, we get a type A. Similarly, we get a type B if m is even.

We divide the chain as a sum of subchains $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$, $\mathcal{G}_5$, where

$$\mathcal{G}_1 = \{E_{(i)+l} \mid i : \text{even} ,\ 1 \leq l \leq r_{i+1}\}$$

$$\mathcal{G}_2 = \{E_{(i)+l} \mid i : \text{odd} ,\ (i)+l \geq (m)+c_{m+1}+1,\ 1 \leq l \leq r_{i+1}\}$$

$$\mathcal{G}_3 = \{F_{[i]+l} \mid i : \text{even},\ 1 \leq l \leq c_{i+1}\}$$

$$\mathcal{G}_4 = \{F_{[i]+l} \mid i : \text{odd},\ 1 \leq l \leq c_{i+1}\}$$

$$\mathcal{G}_5 = \{E_{(i)+l} \mid i : \text{odd},\ (i)+l \leq (m)+c_{m+1},\ 1 \leq l \leq r_{i+1}\}$$

Note that $\mathcal{G}_1$ and $\mathcal{G}_2$ is connected by $E_{(k+1)}$ and that $\mathcal{G}_3$ and $\mathcal{G}_4$ by $F_{[h+1]}$. When we compute the greatest common divisors of two adjacent exceptional curves, we use the easy fact

$$(a+bc, b) = (a, b), \quad \text{for every integer } a, b, c.$$

For two adjacent exceptional curves in $\mathcal{G}_1$ with $i \leq m$,

$$\begin{aligned}
(m(E_{(i)+l}), m(E_{(i)+l+1})) &= (m(E_{(i-1)}), m(E_{(i)}) + s_i + d_i) \\
&= (pp_{i-1} + aa_{i-1}, pp_i + aa_i) \\
&= (p+a)(p_{i-1}, p_i) \\
&= p+a; \\
(m(E_{(i)+l}), m(E_{(i)+l+1})) &= (m(E_{(i-1)}), m(E_{(i)}) + s_i) \\
&= (pp_{i-1} + ap_{i-1}, pp_i + ap_i) \\
&= (p+a)(p_i, p_{i-1}) \\
&= p+a.
\end{aligned}$$

For two adjacent components of $\mathcal{G}_2$,

$$
\begin{aligned}
&(m(E_{(i)+l}), m(E_{(i)+l+1})) = (m(E_{(i-1)}), m(E_{(i)}) + s_i) \\
&= (qq_{i-1} + ap_{i-1}, qq_i + ap_i) = (qq_{i-1} + ap_{i-1}, qq_{i-2} + ap_{i-2}) \\
&= \cdots = (qq_0 + ap_0, qq_{-1} + ap_{-1}) = (a, q); (m(E_{(m)+l}), m(E_{(m)+l+1})) = (a, q).
\end{aligned}
$$

Similarly, one can show

$$
\begin{aligned}
(m(F_l), m(F_{l+1})) &= (q, a) \quad \text{if} \quad F_l, F_{l+1} \in \mathcal{G}_3 \\
(m(F_l), m(F_{l+1})) &= q + b \quad \text{if} \quad F_l, F_{l+1} \in \mathcal{G}_4 \\
(m(E_l), m(E_{l+1})) &= q + b \quad \text{if} \quad E_l, E_{l+1} \in \mathcal{G}_5
\end{aligned}
$$

by (3)-(9). Also, one has to compute $(m(E_{(i)}, m(E_{(i+1)+1}))$ whenever they meet. Finally, we know, by Lemma 1, that the proper transforms of X_0 and Y_0 meet only $E_{(k+1)}$ and $F_{[h+1]}$ respectively. Put

$$E = E_{(k+1)}, \quad F = F_{[h+1]}.$$

Proof of Theorem. We explain only type A. Let $\pi' : S \to S_0 \to \Delta$ be a composition of π and all blow-ups we have taken for the resolution of C_0. Note that in this theorem C_0 is connected and $\pi^{-1}(0) = C + \sum_{E \in \mathcal{E}} m(E)E$. Note that C is the proper transform of C_0 which is the normalization of C_0. Since a new central fiber is not reduced, we take a base change of order of the least common multiple of all components of $\mathcal{E}$ and a normalization S' to make the central fiber reduced. Note that as far as the multiplicities of two adjacent components are relatively prime the intersection points are always ramified. Because of this reason, S' is ramified over C and, on each component G of $\mathcal{E}$, only the base change of degree $m(G)$ will make something happen to G. Therefore, except the components E and F, we have $p+a$ copies of $\mathcal{E}_1$, (q, a) copies of $\mathcal{G}_2$ and $\mathcal{G}_3$, and $q+b$ copies of $\mathcal{G}_4$ and $\mathcal{G}_5$; while each copy of $\mathcal{E}_1$ meets the curve $\bar{E}$ over E, one end of each copy of $\mathcal{G}_2$ meets $\bar{E}$, one end of each copy of $\mathcal{G}_3$ meets $\bar{F}$ over F, and one end of each copy of $\mathcal{G}_4$ meets $\bar{F}$. We always use Hurwitz formula to compute $g(X)$ of a finite morphism $f : X \to Y$ of curves. Remember that X is rational if $f : X \to \mathbb{P}^1$ is completely branched at two points.

Since C is ramified under this base change, we have $\bar{E} \to E$ is a degree $q'(p+a)$ morphism totally ramified at s_k points, evenly $(p+a)$-ramified at one point, evenly (q, a)-ramified at one point; $\bar{F} \to F$ is a degree $a'(q+b)$ morphism totally ramified at d_h points, evenly $(q+b)$-ramified at one point, evenly (q, a)-ramified at one point

Therefore, by Hurwitz formula, we have

$$g(\bar{E}) = \frac{1}{2}\{pq + aq - s_k - p - a - (q,a)\} + 1$$
$$g(\bar{F}) = \frac{1}{2}\{ab + aq - d_k - q - b - (q,a)\} + 1$$

For the formula of $\delta(P_0)$, we recall the genus formula ([2], p.48) of a connected nodal curve which extends the arithmetic genus of an irreducible nodal curve: if D has δ nodes and ν irreducible components $D_1, D_2, \cdots, D_\nu$ of geometric genera $g_1, g_2, \cdots, g_\nu$, then

$$g(D) = \Big(\sum_{i=1}^{\nu} g_i\Big) + \delta - \nu + 1.$$

Since genera of all fibers of a flat family of curves are constant, we have

$$g = g(\tilde{\pi}^{-1}(0)) = g(C) + g(\bar{E}) + g(\bar{F}) + (p,q) + (a,b) + (q,a) - 3 + 1$$

Since $g(C) = p_a(C_0) - \delta(P_0) = g - \delta(P_0)$, we have

$$\begin{aligned}\delta(P_0) &= g(\bar{E}) + g(\bar{F}) + (p,q) + (a,b) + (q,a) - 2\\ &= \frac{1}{2}\{ab - a - b + (a,b)\} + \frac{1}{2}\{pq - p - q + (p,q)\} + aq\end{aligned}$$

Contracting all copies of rational exceptional curves from S', we get $\tilde{\pi} : \tilde{S} \to \Delta$ in Theorem. Note that the contraction of (q,a) copies of $\mathcal{G}_2$ and $\mathcal{G}_3$ makes that $\bar{E}$ and $\bar{F}$ intersect at (q,a) points.

Similarly, we get type B if m is even. If this happens, we have that $\bar{E} \to E$ is a degree $q'(p+a)$ morphism totally ramified at s_k points, evenly $(q+b)$-ramified at one point, evenly (p,b)-ramified at one point; $\bar{F} \to F$ is a degree $a'(q+b)$ morphism totally ramified at d_h points, evenly $(p+a)$-ramified at one point, evenly (p,b)-ramified at one point. So, Hurwitz gives the answer.

Remark. As in [4], one can describe tails $\bar{E}$ and $\bar{F}$ as plane curves from the information of ramifications. If we have a singular point that is a union of several toric singularities, we get the similar formulas according when each component becomes separated from the others.

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