TAIWANESE JOURNAL OF MATHEMATICS
Vol. 8, No. 2, pp. 337-341, June 2004
This paper is available online at http://www.math.nthu.edu.tw/tjm/

ON THE PRIME RADICAL OF A MODULE OVER A NONCOMMUTATIVE RING

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Abstract. Let R be a ring and M a left $R-$module. The radical of M is the intersection of all prime submodules of $M.$ It is proved that if R is a hereditary, noetherian, prime and non right artinian and M a finitely generated $R-$module then the radical of M has a certain form.

Throughout this note, all rings are associative with identity and all modules are unital left modules. Let M be a left $R-$module. Then a proper submodule N of M is prime if, for any $r \in R$ and $m \in M$ such that $rRm \subseteq N,$ either $rM \subseteq N$ or $m \in N.$ Prime submodules have been studied in a number of papers, for example $[2], [3].$ In particular, a number of papers have been devoted to describing the radical of a module over a commutative ring. It is natural therefore to ask whether the radical of a module over noncommutative ring has a simple description. A ring R is called hereditary if all left and right ideals are projective $R-$modules. A ring R is called Noetherian if R is left and right Noetherian and R is called prime if every product of non-zero(2-sided) ideals is again non-zero. A ring R is called HNP-ring if it is hereditary, noetherian, prime and non right artinian. We shall define the prime radical of M to be intersection of all prime submodules of $M.$ We shall denote the radical of M by $radM.$ In $[3],$ James Jenkins and Patrick F.Smith proved that if R is a Dedekind domain and M an $R-$module then the radical of M has a certain form. We shall prove that if R is a HNP-ring and M a left $R-$module then the radical of M has a certain form.

Definition 1. Let R be a ring and M an $R-$module. Let $r_1 \in R, m_1 \in M.$ The element r_1m_1 of M is called strongly nilpotent if every sequence $r_1m_1, r_2m_1, r_3m_1,$ $...$ such that $r_{i+1}m_1 \in r_iRr_im_1$ and $r_{i+1} \in r_iRr_i$ $(i = 1, 2, 3, ...)$ is ultimately zero. $W(M)$ will denote the submodule of M generated by strongly nilpotent elements.

Received April 15, 2003.
Communicated by S. B. Hsu.
2000 *Mathematics Subject Classification*: 16E60; 16D40.
Key words and phrases: Prime submodule, Hereditary rings, Noetherian rings.

Lemma 1. *Let* M *be an* $R-$*module. Then,* $W(M) \subseteq radM$:

Proof. Let $r_1m_1 \notin radM$ where $r_1 \in R$ and $m_1 \in M$: We will show that r_1m_1 is not strongly nilpotent element of M: Since $r_1m_1 \notin radM$; there exists a prime submodule N of M such that $r_1m_1 \notin N$: Thus, $r_1Rr_1m_1 \nsubseteq N$ and so there exists an element $r_2m_1 \in r_1Rr_1m_1$ such that $r_2m_1 \notin N$: Since N is prime submodule of M, $r_2Rr_2m_1 \nsubseteq N$. There exists $r_3m_1 \in r_2Rr_2m_1$ such that $r_3m_1 \notin N$: Therefore, there exists a sequence $r_1m_1; r_2m_1; r_3m_1; :::$ such that $r_{i+1}m_1 \in r_iRr_im_1$ and $r_{i+1} \in r_iRr_i$ $(i = 1; 2; 3; :::)$ but is not ultimately zero.Then r_1m_1 is not strongly nilpotent. So $W(M) \subseteq radM$: ■

For any submodule N of M; $(N : M) = \{r \in R : rM \subseteq N\}$ is the annihilator of the module $M=N$ and $Ann(m) = \{r \in R : rm = 0\}$ is the annihilator of the element $m \in M$:

Lemma 2. *Let* R *be a ring and* M *a cyclic module such that* $M = Rm$ *for some* $m \in M$: *Suppose that* P *is a prime ideal of* R *and* $Ann(m) \subseteq P$: *Then* Pm *is a prime submodule of* M *and* $P = (Pm : M)$:

Proof. Let $e \in M$ and $r \in R$: Let $rRe \subseteq Pm$ and $e = sm$ for some $s \in R$: Then $rRsm \subseteq Pm$: Since $Ann(m) \subseteq P$; $rRs \subseteq P$ and so $r \in P$ or $s \in P$: Therefore, $rM = rRm \subseteq Pm$ or $e = sm \in Pm$: Pm is a prime submodule of M: It is clear that $P \subseteq (Pm : M)$: Let $r \in (Pm : M)$: Then $rRm \subseteq Pm$: Since $Ann(m) \subseteq P$; $rR \subseteq P$ and so $r \in P$: As a result, $P = (Pm : M)$: ■

Theorem 1. *Let* R *be a ring and* M *a cyclic module such that* $M = Rm_1$ *for some* $m_1 \in M$: *Let* $Ann(m_1) \subseteq radR$: *Then* $radM = W(M)$:

Proof. By Lemma 1, $W(M) \subseteq radM$: We will show that $radM \subseteq W(M)$: We have $radM \subseteq \bigcap_{i \in I} (P_im_1) = (\bigcap_{i \in I} P_i)m_1 = (radR)m_1$ where P_i are the prime ideals of R and P_im_1 $(i \in I)$ are prime submodules of M by Lemma 2. Since $radR$ is precisely the set of strongly nilpotent elements of R; then every element of $(radR)m_1$ is strongly nilpotent element of M: Then $(radR)m_1 \subseteq W(M)$: Therefore $radM \subseteq W(M)$: ■

Proposition 1. *Let* N *be any submodule of an* $R-$*module* M: *Then* $W(N) \subseteq W(M)$:

Proof. Elementary. ■

Lemma 3. *Let* N *be a submodule of an* $R-$*module* M: *Then,* $radN \subseteq radM$:

Proof. Let P be a prime submodule of M: If $N \subseteq P$; then $radN \subseteq P$: If $N \nsubseteq P$; then $N \cap P$ is a prime submodule of N: Indeed, let $rRn \subseteq N \cap P$ and $r \notin$

$(N \cap P : N) = (P : N)$ where $r \in R$ and $n \in N$. Since P is a prime submodule of M, then $n \in P$. Therefore $n \in N \cap P$. Consequently, $\mathrm{rad}N \subseteq N \cap P \subseteq P$ and so $\mathrm{rad}N \subseteq \mathrm{rad}M$. ■

Lemma 4. *Let* R *be a ring and* M *an* R*-module such that* $M = \bigoplus_{i \in I} N_i$ *is a direct sum of submodules* $N_i\,(i \in I)$. *Then* $\mathrm{rad}M = \bigoplus_{i \in I} \mathrm{rad}N_i$.

Proof. By Lemma 3, $\mathrm{rad}N_i \subseteq \mathrm{rad}M$ for all $i \in I$. Then, we obtain $\bigoplus_{i \in I} \mathrm{rad}N_i \subseteq \mathrm{rad}M$. Let $m \in M$ and suppose that $m = \bigoplus_{k \in I} m_k \notin \bigoplus_{i \in I} \mathrm{rad}N_i$. There exists $k \in I$ such that $m_k \notin \mathrm{rad}N_k$ and so $m_k \notin N_k^*$ where N_k^* is a prime submodule of N_k. Let $K = N_k^* \bigoplus (\bigoplus_{i \neq k} N_i)$. K is a prime submodule of M. Indeed, let $rRs \subseteq K$ where $r \in R$ and $s = \bigoplus_{k \in I} s_k \in M$. Then $rRs_k \subseteq N_k^*$. Since N_k^* is a prime submodule of N_k, $s_k \in N_k^*$ or $rN_k \subseteq N_k^*$. Therefore, $s \in K$ or $rM \subseteq K$. Since $m \notin K$, then $m \notin \mathrm{rad}M$. It follows that $\mathrm{rad}M = \bigoplus_{i \in I} \mathrm{rad}N_i$. ■

Lemma 5. *Let* R *be a ring and* M *an* R*-module such that* $\mathrm{rad}M = W(M)$. *Then* $\mathrm{rad}N = W(N)$ *for any direct summand* N *of* M.

Proof. Suppose that $M = N \bigoplus K$ for some submodule K of M. We know that $W(N) \subseteq \mathrm{rad}N$ by lemma 1. Suppose that $m \in \mathrm{rad}N$. Then $m \in \mathrm{rad}M$ by Lemma 3. By hypothesis, $m = a_1 r_1 m_1 + \ldots + a_n r_n m_n$ where $a_i \in R$ and $r_i m_i$ are strongly nilpotent elements of M and $r_i m_i = r_i x_i + r_i y_i$ for all $1 \leq i \leq n$. Clearly, $r_i x_i$ are strongly nilpotent elements of N. Then $m - (a_1 r_1 x_1 + \ldots + a_n r_n x_n) = a_1 r_1 y_1 + \ldots + a_n r_n y_n \in N$, and so $m = a_1 r_1 x_1 + \ldots + a_n r_n x_n \in W(N)$. It follows that $\mathrm{rad}N \subseteq W(N)$. ■

Lemma 6. *Let* R *be a ring and* M *any projective* R*-module. Suppose that* $\mathrm{Ann}(m) \subseteq \mathrm{rad}R$ *for all* $m \in M$. *Then,* $\mathrm{rad}M = W(M)$.

Proof. There exists a free R-module F such that M is a direct summand of F. There exist an index I and cyclic free submodules $F_i\,(i \in I)$ of F such that $F = \bigoplus_{i \in I} F_i$. Then $\mathrm{rad}F = \bigoplus_{i \in I} \mathrm{rad}F_i$ by Lemma 4. But by Theorem 1, $\mathrm{rad}F_i = W(F_i) \subseteq W(F)$ for each $i \in I$. Hence $\mathrm{rad}F = W(F)$. Consequently, $\mathrm{rad}M = W(M)$ by Lemma 5. ■

Lemma 7. *Let* R *be a HNP-ring and* M *a finitely generated* R*-module. Then,* $M = \bigoplus_{i \in I} M_i$ *where submodules* M_i *is either projective or cyclic.*

Proof. By [1; Lemma 7.4], $M = \tau(M) \bigoplus M/\tau(M)$ where $\tau(M)$ is a torsion submodule of M. Moreover, $\tau(M)$ has finite length and $M/\tau(M)$ is projective. $\tau(M)$ is cyclic or a direct sum of cyclics by [1; Lemma 7.3]. Indeed, let N be a submodule of M such that $\tau(M)/N$ is cyclic. We use induction on the length of N. If $N = 0$, it is trivial. Otherwise, choose a simple submodule L of N. By induction $\tau(M)/L$ is cyclic, and if the sequence $0 \to L \to \tau(M) \to \tau(M)/L \to 0$ is nonsplit, then by [1; lemma 7.3 (a)], $\tau(M)$ is cyclic. So, suppose the sequence is split, and then $\tau(M) = L \bigoplus \tau(M)/L$. ■

Theorem 2. *Let* R *be a HNP-ring and* M *a finitely generated* $R-$*module. Suppose that* $Ann(m) \subseteq radR$ *for all* $m \in M$. *Then* $radM = W(M)$.

Proof. We know that $W(M) \subseteq radM$ by Lemma1. Now $M = \bigoplus_{i \in I} M_i$ where submodules M_i is either projective or cyclic by Lemma 7. Then $radM = \bigoplus_{i \in I} radM_i = \bigoplus_{i \in I} W(M_i) \subseteq W(M)$. As a result, $radM = W(M)$. ■

Lemma8. *If* $f : M \to S$ *is an epimorphism of* $R-$*modules with kernel* K *then there is a one-to-one correspondence between the set of prime submodules of* M *which contain* K *and the set of prime submodules of* S.

Proof. Let N be a prime submodule of M containing K. Let $r \in R$ and $m \in M$ such that $rRf(m) \subseteq f(N)$ and $f(m) \notin f(N)$. We will show that $rS \subseteq f(N)$. As $f(rRm) \subseteq f(N)$, $rRm \subseteq K + N = N$ and $m \notin N$ which implies that $rM \subseteq N$. Hence $rS = rf(M) = f(rM) \subseteq f(N)$. Let L be a prime submodule of S. Let $rRm^* \subseteq f^{-1}(L)$ and $m^* \notin f^{-1}(L)$ where $m^* \in M$, $r \in R$. Then $f(rRm^*) \subseteq f\left(f^{-1}(L)\right) \subseteq L$ and so $rRf(m^*) \subseteq L$. Since L is a prime submodule of S and $f(m^*) \notin L$, then $rS \subseteq L$ and so $rf(M) \subseteq L$. Consequently, $rM \subseteq f^{-1}(L)$. ■

M satisfies the radical formula if $rad(M/N) = W(M/N)$ for any submodule N of M. A proper submodule N of a module M is called semiprime if, for any $r \in R$ and $m \in M$ such that $rRrm \subseteq N$, $rm \in N$. If N is a submodule of M such that N is an intersection of prime submodules of M, then N is semiprime. We don't know if the converse is true in general, but it is true in the following special case. (see 2, for more detail)

Theorem 3. *Let* R *be a ring and* M *an* $R-$*module. If* M *satisfies the radical formula, then every semiprime submodule of* M *is an intersection of prime submodules of* M *and* $W(M/W(M)) = \overline{0}$.

Proof. Let N be a semiprime submodule of M. Then, $W(M/N) = \overline{0}$. Indeed, if $r_1 m_1 \notin N$ where $r_1 \in R$ and $m_1 \in M$, then there exists a chain

$r_1m_1; r_2m_1; \ldots$ such that $r_{i+1} \in r_iRr_i$ and $r_im_1 \notin N$ for all $i = 1;2;3;\ldots$ as N is a semiprime submodule. Then $r_1\overline{m_1}$ is not strongly nilpotent element of M/N. By hypothesis, $rad(M/N) = \overline{0}$. Hence N is an intersection of prime submodules of M by Lemma 8. Moreover, it is clear that $rad(M/radM) = \overline{0}$, so $W(M/W(M)) = rad(M/radM) = \overline{0}$. ■

Acknowledgment

We thank the referee for the helpful suggestions and comments.

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