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SPECTRAL DISTRIBUTIONS AND GELFAND'S THEOREM

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Abstract. This paper highlights a new short proof of a generalization of Gelfand's theorem through the use of spectral distributions. But above all it aims at studying, when the spectrum is discrete, if the operator admits a spectral resolution of the identity.

1. Introduction

Gelfand showed in 1941 the following theorem (see [6]): Let A be a linear bounded operator satisfying $\mathrm{Sup}_{n\in Z}\|A^n\| < +\infty$. If $\sigma(A) = \{1\}$, then $A = I$.

The semigroup version of this theorem is as follows: Let A be a linear (unbounded) operator that generates a C_0-group satisfying $\sup_{t\in R}\|e^{tA}\| < +\infty$. Assume $\sigma(A) = \{0\}$, then $A = 0$.

This is probably the first result that highlights a very particular class of operators: linear (unbounded) operators that generate bounded groups.
Results found in [5], [7] and [8] permit to say that this class can be generalized to linear (unbounded) operators that generate temperate integrated groups, where a complete study of a symbolic calculi for such operators using spectral distributions is done.

Recall that a linear operator A generates a k-times integrated semigroup if there exists a family of bounded operators $\{G(t)\}_{t\in R}$ satisfying

$$(\lambda - A)^{-1} = \lambda^k \int_0^{+\infty} e^{-\lambda t} G(t)\, dt \qquad \text{for all } \lambda > w;$$

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for some positive real number w (see [2] and [9]). Of course, a 0-times integrated semigroup is a continuous semigroup. $\{G(t)\}$ is said to be temperate if there exists a positive constant C such that $\|G(t)\| \leq C|t|^k$ for all $t > 0$.

In the second section, we recall definitions and results on spectral distributions that are needed.

In the third section, the following generalization of Gelfand's theorem will be shown: Let A be a linear (unbounded) operator that generates a temperate integrated group. If $\sigma(A) = \{0\}$, then $A = 0$.

In the last section, it will be shown that in case of discrete spectrum: $\sigma(A) = \{\lambda_n;\ n \in Z\}$, then A admits a resolution of the identity. More precisely, there exist projectors P_n satisfying $\sum P_n = I$ and $A = \sum \lambda_n P_n$. This is related to almost periodicity. See for example [3], [4] and the references therein.

2. Spectral Distributions

Let X be a Banach space and $\mathcal{L}(X)$ the algebra of bounded linear operators with uniform operator topology. In the following we recall basic results and definitions on spectral distributions (for more details see [5] or [7]).

Definition 1. [5, Definition 1.1] *By a spectral distribution* we mean a linear mapping $\mathcal{E}$ from $\mathcal{D}$ (the space of all functions in $\mathcal{C}^\infty(R)$ with compact support) into $\mathcal{L}(X)$ which satisfies:

(i) $\mathcal{E}(\varphi\psi) = \mathcal{E}(\varphi)\mathcal{E}(\psi)$, for all $\varphi, \psi \in \mathcal{D}$.

(ii) For any function $\varphi \in \mathcal{D}$ such that $\varphi(0) = 1$, $\mathcal{E}(\varphi_n)$ converges strongly to the identity I, where $\varphi_n(t) := \varphi(t/n)$.

Lemma 1. [5. Lemma 1.2] *Let $\mathcal{E}$ be a spectral distribution, then we have*

(a) $\mathcal{N} := \bigcap_{\rho\in\mathcal{D}} Ker\mathcal{E}(\rho) = 0$.

(b) $\mathcal{R} := \bigcup_{\rho\in\mathcal{D}} Im\mathcal{E}(\rho)$ *is everywhere dense in* X.

Definition 2. [5. Definition 1.3] *For any* $f \in \mathcal{C}^\infty(R)$ *define* $\mathcal{E}(f)$ *as follows;*

$$D(\mathcal{E}(f)) := \{x \in X \mid \lim_{n\to\infty} \mathcal{E}(f\rho_n)x \text{ exists for any } \rho \in \mathcal{D};\ \rho(0) = 1\}$$
$$\mathcal{E}(f)x := \lim_{n\to\infty} \mathcal{E}(f\rho_n)x \quad \text{for } x \in D(\mathcal{E}(f)).$$

Proposition 1. [5, Proposition 1.4] *For any* $f \in \mathcal{C}^\infty(R)$, $\mathcal{E}(f)$ *is a densely defined closable linear operator.*

In the sequel for any $f \in \mathcal{C}^\infty(R)$ we denote by $\mathcal{E}(f)$ the smallest closed extension of $\mathcal{E}(f)$.

Corollary 1. [5, Corollary 1.5] *Let $\mathcal{E}$ be a spectral distribution on X. Then $\mathcal{E}(1) = I$ (the identity operator on X).*

Definition 3. [5, Definition 1.6] We say that an unbounded linear operator B *admits the spectral distribution $\mathcal{E}$*, or B is the *momentum* of $\mathcal{E}$, if there is a spectral distribution $\mathcal{E}$ such that $B = \mathcal{E}(t)$. Here t denotes the identity function in $\mathbb{R}$.

Now we introduce an integer which measures in some sense the regularity of a spectral distribution. This integer which we will call the *order of $\mathcal{E}$* is by Fourier transformation the distributional order of $\mathcal{E}$. For a precise definition we have to introduce the following distribution spaces.
For $\ell \in \mathbb{N}$, let p_ℓ be the following norm on $\mathcal{D}$:

$$p_\ell(\varphi) := \sum_{k=0}^{\ell} \|t^k \frac{d^k \varphi}{dt^k}\|_{L^1}.$$

Let $\mathcal{T}_\ell$ denote the completion of $\mathcal{D}$ for p_ℓ. We designate by

$$[\mathcal{F}f](t) := \int_{\mathbb{R}} e^{-2i\pi st} f(s)ds$$

the Fourier transformation and by

$$\check{\mathcal{T}}_\ell := \mathcal{F}^{-1}\mathcal{T}_\ell = \{f \in \mathcal{S}' \mid \mathcal{F}f \in \mathcal{T}_\ell\},$$

where $\mathcal{S}'$ is the space of temperate distributions.

Definition 4. [5, Definition 1.8] We say that a spectral distribution $\mathcal{E}$ *is of order* ℓ if $\mathcal{E}$ can be extended as a linear continuous mapping on $\check{\mathcal{T}}_\ell$ equipped with the norm

$$\Pi_\ell(f) := \sum_{k=0}^{\ell} \|t^k \frac{d^k}{dt^k} \mathcal{F}f\|_{L^1}.$$

By virtue of Peetre's inequality, we have

$$\ell! p_k(\varphi) \leq k! p_\ell(\varphi), \quad \text{for any } k \leq \ell,$$

so $\check{\mathcal{T}}_\ell \hookrightarrow \check{\mathcal{T}}_k$.
The following lemma shows the particularity of these norms. The proof is a direct change of variable calculation.

Lemma 2. *For all $\varphi \in \check{\mathcal{T}}_k$, $s > 0$, define $\varphi_s(t) := \varphi(st)$. Then $\Pi_k(\varphi_s)$ is independent of s.*

Theorem 1. (Stone's generalized theorem) [5, Theorem 3.4] *Let A be a linear densely defined operator and k a nonnegative integer. The following are equivalent:*

(1) A *generates a* k*-times integrated temperate group.*

(2) iA *is the momentum of a spectral distribution of order* k.

Theorem 2. [5, Theorem 4.2] *Let* B *be the momentum of a spectral distribution* $\mathcal{E}$ *of order* k. *Then* $\operatorname{supp}\mathcal{E} = \sigma(B)$.

We deduce directly:

Corollary 2. *Let* f, g *be two* $\mathcal{C}^\infty$ *functions satisfying* $f \equiv g$ *on a neighborhood of* $\sigma(B)$. *Then as unbounded operators* $\mathcal{E}(f) = \mathcal{E}(g)$.

3. Gelfand's Generalized Theorem

In 1941, Gelfand showed that if a bounded operator whose spectrum is $\{1\}$ and doubly bounded then it is the identity operator:

Theorem 3. (Gelfand's theorem) *Let* A *be a a bounded operator satisfying* $\sigma(A) = \{1\}$. *If* $\sup_{n\in\mathbb{Z}} \|A^n\| < +\infty$, *then* $A = I$.

The original proof was not as simple as the statement above. This theorem can be proved in different ways (see [1], [6] and [10] and the references given in there). The following is the semigroup version of Gelfand's theorem:

Theorem 4. (Gelfand's theorem) *Let* A *be an operator that generates a uniformly bounded group* $(G(t))$ *(satisfying* $\|G(t)\| \le C$ *for some positive constant* C*). If* $\sigma(A) = \{0\}$, *then* $A = 0$.

Corollary 3. *If* $\sigma(A) = \{\lambda\}$ *then* $A = \lambda I$.

Proof. First of all notice that $\lambda \in i\mathbb{R}$. Now set $A_\lambda := A - \lambda I$. A_λ generates the uniformly bounded group $(e^{-\lambda t}G(t))$ and $\sigma(A_\lambda) = \{0\}$.
Therefore $A_\lambda = 0$. ∎

We will need the following lemma:

Lemma 3. *There exists a positive constant* C *such that for all* $f \in \check{\mathcal{T}}_k$,

$$\Pi_k(f) \le C \sum_{0\le j\le k} \|t^j f^{(j)}\|_{H^1}.$$

Proof. We have

$$\begin{aligned} x^k(\mathcal{F}f)^{(k)}(x) &= (-1)^k \mathcal{F}\left[(t^k f)^{(k)}\right](x) \\ &= (-1)^k \sum_{0\le j\le k} \binom{k}{j} \frac{k!}{(k-j)!} \mathcal{F}\left(t^{k-j} f^{(k-j)}\right)(x). \end{aligned}$$

We terminate using the well known inequality $\|\mathcal{F}f\|_{L^1} \le C\|f\|_{H^1}$. ■

The following is the natural generalization of Gelfand's theorem for momentum of a spectral distribution of order k. Remark that the case $k = 0$ is Gelfand's theorem. However it is a new approach and the proof is very simple.

Theorem 5. (Gelfand's generalized theorem) *Let A be the generator of a k-times integrated group $(G(t))$ satisfying $\|G(t)\| \le C|t|^k$ for some positive constant C. If $\sigma(A) = \{0\}$, then $A = 0$.*

Proof. By Theorem 1, $B = -iA$ admits a spectral distribution $\mathcal{E}$ of order k. Let $\varphi \in \mathcal{D}$ satisfying $\varphi(t) = 1$ on a neighborhood of zero, and for $z \in \mathbb{C}$ set $G(z) := \mathcal{E}(t \longmapsto e^{zt}\varphi(t)) \in \mathcal{L}(X)$. Let's show that $(G(z))$ is an entire group generated by B. By corollary 2, $G(0) = \mathcal{E}(t \longmapsto \varphi(t)) = I$. For $z, u \in \mathbb{C}$, using Definition 1(i) we have: $G(u)G(z) = \mathcal{E}(t \longmapsto e^{ut}\varphi(t))\mathcal{E}(t \longmapsto e^{zt}\varphi(t)) = \mathcal{E}(t \longmapsto e^{(u+z)t}\varphi(t)) = G(u + z)$. Remainder to show that the group is analytic. Writing $e^{zt}\varphi(t) = \sum_{n\ge 0} \frac{z^n t^n}{n!}\varphi(t)$, and since $\mathcal{E}$ is continuous from $\check{\mathcal{T}}_k$ into $\mathcal{L}(X)$, it suffices to show that the series $\sum_{n\ge 0} \frac{z^n t^n}{n!}\varphi(t)$ converges, in $\check{\mathcal{T}}_k$, to $e^{zt}\varphi(t)$. For this, using lemma 3, we have

$$\begin{aligned}\Pi_k(t^n\varphi) &\le C \sum_{j=0}^{k} \left\| t^j (t^n\varphi)^{(j)} \right\|_{H^1} \\ &\le C_1 n(n-1)\cdots(n-k+1)a^n,\end{aligned}$$

where $a := \max(1, \sup\{|x|;\ x \in \operatorname{supp}\varphi\})$ and C, C_1 are positive constants. Since $\mathcal{E}$ is continuous on $\check{\mathcal{T}}_k$ and

$$\mathcal{E}\left(t \longmapsto e^{zt}\varphi(t)\right) = \sum_{n=0}^{m-1} \frac{z^n}{n!}\mathcal{E}\left(t \longmapsto t^n\varphi(t)\right) + \mathcal{E}\left(t \longmapsto e^{zt}\varphi(t) - \sum_{n=0}^{m-1} \frac{z^n t^n}{n!}\varphi(t)\right)$$

for $m = 1, 2, \cdots$ and $z \in \mathbb{C}$, it suffices to show that

$$\left\| \mathcal{E}\left(t \longmapsto \sum_{n\ge m} \frac{z^n t^n}{n!}\varphi(t)\right) \right\| \to 0 \qquad \text{as } m \to \infty$$

uniformly on z in compact subsets of $\mathbb{C}$.

We have

$$\begin{aligned}\left\| \mathcal{E}\left(t \longmapsto \sum_{n\ge m} \frac{z^n t^n}{n!}\varphi(t)\right) \right\| &\le C\Pi_k\left[\sum_{n\ge m} \frac{z^n t^n}{n!}\varphi(t)\right] \\ &\le C_1 \sum_{0\le j\le k} \left\| t^j \left[\sum_{n\ge m} \frac{z^n t^n}{n!}\varphi(t)\right]^{(j)} \right\|_{H^1}\end{aligned}$$

$$\leq C_1 \sum_{0\leq j\leq k} \sum_{n\geq m} \frac{|z|^n}{n!} \left\| t^j [t^n \hat{A}(t)]^{(j)} \right\|_{H^1}$$
$$\leq C_2 \sum_{n\geq m} \frac{|z|^n}{n!} n(n-1)\cdots(n-k+1)$$

that tends to zero uniformly on z in compact subsets of C.
This proves

$$\mathcal{E}\left(t \longmapsto e^{zt}\hat{A}(t)\right) = \sum_{n\geq 0} \frac{z^n}{n!}\mathcal{E}\left(t \longmapsto t^n \hat{A}(t)\right)$$

and hence $\{G(z)\}$ is entire.

From the other hand, writing $z = r®$, $® \in C$ and $|®| = 1$, we get $G(z) = \mathcal{E}(f_r)$, where $f_r(t) := f^{®}(rt)$ with $f^{®}(t) = e^{®t}\hat{A}(t)$ (since $f_r(t) = e^{r®t}\hat{A}(rt) = e^{r®t}\hat{A}(t)$ on a neighborhood of zero). Now using Lemma 3, and since $\operatorname{supp} f^{®}$ is bounded, we see that $\Pi_k(f^{®})$ can be bounded by a constant independent of ®, $|®| = 1$. Hence by Lemma 2, $\|G(z)\| \leq K\Pi_k(f_r) \leq K\Pi_k(f^{®}) \leq M$, where M is a constant. Thus the entire group $(G(z))$ is bounded hence constant, i.e. $B = 0$. ∎

Remark 1. The main idea of the proof lies on the basic property of the semi-norms Π_k: invariance by homothety, see Lemma 2.

The following proposition shows that in this case the operator is bounded:

Proposition 2. *Let A be the generator of a k-times integrated group $(G(t))$ satisfying $\|G(t)\| \leq C|t|^k$ for some positive constant C. If ¾(A) is bounded then A is a bounded operator.*

Proof. $B := iA$ admits a spectral distribution of order k and ¾(B) is in a bounded interval of R. Let $' \in \mathcal{D}$ satisfying $' := 1$ on a neighborhood of ¾(B). By corollary 2, $B = \mathcal{E}(t) = \mathcal{E}(t\,')$ which is a bounded operator. ∎

The same Gelfand's theorem still valid in the case where k is any real positive number.

The next step is to show that if ¾$(A) = \{i\}$ then $A = iI$. Unfortunately rescaling the generator of a temperate k-times integrated group is in general no more temperate. The integrated group is bounded by a polynomial of degree $2k$. One expect to obtain spectral decomposition with nilpotent remainder. See [5, Th. 5.1].

By the following example we will show that ¾$(A) = \{¸\}$ does, in general, not imply $A = ¸I$. In the general case, it can be shown that $A - ¸I$ is nilpotent of order $k+1$.

Example In the following, we give a bounded operator A, in the finite dimensional case, with $A \neq iI$, ¾$(A) = \{i\}$, while A generates a temperate once

integrated group: Let $A = \begin{pmatrix} i & 1 \\ 0 & i \end{pmatrix}$, then $e^{tA} = \begin{pmatrix} e^{it} & te^{it} \\ 0 & e^{it} \end{pmatrix}$.
We have $\sigma(A) = \{i\}$, while the once integrated group

$$G(t) = \begin{pmatrix} \frac{e^{it}-1}{i} & (1-it)e^{it}-1 \\ 0 & \frac{e^{it}-1}{i} \end{pmatrix}$$

is temperate: $\|G(t)\| \leq M\,|t|$.
Notice that here we had to choose "double eigenvalue".

4. Resolution of the Identity in the Case of Discrete Spectrum

In the finite dimensional case, let A be a matrix whose spectrum is purely imaginary, then it is easy to see that if A is diagonalizable then $\sup_{t\in\mathbb{R}} \|e^{tA}\|$ is finite. The converse is true also: If $\sup_{t\in\mathbb{R}} \|e^{tA}\| < \infty$ then A is diagonalizable and the spectrum is purely imaginary). Of course the same hold in Hilbert situation.

One of aims of spectral theory is to generalize this setting to the infinite dimensional Banach situation.

In the following theorem we will show that if the spectrum is discrete (infinite) then the operator admits a resolution of the identity.

We start by the following general results:

Lemma 4. *Let* $x_m \to x$, (A_m) *bounded operators satisfying:*

(1) *For every* $y \in X$, $A_m y \to Ay$ *where* A *is a bounded operator;*

(2) *there is a constant* C *such that* $\|A_m\| \leq C$ *for every* m.

Then $A_m x_m \to Ax$.

Proof. Writing $A_m x_m - Ax = A_m x_m - A_m x + A_m x - Ax$ we get
$\|A_m x_m - Ax\| \leq \|A_m\| \|x_m - x\| + \|A_m x - Ax\|$. ■

Proposition 3. *Let* X *be a Banach space and* $\mathcal{E}$ *a spectral distribution on* X. *Suppose that* $(h_n)_{n\in\mathbb{Z}}$ *is a two-sided sequence of elements of* $\mathcal{D}$ *satisfying the following conditions:*

(i) $h_n h_m = 0$ *for all* $n \neq m$;

(ii) $\inf |supp\, h_n| \longrightarrow +\infty$ *as* $|n| \to +\infty$;

(iii) $h := \sum_{n\in\mathbb{Z}} h_n \in \mathcal{C}^\infty$.

Define the (unbounded) operator $Q\colon D(Q)\subset X\longrightarrow X$ *by*

$$D(Q):=\{x\in X;\ \sum_n \|\mathcal{E}(h_n)x\|<\infty\}$$
$$Qx:=\sum_n \mathcal{E}(h_n) \textit{ for } x\in D(Q).$$

Then Q *is closable and* $\mathcal{E}(h)=\overline{Q}$.

Proof. First notice that if $\varphi\in\mathcal{D}$, then $h\varphi=\sum_{|n|\le N_\varphi}\varphi h_n$ for some positive integer N_φ. Hence for all $x\in X$, $\mathcal{E}(\varphi)x\in D(Q)$ and

$$Q\mathcal{E}(\varphi)x=\sum_{|n|\le N_\varphi}\mathcal{E}(\varphi h_n)x.$$

Thus $\mathcal{R}\subset D(Q)$. Let's show that $D(Q)\subset D(\mathcal{E}(h))$ and for all $x\in D(Q)$, $Qx=\mathcal{E}(h)x$. Let φ_j be as in Definition 1(ii). For all j denote by N_j a positive integer so that for all $|n|\ge N_j$, $\operatorname{supp} h_n\cap\operatorname{supp}\varphi_j=\emptyset$. Let $x\in D(Q)$, we have

$$\mathcal{E}(h\varphi_j)x=\sum_{|n|<N_j}\mathcal{E}(h_n\varphi_j)x=\mathcal{E}(\varphi_j)\sum_{|n|<N_j}\mathcal{E}(h_n)x.$$

Observing that $N_j\to+\infty$ as $|j|\to+\infty$ and using Lemma 4, we deduce that $\lim_j\mathcal{E}(h\varphi_j)x$ exists and is equal to Qx, i.e. $x\in D(\mathcal{E}(h))$ and $D(Q)x=\mathcal{E}(h)x$. Therefore $\mathcal{E}(h)$ is an extension of Q. Since by construction (proposition 1) $\mathcal{E}(h)$ is the smallest closed extension and $\mathcal{E}(h)=Q$ on $\mathcal{R}$, remainder to show that Q is closable. For this assume that $y_j\to 0$, $y_j\in D(Q)$ and $Qy_j\to z$. Then, using the lemma 4 and the closedness of $\mathcal{E}(\varphi_m)$ and $\mathcal{E}(h_i)$, we have

$$\begin{aligned}
z&=\lim_m\mathcal{E}(\varphi_m)z=\lim_m\mathcal{E}(\varphi_m)\lim_j Qy_j\\
&=\lim_m\lim_j\mathcal{E}(\varphi_m)Qy_j=\lim_m\lim_j Q\mathcal{E}(\varphi_m)y_j\\
&=\lim_m\lim_j\sum_{|i|<N_m}\mathcal{E}(h_i\varphi_m)y_j=\lim_m\sum_{|i|<N_m}\lim_j\mathcal{E}(h_i\varphi_m)y_j\\
&=\lim_m\mathcal{E}(\varphi_m)\sum_{|i|<N_m}\lim_j\mathcal{E}(h_i)y_j=0.
\end{aligned}$$

Theorem 6. *Let* A *be the generator of a bounded group* $(G(t))$. *Assume that* $\sigma(A)=\{\lambda_n;\ n\in\mathbb{Z}\}$ *is a discrete set. Then there exist projectors* P_n, $n\in\mathbb{Z}$, *satisfying* $\overline{\sum_n P_n}=I$ *and* $A=\overline{\sum_n\lambda_n P_n}$.

Remark 2.

(1) The operators $\sum_n P_n$ and $\sum_n\alpha_n P_n$ are defined as in Proposition 3.
(2) In the finite dimensional case, this correspond to simple eigenvalues:

Using the notations of the last example we see that $\{e^{tA}\}$ is not bounded. While if we consider a matrix with simple eigenvalues then it is diagonalizable:

For example, let $A = \begin{pmatrix} i & 1 \\ 0 & 2i \end{pmatrix}$, then $\sigma(A) = \{i; 2i\}$, and

$e^{tA} = \begin{pmatrix} e^{it} & \frac{1}{i}(e^{2it} - e^{it}) \\ 0 & e^{2it} \end{pmatrix}$ is bounded. It is not difficult to see that setting $P_1 = \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 0 & -i \\ 0 & 1 \end{pmatrix}$ we have: $P_1P_2 = 0$, $P_1 + P_2 = I$, $iP_1 + 2iP_2 = A$.

Proof of the theorem The proof is in the four following steps.

1: Let $B := iA$. B admits a spectral distribution of order k and $\sigma(B) = \{\alpha_n;\ n \in Z\} \subset R$, with $\alpha_n := i\lambda_n$. Without loss of generality we can assume that the sequence (α_n) is increasing. Since (α_n) is discrete, let, for every n, $h_n \in \mathcal{D}(R)$ satisfying $h_n := 1$ on a neighborhood of α_n and $h_n := 0$ on a neighborhood of α_m for $m \neq n$, and $P_n := \mathcal{E}(h_n)$. Since $h_n^2 = h_n$ on a neighborhood of $\sigma(B)$, by Corollary 2, , $P_n^2 = P_n$. Now set $h(t) = \sum h_n(t)$, clearly h is a $\mathcal{C}^\infty$ function and $h := 1$ on a neighborhood of $\sigma(B)$ hence, by corollary 2, $\mathcal{E}(h) = I$.

2: By the proposition **??**, the operator $E := \sum P_n$, with maximal domain, is a closable operator and its closure is the identity operator.

3: $\underline{BP_n = \alpha_n P_n}$: For $n \in N$, setting $Q_n = I - P_n = \mathcal{E}(g_n)$, with $g_n = 1 - h_n$. Q_n is a projector and since for $m \neq n$, α_m is not in the enclosure of the set $\{th_n(t);\quad t \in R\}$, then by [5, Theorem 4.3], $\alpha_m \notin \sigma(BP_n)$ therefore $\sigma(BP_n) = \{\alpha_n\}$. On the space $X_n := P_nX$, define $\mathcal{E}_n(\varphi) := \mathcal{E}(\varphi h_n)$, for $\varphi \in \mathcal{D}$. Let's show that $\mathcal{E}_n$ is a spectral distribution of order 0 on X_n generated by BP_n: By corollary 2, $\mathcal{E}_n(\varphi\psi) = \mathcal{E}(\varphi\psi h_n) = \mathcal{E}(\varphi\psi h_n^2) = \mathcal{E}(\varphi h_n)\mathcal{E}(\psi h_n) = \mathcal{E}_n(\varphi)\mathcal{E}_n(\psi)$. If (φ_j) is as in Definition 1(ii), then $\mathcal{E}_n(\varphi_j) = \mathcal{E}(\varphi_j h_n) \to \mathcal{E}(h_n) = I_{X_n}$, and $\mathcal{E}_n(t) = \mathcal{E}(t \longmapsto th_n(t)) = BP_n$. Finally, for $f \in \check{\mathcal{T}}_0$, the inequality

$$\|\mathcal{E}_n(f)\| \leq C\Pi_0(fh_n) = C\|\mathcal{F}(fh_n)\|_{L^1} = C\|\mathcal{F}f * \mathcal{F}h_n\|_{L^1} \leq C\Pi_0(h_n)\Pi_0(f);$$

shows that the spectral distribution $\mathcal{E}_n$ is of order 0. Now using corollary 3 we get $BP_n = \alpha_n I_{X_n} = \alpha_n P_n$.

4: $A = \overline{\sum_n \lambda_n P_n}$. Applying the proposition 3 to the functions $t \longmapsto th_n(t)$ we see that $B = \mathcal{E}(t)$ is the closure of $\sum_n \mathcal{E}(th_n) = \sum_n \alpha_n P_n$. ∎

Remark 3. Using the technique of spectral distributions, we cannot give more information about the convergence of the sums $\sum P_n x$ and $\sum \lambda_n P_n x$. By the last proof, we showed that we have convergence in some dense subset. It would be interesting to remedy this.

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