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NOTES ON SINGULAR INTEGRALS ON SOME INHOMOGENEOUS HERZ SPACES

Yasuo Komori

Abstract. We consider the singular integral operators which are more singular than Calderón-Zygmund operator and include pseudo-differential operators. We obtain the boundedness of these operators on inhomogeneous Herz spaces and Herz-type Hardy spaces.

1. Introduction

Alvarez, Guzmán-Partida and Lakey [1] generalized the Calderón-Zygmund singular integrals and introduced (q, λ)-central singular integrals which are more singular than Calderón-Zygmund operators and include pseudo-differential operators (see section 2). They obtained the boundedness of these operators on some Herz spaces $B^{q,\lambda}$. In this paper we refine their results and correct some mistakes. Furthermore we study the boundedness of these operators on Herz-type Hardy spaces. In section 4, we shall define another Herz-type Hardy space and consider the estimate of another type.

2. Definitions and Notations

The following notation is used: For a set $E \subset \mathbb{R}^n$ we denote the Lebesgue measure of E by $|E|$. We denote a characteristic function of E by χ_E. We write a ball of radius R centered at x by $B(x, R) = \{y; |x-y| < R\}$ and write $C_j(0, R) = B(0, 2^{j+1}R) \setminus B(0, 2^j R)$.

Following [1] and [5], we define some function spaces which we shall consider in section 3.

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Definition 1 $(B^{q,\lambda})$**.** Let $\lambda \in \mathbb{R}^1$ and $1 < q < \infty$.

$$B^{q,\lambda}(\mathbb{R}^n) = \{f; \|f\|_{B^{q,\lambda}} < \infty\},$$

where

$$\|f\|_{B^{q,\lambda}} = \sup_{R \geq 1} \left(\frac{1}{|B(0,R)|^{1+\lambda q}} \int_{B(0,R)} |f(x)|^q dx \right)^{1/q}.$$

Remark. If $\lambda < -1/q$ then $B^{q,\lambda} = \{0\}$, and $B^{q,-1/q} = L^q$.

Definition 2 $(CMO^{q,\lambda})$**.** Let $\lambda < 1/n$ and $1 < q < \infty$.

$$CMO^{q,\lambda}(\mathbb{R}^n) = \{f; \|f\|_{CMO^{q,\lambda}} < \infty\},$$

where

$$\|f\|_{CMO^{q,\lambda}} = \sup_{R \geq 1} \left(\frac{1}{|B(0,R)|^{1+\lambda q}} \int_{B(0,R)} |f(x) - f_R|^q dx \right)^{1/q},$$

and $f_R = \frac{1}{|B(0,R)|} \int_{B(0,R)} f(x)dx$.

Remark. If $\lambda < -1/q$ then the space $CMO^{q,\lambda}$ reduces to the constant functions, and if $\lambda = -1/q$ then $CMO^{q,\lambda}$ coincides with L^q modulo constants (see [1, p. 5]).

Remark. By using the notation of Herz space, we can write $B^{q,\lambda} = K_q^{-1/q-\lambda,\infty}$ (see [7]). When $-1/q < \lambda < 0$, we can consider $B^{q,\lambda}$ as a local version of inhomogeneous Morrey space (see [1]).

Next we define singular integrals.

Definition 3. Let $T : \mathcal{D} \to \mathcal{D}'$ be a linear continuous operator. We say T is a Calderón-Zygmund operator if T satisfies the following:

(0) If $f, g \in \mathcal{D}$ and $\text{supp}(f) \cap \text{supp}(g) = \emptyset$, T has the integral representation
$$(Tf, g) = \int \int K(x,y) f(y) g(x) dy dx.$$

(1) $|K(x,y)| \leq \dfrac{C}{|x-y|^n}$.

(2) $|\nabla_x K(x,y)| + |\nabla_y K(x,y)| \leq \dfrac{C}{|x-y|^{n+1}}$.

(3) T is bounded on $L^2(\mathbb{R}^n)$.

Following [1], we generalize Calderón-Zygmund operator.

Definition 4. Let $T : \mathcal{D} \to \mathcal{D}'$ be a linear continuous operator and we assume T has the same integral representation (0) as above. We say T is a (q, λ, N)-central singular integral if T satisfies the following:

(1') $|K(0, y)| \le \dfrac{C}{|y|^N}$ where $|y| \ge 1$.

(2') $\sup_{R\ge 1} \sup_{|x|<R} \left(|C_j(0, R)|^{q'-1} \int_{C_j(0,R)} |K(x, y) - K(0, y)|^{q'} dy \right)^{1/q'} \le d_j$

with $\sum_{j=1}^{\infty} 2^{jn\lambda} d_j < \infty$ where $1/q + 1/q' = 1$.

(3') T is bounded on $L^q(\mathbb{R}^n)$.

Remark. Compare with Def. 5.1 in [1]. We add the condition (1').

Examples 1. 1. Calderón-Zygmund operator is a (q, λ, n)-central singular integral for $1 < q < \infty$ and $\lambda < 1/n$.

2. Weakly-strongly singular integral operator of Fefferman [3], [4]

$$T_{\alpha,\beta} f(x) = \frac{e^{i|x|^{-\alpha}}}{|x|^{n+\beta}} * f(x), \quad \alpha, \beta > 0$$

is a $(q, \lambda, n + \beta)$-central singular integral for some $1 < q < \infty$ and $\lambda < \alpha q'/n$.

3. Pseudo-differential operator [1]

$$Tf(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} p(x, \xi) \hat{f}(\xi) d\xi$$

where the symbol $p(x, \xi)$ belongs to the class $S^m_{\rho,\delta}$, is a (q, λ, N)-central singular integral for some $1 < q < \infty, \lambda$ and N.

3. Singular Integrals on $B^{q,\lambda}$

Alvarez, Guzmán-Partida and Lakey [1] proved the next theorem.

Theorem A. *Let* $1 < q < \infty$. *If* T *satisfies (2') and (3') in Def.* 4, *then* T *is bounded from* $B^{q,\lambda}(\mathbb{R}^n)$ *to* $CMO^{q,\lambda}(\mathbb{R}^n)$.

Our results are the following:

Proposition 1. *Let T be a (q, λ, N)-central singular integral where $1 < q < \infty$, $N \geq n$ and $N > n(1+\lambda)$. Then T is bounded on $B^{q,\lambda}(\mathbb{R}^n)$.*

Remark. Let $N = n$ and $\lambda = -1/q$ in Prop. 1. Then the proposition says that a Calderón-Zygmund operator is bounded on L^q.

Definition 5 (The commutator of Coifman, Rochbeg and Weiss)**.** We define the commutator operator $[b, T]$ by

$$[b, T]f = b \cdot Tf - T(bf).$$

Remark. If b is in BMO (John-Nirenberg space, see [11]) and T is a Calderón-Zygmund operator, then $[b, T]$ is bounded on L^q where $1 < q < \infty$ (see [2]).

Proposition 2. *Let $1 < p < q < \infty,\ 1/s = 1/p - 1/q,\ 0 < \mu < 1/n,\ N \geq n$ and $N > n(1+\lambda)$. If b is in $CMO^{s,\mu}(\mathbb{R}^n)$ and T is a (p, λ, N)-central singular integral, and we assume T is bounded on $L^q(\mathbb{R}^n)$, then $[b, T]$ is bounded from $B^{q,\lambda-\mu}(\mathbb{R}^n)$ to $B^{p,\lambda}(\mathbb{R}^n)$ and*

$$\|[b, T]f\|_{B^{p,\lambda}} \leq C\, \|b\|_{CMO^{s,\mu}} \|f\|_{B^{q,\lambda-\mu}}$$

where C is a positive constant which is independent of f.

Remark. Prop. 5.4 in [1] is incorrect. The estimate of I_1 on p. 36 is wrong.

The proof of two propositions are essentially same as in [1], so we show only outline of the proofs and point out the differences.

Proof of Proposition 1. Let $R \geq 1$. We write

$$f(x) = f(x)\chi_{B(0,2R)} + f(x)(1 - \chi_{B(0,2R)}) = f_1(x) + f_2(x),$$

and write

$$\begin{aligned}&\left(\frac{1}{|B(0,R)|^{1+\lambda q}} \int_{B(0,R)} |Tf(x)|^q dx\right)^{1/q} \leq \left(\frac{1}{|B(0,R)|^{1+\lambda q}} \int_{B(0,R)} |Tf_1(x)|^q dx\right)^{1/q} \\ &+ \left(\frac{1}{|B(0,R)|^{1+\lambda q}} \int_{B(0,R)} |Tf_2(x) - Tf_2(0)|^q dx\right)^{1/q} + \frac{|Tf_2(0)|}{|B(0,R)|^{\lambda}} \\ &= I_1 + I_2 + I_3.\end{aligned}$$

As in the proof of [1, p. 34], we have

$$I_1 + I_2 \leq C\|f\|_{B^{q,\lambda}}.$$

We estimate I_3. By using the condition (1') and $N > n(1+\lambda)$, we have

$$|Tf_2(0)| \le CR^{n(1+\lambda)-N}\|f\|_{B^{q,\lambda}}.$$

So we obtain $I_3 \le C\|f\|_{B^{q,\lambda}}$, because $N \ge n$ and $R \ge 1$.

Proof of Proposition 2. Let $R \ge 1$. We write

$$f(x) = f(x)\chi_{B(0,2R)} + f(x)(1-\chi_{B(0,2R)}) = f_1(x) + f_2(x),$$

and

$$[b,T]f(x) = (b(x)-b_{2R})Tf(x) - T((b-b_{2R})f_1)(x) - T((b-b_{2R})f_2)(x),$$

where $b_{2R} = \frac{1}{|B(0,2R)|}\int_{B(0,2R)} b(x)dx$.

Thus

$$\begin{aligned}
&\left(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)} |[b,T]f(x)|^p dx\right)^{1/p}\\
&\le \left(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)} |(b(x)-b_{2R})Tf(x)|^p dx\right)^{1/p}\\
&+ \left(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)} |T((b-b_{2R})f_1)(x)|^p dx\right)^{1/p}\\
&+ \left(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)} |T((b-b_{2R})f_2)(x) - T((b-b_{2R})f_2)(0)|^p dx\right)^{1/p}\\
&+ \frac{|T((b-b_{2R})f_2)(0)|}{|B(0,R)|^{\lambda}} = I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

By the estimates in [1, p. 38], we have

$$I_2 + I_3 \le C\|b\|_{CMO^{s,\mu}}\|f\|_{B^{q,\lambda-\mu}}.$$

By using the condition (1'), $N > n(1+\lambda)$ and $N \ge n$, we have

$$I_4 \le C\|b\|_{CMO^{s,\mu}}\|f\|_{B^{q,\lambda-\mu}}.$$

Finally we estimate I_1. We write

$$\begin{aligned}
I_1 &\le \left(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)} |(b(x)-b_{2R})Tf_1(x)|^p dx\right)^{1/p}\\
&+ \left(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)} |(b(x)-b_{2R})Tf_2(x)|^p dx\right)^{1/p}\\
&= I_{11} + I_{12}.
\end{aligned}$$

Because T is bounded on L^q, we have $I_{11} \le C\|b\|_{CMO^{s,\mu}}\|f\|_{B^{q,\lambda-\mu}}$.
By the condition (1'), we have

$$|Tf_2(x)| \le CR^{n(1+\lambda-\mu)-N}\|f\|_{B^{q,\lambda-\mu}} \quad \text{where} \quad |x| \le R.$$

So we obtain $I_{12} \le C\|b\|_{CMO^{s,\mu}}\|f\|_{B^{q,\lambda-\mu}}$.

4. Singular Integrals on Herz-Type Hardy Space

Now we define some function spaces which are main objects of this paper.

4.1. Herz spaces

First we define the Herz spaces and the Hardy spaces associated with the Herz spaces (see [6, 7, 9] and [10]). Following Lu and Yang [10], we define central atoms and blocks.

Definition 6. Let $0 < p \le 1 < q < \infty$. A function $a(x)$ is a central (p, q)-block if there exists $R \ge 1$ such that the following conditions are satisfied

(i) $$\text{supp}(a) \subset B(0, R),$$

(ii) $$\|a\|_{L^q} \le |B(0, R)|^{1/q-1/p}.$$

Definition 7. Let $0 < p \le 1 < q < \infty$. A function $a(x)$ is a central (p, q)-atom if there exists $R \ge 1$ such that the following conditions are satisfied (i), (ii) and

(iii) $$\int a(x)dx = 0.$$

We define inhomogeneous Herz spaces and Hardy spaces associated with the Herz spaces (see [10]).

Definition 8. Let $n/(n+1) < p \le 1 < q$. We denote, by $K^p_q(\mathbb{R}^n)$, the family of distributions f that, in the sense of distributions, can be written as $f = \sum_{k=1}^{\infty} \lambda_k a_k$, where a_k is a (p, q)-block and $\sum_{k=1}^{\infty} |\lambda_k|^p < \infty$.

We define $\|f\|^p_{K^p_q} = \inf \sum_{k=1}^{\infty} |\lambda_k|^p$, where the infimum is taken over all representations of f.

Definition 9. Let $n/(n+1) < p \le 1 < q < \infty$. We denote, by $HK^p_q(\mathbb{R}^n)$, the family of distributions f that, in the sense of distributions, can be written as $f = \sum_{k=1}^{\infty} \lambda_k a_k$, where a_k is a (p, q)-atom and $\sum_{k=1}^{\infty} |\lambda_k|^p < \infty$.

We define $\|f\|^p_{HK^p_q} = \inf \sum_{k=1}^{\infty} |\lambda_k|^p$, where the infimum is taken over all representations of f.

Remark. In [10], these spaces are written by $K_q^{n(1/p-1/q),p}$ and $HK_q^{n(1/p-1/q),p}$ respectively.

4.2. Singular integrals

Following [1], we introduce new class of singular integral operators.

Definition 10. Let $T : \mathcal{D} \to \mathcal{D}'$ be a linear continuous operator and we assume T has the same integral representation (0) as in Def. 3. We say T is a $(q, \theta)^t$-central singular integral if T satisfies the following:

$$\sup_{R\geq 1} \sup_{|y|<R} R^{n(q-1)} \int_{C_j(0,R)} |K(x, y) - K(x, 0)|^q dx \leq e_j \quad \text{with} \quad \sum_{j=1}^{\infty} 2^{j\theta q} e_j < \infty. \tag{4}$$

$$T \text{ is bounded on } L^q(\mathbb{R}^n). \tag{5}$$

Example. Calderón-Zygmund operator is a $(q, \theta)^t$-central singular integral for $1 < q < \infty$ and $0 < \theta < n(1 - 1/q) + 1$.

Alvarez, Guzmán-Partida and Lakey [1] proved the following:

Theorem B. *Let $n/(n + 1) < p \leq 1 < q < \infty$. If T is a $(q, \theta)^t$-central singular integral where $\theta > n(1/p - 1/q)$, then T is bounded from $HK^p_q(\mathbb{R}^n)$ to $K^p_q(\mathbb{R}^n)$.*

Theorem C. *Let p and q be same as above, and T is a $(q, \theta)^t$-central singular integral where $\theta > n(1/p - 1/q)$. Furthermore we assume $T^t(1) = 0$ where T^t is an adjoint operator of T. Then T is bounded from $HK^p_q(\mathbb{R}^n)$ to $HK^p_q(\mathbb{R}^n)$.*

However the condition $T^t(1) = 0$ is very strong, so we shall consider intermediate spaces between K^p_q and HK^p_q.

Definition 11. Let $0 < p \leq 1 < q < \infty$ and $\varepsilon < 1$. A function $a(x)$ is a central (p, q, ε)-block if there exists $R \geq 1$ such that the following conditions are satisfied (i), (ii) and

$$\Big|\int a(x)dx\Big| \leq |B(0, R)|^{\varepsilon - 1/p}. \tag{iii'}$$

Remark. If a function $a(x)$ is a central (p,q)-block supported on $B(0,R)$, then $\|a\|_{L^1} \leq |B(0,R)|^{1-1/p}$.

Definition 12. Let $n/(n+1) < p \leq 1 < q < \infty$ and $\varepsilon < 1$. We say $f \in K_q^{p,\varepsilon}(\mathbb{R}^n)$ if f can be represented as

$$f = \sum_{k=1}^{\infty} \lambda_k a_k, \text{ where } a_k \text{ is a } (p,q,\varepsilon)\text{-block},$$

and we define $\|f\|_{K_q^{p,\varepsilon}}^p = \inf \sum_{k=1}^{\infty} |\lambda_k|^p$.

Remark. $HK_q^p \subset K_q^{p,\varepsilon}$.

Our result is the following:

Theorem D. *Let $n/(n+1) < p \leq 1 < q < \infty$, $q/(q-1) \leq s$, $\lambda \leq \varepsilon - 1$, and T is a $(q,\theta)^t$-central singular integral where $\theta > n(1/p - 1/q)$. Furthermore we assume that $T^t(1) \in CMO^{s,\lambda}(\mathbb{R}^n)$. Then T is bounded from $HK_q^p(\mathbb{R}^n)$ to $K_q^{p,\varepsilon}(\mathbb{R}^n)$.*

5. Proof of Theorem

5.1. Lemmas

First we define molecules on $K_q^{p,\varepsilon}(\mathbb{R}^n)$.

Definition 13. Let $n/(n+1) < p \leq 1 < q < \infty, \theta > n(1/q - 1/p)$ and $\delta \leq n(\varepsilon - 1/p)$. We say a function $M(x)$ is a (p,q,θ,δ,R)-molecule, if there exists $R \geq 1$ such that the following conditions are satisfied

$$(\mathrm{M}_1) \qquad \left(\int_{|x|\leq 2R} |M(x)|^q dx\right)^{1/q} \leq R^{n(1/q-1/p)},$$

$$(\mathrm{M}_2) \qquad \left(\int_{|x|>2R} |M(x)|^q |x|^{q\theta} dx\right)^{1/q} \leq R^{n(1/q-1/p)+\theta},$$

$$(\mathrm{M}_3) \qquad \Big|\int M(x)dx\Big| \leq R^{\delta}.$$

Remark. For the definition of molecule on HK_q^p, see [10].

Lemma 1. *Let Let* $n/(n+1) < p \leq 1 < q < \infty, \theta > n(1/q - 1/p)$ *and* $\delta \leq n(\varepsilon - 1/p)$. *If a function* $M(x)$ *is a* $(p, q, \theta, \delta, R)$*-molecule, then we have* $\|M\|_{K_q^{p,\varepsilon}} \leq C$, *where* C *is a positive constant which is independent of* R.

Proof. The proof is essentially same as in [8]. So we show only outline of the proof and point out the difference.

Let $E_0 = \{x; |x| < 2R\}$ and $E_k = \{x; 2^k R \leq |x| < 2^{k+1}R\}, k = 1, 2, 3, \ldots,$ and let

$$\chi_k(x) = \chi_{E_k}(x), \quad \tilde{\chi}_k(x) = \frac{1}{|E_k|}\chi_{E_k}(x),$$

$$m_k = \frac{1}{|E_k|}\int_{E_k} M(y)dy, \quad \tilde{m}_k = \int_{E_k} M(y)dy$$

and $M_k(x) = (M(x) - m_k)\chi_k(x)$.

We write

$$M(x) = \sum_{k=0}^{\infty} M_k(x) + \sum_{k=0}^{\infty} m_k\chi_k(x) = \sum_{k=0}^{\infty} M_k(x) + \sum_{k=0}^{\infty} \tilde{m}_k\tilde{\chi}_k(x).$$

Let $N_k = \sum_{j=k}^{\infty} \tilde{m}_j$ and we write

$$\begin{aligned} M(x) &= \sum_{k=0}^{\infty} M_k(x) + \sum_{k=1}^{\infty} N_k(\tilde{\chi}_k(x) - \tilde{\chi}_{k-1}(x)) + N_0\tilde{\chi}_0(x) \\ &= I(x) + II(x) + III(x). \end{aligned}$$

Because $\int M_i(x)dx = \int(\tilde{\chi}_k(x) - \tilde{\chi}_{k-1}(x))dx = 0$, we can show $\|I\|_{HK_q^p} \leq C$ and $\|II\|_{HK_q^p} \leq C$.

By the condition (M_3), we have $|III(x)| \leq CR^{n(\delta-n)}\chi_{\{|x|\leq 2R\}}(x)$ and $\|III\|_{L^q} \leq CR^{n(1/q-1/p)}$. Furthermore we have $|\int III(x)dx| \leq CR^{n(\varepsilon-1/p)}$. So $III(x)$ is a constant multiple of a (p, q, ε)-block and we obtain $\|III\|_{K_q^{p,\varepsilon}} \leq C$. ∎

The following lemma is trivial from the definition.

Lemma 2. *Let* $f \in CMO^{s,\lambda}(R^n)$ *and we assume that supp*$(a) \subset B(0, R)$ *for some* $R \geq 1$ *and* $\int a(x)dx = 0$. *Then we have*

$$\left|\int a(x)f(x)dx\right| \leq |B(0, R)|^{1/s+\lambda}\|a\|_{L^{s'}}\|f\|_{CMO^{s,\lambda}},$$

where $1/s + 1/s' = 1$.

5.2. Proof of Theorem D.

By Lemma 1, it suffices to show that if a function a is a central (p,q)-atom such that supp$(a) \subset B(0,R)$, then Ta is a constant multiple of a $(p,q,\theta,n(\varepsilon-1/p),R)$-molecule.

The condition (M_1) and (M_2) are easily verified (see [1] and [8]). So we only need to check the condition (M_3). By Lemma 2, we have

$$\Big|\int Ta(x)dx\Big| = |(Ta,1)| = |(a,T^t(1))| \le CR^{n(1/s+\lambda)}\|a\|_{L^{s'}}\|T^t(1)\|_{CMO^{s,\lambda}}$$

$$\le C\|T^t(1)\|_{CMO^{s,\lambda}}R^{n(1-1/p+\lambda)} \le C\|T^t(1)\|_{CMO^{s,\lambda}}R^{n(\varepsilon-1/p)}.$$

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Yusuo Komori
School of High Technology for Human Welfare, Tokai University,
317, Nishino Numazu, Shizuoka 410-0395,
Japan
E-mail: komori@wing.ncc.u-tokai.ac.jp