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PERIODIC SOLUTIONS OF A RATIO-DEPENDENT FOOD CHAIN MODEL WITH DELAYS

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Abstract. By using the continuation theorem base on Gaines and Mawhin's coincidence degree, sufficient and realistic conditions are obtained for the global existence of positive periodic solutions for a delayed food chain model. Indeed, our result are applicable to distributed delays.

1. INTRODUCTION

Recently, there has been considerable interest in ratio-dependent predator-prey model; see [1], [5], [6], [7], [8], and the references therein. In their paper [7], Hsu, Hwang and Kuang considered the following ratio-dependent food chain model

(1)
$$\begin{cases} \dfrac{dx}{dt} = rx\left(1 - \dfrac{x}{K}\right) - \dfrac{1}{\gamma_1}\dfrac{m_1xy}{a_1y + x}; \\ \dfrac{dy}{dt} = \dfrac{m_1xy}{a_1y + x} - d_1y - \dfrac{1}{\gamma_2}\dfrac{m_2yz}{a_2z + y}; \\ \dfrac{dz}{dt} = \dfrac{m_2yz}{a_2z + y} - d_2z: \end{cases}$$

where $x; y$ and z represent the population density of prey, predator and top predator, respectively. Observe that the simple relation of these three species: z consumes y and y consunes on x and nutrient recycling is not accounted for. They show that this model is rich in boundary dynamics and is capable of generating extinction dynamics. Specifically, they provide partial answers to question such as: under what scenarios a potential biological control may be successful, and when it may fail.

Since the variation of the environment plays an important role in many biological and ecological systems. In particular, the effects of a periodically varing

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environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a steady environment. Thus, the assumption of periodicity of the parameters in the way (in a way) incorporates the periodicity of the environment (e.g., seasonal affects of weather, food supplies, mating habits, etc.). Therefore, it is interesting and important to study the following nonautonomous delayed ratio-dependent food chain model

$$(2)\quad \begin{cases} \dfrac{dx(t)}{dt} = x(t)\left[r(t) - b(t)x(t-\tau_1(t)) - \dfrac{c_1(t)y(t)}{a_1y(t)+x(t)}\right]; \\ \dfrac{dy}{dt} = y(t)\left[\dfrac{m_1(t)x(t-\tau_2(t))}{a_1y(t-\tau_2(t))+x(t-\tau_2(t))} - d_1(t) - \dfrac{c_2(t)z(t)}{a_2z(t)+y(t)}\right]; \\ \dfrac{dz}{dt} = z(t)\left[\dfrac{m_2(t)y(t-\tau_3(t))}{a_2z(t-\tau_3(t))+y(t-\tau_3(t))} - d_2(t)\right]; \end{cases}$$

where $r(t); b(t); c_1(t); c_2(t); d_1(t); d_2(t); m_1(t); m_2(t) \in C(R; R^+); R^+ = (0; +\infty)$ are ω-periodic function; $\tau_i(t); i = 1; 2; 3 \in C(R; R)$ are ω-periodic function; a_1 and a_2 are positive constants.

An important ecological problem associated with the study of multispecies population interaction in a periodic environment is the global existence of periodic solution. The main purpose of this paper is to derive sufficient conditions for the global existence of positive periodic solutions of systems (2). The method used here will be the coincidence degree theory developed by Gaines and Mawhin [3]. Such approach was adopted in [2], [4], [9] and [10].

2. Periodic Solutions of a Ratio-dependent Food Chain Model with Delays

In order to obtain the existence of a positive periodic solution of the system (2), we first introduce the followings.

Let X and Z be two Banach spaces. Consider an operator equation

$$Lx = \lambda Nx; \ \lambda \in (0; 1);$$

where $L : \mathrm{Dom}L \cap X \to Z$ is a linear operator and λ is a parameter. Let P and Q denote two projectors such that

$$P : X \cap \mathrm{Dom}L \to \mathrm{Ker}L \text{ and } Q : Z \to Z/\mathrm{Im}L.$$

Let $J : \mathrm{Im}Q \to \mathrm{Ker}L$ be an isomorphism of $\mathrm{Im}Q$ onto $\mathrm{Ker}L$. In the sequel, we will use the following result of Mawhin [3, p.40]

Lemma 2.1. *Let X and Z be two Banach spaces and L a Fredholm mapping of index zero. Assume that $N : \overline{\Omega} \to Z$ is L-compact on $\overline{\Omega}$ with Ω open bounded in X. Furthermore we assume:*

(a) *for each* $\lambda \in (0,1)$, $x \in \partial\Omega \cap \mathrm{Dom}L$,

$$Lx \neq \lambda N x;$$

(b) *for each* $x \in \partial\Omega \cap \mathrm{Ker}L$,

$$QNx \neq 0$$

and

$$\deg\{JQN, \Omega \cap \mathrm{Ker}L, 0\} \neq 0.$$

Then the equation $Lx = Nx$ *has at least one solution in* $\overline{\Omega}$.

Recall that a linear mapping $L : \mathrm{Dom}L \cap X \to Z$ with $\mathrm{Ker}L = L^{-1}(0)$ and $\mathrm{Im}L = L(\mathrm{Dom}L)$, is called a Fredholm mapping if the following two conditions hold:

(i) $\mathrm{Ker}L$ has a finite dimension;

(ii) $\mathrm{Im}L$ is closed and has a finite codimension.

We also note that the codimension of $\mathrm{Im}L$ is the dimension of $Z/\mathrm{Im}L$, i.e., the dimension of the cokernel coker L of L.

When L is a Fredholm mapping, its index is the integer $\mathrm{Ind}L = \dim\ker L - \mathrm{codim}\,\mathrm{Im}L$.

We say that a mapping N is L-compact on Ω if the mapping $QN : \overline{\Omega} \to Z$ is continuous, $QN\left(\overline{\Omega}\right)$ is bounded, and $K_p(I-Q)N : \overline{\Omega} \to X$ is compact, i.e., it is continuous and $K_p(I-Q)N\left(\overline{\Omega}\right)$ is relatively compact, where $K_p : \mathrm{Im}L \to \mathrm{Dom}L \cap \mathrm{Ker}P$ is a inverse of the restriction L_p of L to $\mathrm{Dom}L \cap \mathrm{Ker}P$, so that $LK_p = I$ and $K_pL = I - P$.

For convenience, we shall introduce the notation:

$$\overline{u} = \frac{1}{\omega}\int_0^{\omega} u(t)dt.$$

where u is a periodic continuous function with period ω.

Now we state our first theorem for the existence of a positive ω-periodic solution of system (2).

Theorem 2.1. *If*

$$a_1\overline{r} - \overline{c}_1 > 0, \overline{m}_1 a_2 - \overline{d}_1 a_2 - \overline{c}_2 > 0 \textit{ and } \overline{m}_2 - \overline{d}_2 > 0,$$

then the system (2) has at least one positive ω*-periodic solution.*

Proof. Let

(3) $$x(t) = \exp\{x_1(t)\}, y(t) = \exp\{x_2(t)\}, z(t) = \exp\{x_3(t)\},$$

Then the system (2) becomes

$$(4)\quad \begin{cases} \dfrac{dx_1(t)}{dt} = r(t) - b(t)\exp\{x_1(t-\tau_1(t))\} - \dfrac{c_1(t)\exp\{x_2(t)\}}{a_1\exp\{x_2(t)\}+\exp\{x_1(t)\}}, \\ \dfrac{dx_2(t)}{dt} = \dfrac{m_1(t)\exp\{x_1(t-\tau_2(t))\}}{a_1\exp\{x_2(t-\tau_2(t))\}+\exp\{x_1(t-\tau_2(t))\}} - d_1(t) \\ \qquad - \dfrac{c_2(t)\exp\{x_3(t)\}}{a_2\exp\{x_3(t)\}+\exp\{x_2(t)\}}, \\ \dfrac{dx_3(t)}{dt} = \dfrac{m_2(t)\exp\{x_2(t-\tau_3(t))\}}{a_2\exp\{x_3(t-\tau_3(t))\}+\exp\{x_2(t-\tau_3(t))\}} - d_2(t). \end{cases}$$

In order to apply Lemma 2.1 to system (2), we take

$$X = Z = \left\{x(t) = (x_1(t), x_2(t), x_3(t))^T \in C(R, R^3) \ : x(t+\omega) = x(t)\right\},$$

and denote

$$||x|| = ||(x_1(t), x_2(t), x_3(t))^T|| = \max_{t\in[0,\omega]}|x_1(t)| + \max_{t\in[0,\omega]}|x_2(t)| + \max_{t\in[0,\omega]}|x_3(t)|.$$

Then X and Z are Banach spaces when they are endowed with the norms $||\cdot||$.

Set

$$Nx = \begin{bmatrix} r(t) - b(t)\exp\{x_1(t-\tau_1(t))\} - \dfrac{c_1(t)\exp\{x_2(t)\}}{a_1\exp\{x_2(t)\}+\exp\{x_1(t)\}} \\ \dfrac{m_1(t)\exp\{x_1(t-\tau_2(t))\}}{a_1\exp\{x_2(t-\tau_2(t))\}+\exp\{x_1(t-\tau_2(t))\}} - d_1(t) \\ -\dfrac{c_2(t)\exp\{x_3(t)\}}{a_2\exp\{x_3(t)\}+\exp\{x_2(t)\}} \\ \dfrac{m_2(t)\exp\{x_2(t-\tau_3(t))\}}{a_2\exp\{x_3(t-\tau_3(t))\}+\exp\{x_2(t-\tau_3(t))\}} - d_2(t) \end{bmatrix},$$

and

$$Lx = x', \ Px = \frac{1}{\omega}\int_0^\omega x(t)dt, \ x \in X, \quad Qz = \frac{1}{\omega}\int_0^\omega z(t)dt, z \in Z.$$

Evidently, $\mathrm{Ker}L = \left\{x | x \in X, x = R^3\right\}$, $\mathrm{Im}L = \left\{z | \ z \in Z, \int_0^\omega z(t)dt = 0\right\}$ are closed in Z and $\dim\mathrm{Ker}L = \mathrm{co}\dim\mathrm{Im}L = 3$. Hence, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse of L, $K_p : \mathrm{Im}L \to \mathrm{Ker}P \cap \mathrm{dom}L$ has the form

$$K_p(z) = \int_0^t z(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^t z(s)dsdt.$$

Thus

$$
QNx = \left[\begin{array}{l}
\frac{1}{!}\int_0^{!}\left[r(-b(t)\exp\{x_1(t-¿_1(t))\} - \frac{c_1(t)\exp\{x_2(t)\}}{a_1\exp\{x_2(t)\}+\exp\{x_1(t)\}}\right]dt \\
\frac{1}{!}\int_0^{!}\left[\frac{m_1(t)\exp\{x_1(t-¿_2(t))\}}{a_1\exp\{x_2(t-¿_2(t))\}+\exp\{x_1(t-¿_2(t))\}} - d_1(t)\right. \\
\qquad\qquad \left. - \frac{c_2(t)\exp\{x_3(t)\}}{a_2\exp\{x_3(t)\}+\exp\{x_2(t)\}}\right]dt \\
\frac{1}{!}\int_0^{!}\left[\frac{m_2(t)\exp\{x_2(t-¿_3(t))\}}{a_2\exp\{x_3(t-¿_3(t))\}+\exp\{x_2(t-¿_3(t))\}} - d_2(t)\right]dt
\end{array}\right];
$$

and

$$
K_p(I-Q)N = \left[\begin{array}{l}
\int_0^{t}\left[r(s) - b(s)\exp\{x_1(s-¿_1(s))\}\right. \\
\qquad\qquad \left. - \frac{c_1(s)\exp\{x_2(s)\}}{a_1\exp\{x_2(s)\}+\exp\{x_1(s)\}}\right]ds \\
\int_0^{t}\left[\frac{m_1(s)\exp\{x_1(s-¿_2(s))\}}{a_1\exp\{x_2(s-¿_2(s))\}+\exp\{x_1(s-¿_2(s))\}} - d_1(s)\right. \\
\qquad\qquad \left. - \frac{c_2(s)\exp\{x_3(s)\}}{a_2\exp\{x_3(s)\}+\exp\{x_2(s)\}}\right]ds \\
\int_0^{t}\left[\frac{m_2(s)\exp\{x_2(s-¿_3(s))\}}{a_2\exp\{x_3(s-¿_3(s))\}+\exp\{x_2(s-¿_3(s))\}} - d_2(s)\right]ds
\end{array}\right]
$$

$$
- \left[\begin{array}{l}
\frac{1}{!}\int_0^{!}\int_0^{t}\left[r(s) - b(s)\exp\{x_1(-¿_1(s))\} - \frac{c_1(s)\exp\{x_2(s)\}}{a_1\exp\{x_2(s)\}+\exp\{x_1(s)\}}\right]dsdt \\
\frac{1}{!}\int_0^{!}\int_0^{t}\left[\frac{m_1(s)\exp\{x_1(s-¿_2(s))\}}{a_1\exp\{x_2(s-¿_2(s))\}+\exp\{x_1(s-¿_2(s))\}} - d_1(s)\right. \\
\qquad\qquad \left. - \frac{c_2(s)\exp\{x_3(s)\}}{a_2\exp\{x_3(s)\}+\exp\{x_2(s)\}}\right]dsdt \\
\frac{1}{!}\int_0^{!}\int_0^{t}\left[\frac{m_2(s)\exp\{x_2(s-¿_3(s))\}}{a_2\exp\{x_3(s-¿_3(s))\}+\exp\{x_2(s-¿_3(s))\}} - d_2(s)\right]dsdt
\end{array}\right]
$$

$$
- \left[\begin{array}{l}
\left(\frac{t}{!} - \frac{1}{2}\right)\int_0^{!}\left[r(t) - b(t)\exp\{x_1(t-¿_1(t))\} - \frac{c_1(t)\exp\{x_2(t)\}}{a_1\exp\{x_2(t)\}+\exp\{x_1(t)\}}\right]dt \\
\left(\frac{t}{!} - \frac{1}{2}\right)\int_0^{!}\left[\frac{m_1(t)\exp\{x_1(t-¿_2(t))\}}{a_1\exp\{x_2(t-¿_2(t))\}+\exp\{x_1(t-¿_2(t))\}} - d_1(t)\right. \\
\qquad\qquad \left. - \frac{c_2(t)\exp\{x_3(t)\}}{a_2\exp\{x_3(t)\}+\exp\{x_2(t)\}}\right]dt \\
\left(\frac{t}{!} - \frac{1}{2}\right)\int_0^{!}\left[\frac{m_2(t)\exp\{x_2(t-¿_3(t))\}}{a_2\exp\{x_3(t-¿_3(t))\}+\exp\{x_2(t-¿_3(t))\}} - d_2(t)\right]dt
\end{array}\right]:
$$

Clearly, QN and $K_p(I-Q)N$ are continuous and, moreover, $QN\left(\overline{\Omega}\right); K_p(I-Q)N\left(\overline{\Omega}\right)$ are relatively compact for any open bounded set $\Omega \subset X$: Hence, N is L-compact on $\overline{\Omega}$, here Ω is any open bounded set in X.

Now we are in a position to search for an appropriate open bounded subset Ω for the application of Lemma 2.1. Corresponding to equation $Lx = ¸Nx$, $¸ \in (0;1)$; we have

$$(5)\quad \begin{cases} \dfrac{dx_1(t)}{dt} = ¸\left[r(t) - b(t)\exp\{x_1(t-¿_1(t))\} - \dfrac{c_1(t)\exp\{x_2(t)\}}{a_1\exp\{x_2(t)\}+\exp\{x_1(t)\}}\right]; \\ \dfrac{dx_2(t)}{dt} = ¸\left[\dfrac{m_1(t)\exp\{x_1(t-¿_2(t))\}}{a_1\exp\{x_2(t-¿_2(t))\}+\exp\{x_1(t-¿_2(t))\}} - d_1(t)\right. \\ \qquad\qquad \left. - \dfrac{c_2(t)\exp\{x_3(t)\}}{a_2\exp\{x_3(t)\}+\exp\{x_2(t)\}}\right]; \\ \dfrac{dx_3(t)}{dt} = ¸\left[\dfrac{m_2(t)\exp\{x_2(t-¿_3(t))\}}{a_2\exp\{x_3(t-¿_3(t))\}+\exp\{x_2(t-¿_3(t))\}} - d_2(t)\right]: \end{cases}$$

Suppose that $x(t) = (x_1; x_2; x_3) \in X$ is a solution of system (5) for a certain $¸ \in (0;1)$. By integrating (5) over the interval $[0;!]$, we obtain

$$\begin{cases} \displaystyle\int_0^! \left[r(t) - b(t)\exp\{x_1(t-¿_1(t))\} - \frac{c_1(t)\exp\{x_2(t)\}}{a_1\exp\{x_2(t)\}+\exp\{x_1(t)\}}\right] dt = 0; \\ \displaystyle\int_0^! \left[\frac{m_1(t)\exp\{x_1(t-¿_2(t))\}}{a_1\exp\{x_2(t-¿_2(t))\}+\exp\{x_1(t-¿_2(t))\}} - d_1(t)\right. \\ \qquad\qquad \left. - \dfrac{c_2(t)\exp\{x_3(t)\}}{a_2\exp\{x_3(t)\}+\exp\{x_2(t)\}}\right] dt = 0; \\ \displaystyle\int_0^! \left[\frac{m_2(t)\exp\{x_2(t-¿_3(t))\}}{a_2\exp\{x_3(t-¿_3(t))\}+\exp\{x_2(t-¿_3(t))\}} - d_2(t)\right] dt = 0: \end{cases}$$

Hence we have the followings:

$$(6)\quad \int_0^! \left[b(t)\exp\{x_1(t-¿_1(t))\} + \frac{c_1(t)\exp\{x_2(t)\}}{a_1\exp\{x_2(t)\}+\exp\{x_1(t)\}}\right] dt = \overline{r}!;$$

$$(7)\quad \int_0^! \left[\frac{m_1(t)\exp\{x_1(t-¿_2(t))\}}{a_1\exp\{x_2(t-¿_2(t))\}+\exp\{x_1(t-¿_2(t))\}} - \frac{c_2(t)\exp\{x_3(t)\}}{a_2\exp\{x_3(t)\}+\exp\{x_2(t)\}}\right] dt = \overline{d}_1!;$$

$$(8)\quad \int_0^! \left[\frac{m_2(t)\exp\{x_2(t-¿_3(t))\}}{a_2\exp\{x_3(t-¿_3(t))\}+\exp\{x_2(t-¿_3(t))\}}\right] dt = \overline{d}_2!:$$

From (5), (6), (7) and (8), we obtain

$$
\begin{aligned}
\int_0^{\omega} |x_1'(t)|dt &\le \int_0^{\omega} [b(t)\exp\{x_1(t-\tau_1(t))\}]\,dt \\
&\quad + \int_0^{\omega} \left[\frac{c_1(t)\exp\{x_2(t)\}}{a_1\exp\{x_2(t)\}+\exp\{x_1(t)\}}\right] dt + \int_0^{\omega} |r(t)|\,dt \\
&= 2\bar{r}\omega,
\end{aligned} \tag{9}
$$

and

$$
\begin{aligned}
\int_0^{\omega} |x_2'(t)|dt &\le \int_0^{\omega} \left| \frac{m_1(t)\exp\{x_1(t-\tau_2(t))\}}{a_1\exp\{x_2(t-\tau_2(t))\}+\exp\{x_1(t-\tau_2(t))\}} \right. \\
&\quad \left. - \frac{c_2(t)\exp\{x_3(t)\}}{a_2\exp\{x_3(t)\}+\exp\{x_2(t)\}} \right| dt + \bar{d}_1\omega \\
&\le 2\bar{d}_1\omega.
\end{aligned} \tag{10}
$$

Note that $(x_1(t), x_2(t), x_3(t))^T \in X$, then for $i = 1, 2, 3$ there exists $\xi_i, \eta_i \in [0, \omega]$, $i = 1, 2, 3$ such that

$$x_i(\xi_i) = \min_{t\in[0,\omega]} x_i(t),\ x_i(\eta_i) = \max_{t\in[0,\omega]} x_i(t), \tag{11}$$

By (6) and (11), we have

$$\bar{r}\omega \ge \bar{b}\omega \exp\{x_1(\xi_1)\},$$

and so

$$x_1(\xi_1) \le \ln\left\{\frac{\bar{r}}{\bar{b}}\right\}.$$

Then

$$x_1(t) \le x_1(\xi_1) + \int_0^{\omega} |x_1'(t)|dt \le \ln\left\{\frac{\bar{r}}{\bar{b}}\right\} + 2\bar{r}\omega. \tag{12}$$

By (6) and (11), we also have

$$\bar{r}\omega \le \bar{b}\omega \exp\{x_1(\eta_1)\} + \frac{\bar{c}_1}{a_1}\omega.$$

and

$$x_1(\eta_1) \ge \ln\left[\frac{a_1\bar{r}-\bar{c}_1}{a_1\bar{b}}\right].$$

Thus

$$\text{(13)} \qquad x_1(t) \ge x_1(\xi_1) - \int_0^\omega |x_1'(t)|dt \ge \ln\left[\frac{a_1\bar{r} - \bar{c}_1}{a_1\bar{b}}\right] - 2\bar{r}\omega.$$

From (12) and (13) it follows that

$$\text{(14)} \qquad \max_{t\in[0,\omega]} |x_1(t)| \le \max\left\{\left|\ln\left\{\frac{\bar{r}}{\bar{b}}\right\} + 2\bar{r}\omega\right|, \left|\ln\left[\frac{a_1\bar{r} - \bar{c}_1}{a_1\bar{b}}\right] - 2\bar{r}\omega\right|\right\} := B_1.$$

Similarly, by (7) and (11), we obtain

$$\begin{aligned} \bar{d}_1\omega &\ge \int_0^\omega \left[\frac{m_1(t)\exp\{x_1(\eta_1)\}}{a_1\exp\{x_2(\xi_2)\} + \exp\{x_1(\eta_1)\}} - \frac{c_2(t)}{a_2}\right]dt \\ &= \frac{\bar{m}_1\omega\exp\{x_1(\eta_1)\}}{a_1\exp\{x_2(\xi_2)\} + \exp\{x_1(\eta_1)\}} - \frac{\bar{c}_2\omega}{a_2}; \end{aligned}$$

The above and (13) imply that

$$\exp\{x_2(\xi_2)\} \ge \frac{(\bar{m}_1a_2 - \bar{d}_1a_2 - \bar{c}_2)(\bar{r}a_1 - \bar{c}_1)}{a_1^2(a_2\bar{d}_1 + \bar{c}_2)\bar{b}}\exp\{-2\bar{r}\omega\}.$$

Then

$$x_2(\xi_2) \ge \ln\left\{\frac{(\bar{m}_1a_2 - \bar{d}_1a_2 - \bar{c}_2)(\bar{r}a_1 - \bar{c}_1)}{a_1^2(a_2\bar{d}_1 + \bar{c}_2)\bar{b}}\exp\{-2\bar{r}\omega\}\right\} := H_1;$$

and consequently

$$\text{(15)} \qquad x_2(t) \ge x_2(\xi_2) - \int_0^\omega |x_2'(t)|dt \ge H_1 - 2\bar{d}_1\omega.$$

In addition, by (7) and (11), we obtain

$$\bar{d}_1\omega \le \frac{\bar{m}_1\omega}{a_1}\frac{\exp\{x_1(\xi_1)\}}{\exp\{x_2(\eta_2)\}}.$$

Thus

$$\begin{aligned} x_2(\eta_2) &\le \ln\left\{\frac{\bar{m}_1\exp\{x_1(\xi_1)\}}{a_1\bar{d}_1}\right\} \\ &\le \ln\left\{\frac{\bar{m}_1\bar{r}\exp\{2\bar{r}\omega\}}{a_1\bar{d}_1\bar{b}}\right\} := H_2; \end{aligned}$$

and so

$$\text{(16)} \qquad x_2(t) \le x_2(\eta_2) + \int_0^\omega |x_2'(t)|dt \le H_2 + 2\bar{d}_1\omega.$$

The inequalities (15) and (16) imply that

$$\max_{t\in[0,\omega]} |x_2(t)| \le \max\left\{\left|H_1 - 2\bar{d}_1\omega\right|, \left|H_2 + 2\bar{d}_1\omega\right|\right\} := B_2. \tag{17}$$

Furthermore, by (8) and (11), we obtain

$$\begin{aligned}\bar{d}_2\omega &\ge \int_0^\omega \left[\frac{m_2(t)\exp\{x_2(\xi_2)\}}{a_3\exp\{x_3(\eta_3)\} + \exp\{x_2(\xi_2)\}}\right] dt\\ &= \frac{\bar{m}_2\omega\exp\{x_2(\xi_2)\}}{a_3\exp\{x_3(\eta_3)\} + \exp\{x_2(\xi_2)\}},\end{aligned}$$

The above and (15) imply that

$$\begin{aligned}\exp\{x_3(\eta_3)\} &\ge \frac{\bar{m}_2 - \bar{d}_2}{a_3\bar{d}_2}\exp\{x_2(\xi_2)\}\\ &\ge \frac{\bar{m}_2 - \bar{d}_2}{a_3\bar{d}_2}\exp\{H_1 - 2\bar{d}_1\omega\},\end{aligned}$$

then

$$x_3(\eta_3) \ge \ln\left\{\frac{\bar{m}_2 - \bar{d}_2}{a_3\bar{d}_2}\exp\{H_1 - 2\bar{d}_1\omega\}\right\} := l_1,$$

and consequently

$$x_3(t) \ge x_3(\eta_3) - \int_0^\omega |x_3'(t)|dt \ge l_1 - 2\bar{d}_2\omega. \tag{18}$$

In addition, from (8) and (11), we also have

$$\bar{d}_2\omega \le \frac{\bar{m}_2\omega}{a_2}\frac{\exp\{x_2(\eta_2)\}}{\exp\{x_3(\xi_3)\}}.$$

That is

$$\begin{aligned}x_3(\xi_3) &\le \ln\left\{\frac{\bar{m}_2}{a_2\bar{d}_2}\exp\{x_2(\eta_2)\}\right\}\\ &\le \ln\left\{\frac{\bar{m}_2}{a_2\bar{d}_2}\exp\{H_2 + 2\bar{d}_1\omega\}\right\} := l_2,\end{aligned}$$

then

$$x_3(t) \le x_3(\xi_3) + \int_0^\omega |x_3'(t)|dt \le l_2 + 2\bar{d}_2\omega. \tag{19}$$

The inequalities (18) and (19) imply that

$$\max_{t\in[0,\omega]}|x_3(t)| \leq \max\left\{\left|l_1 - 2\bar{d}_2\omega\right|, \left|l_2 + 2\bar{d}_2\omega\right|\right\} := B_3.$$

Clearly, H_i, l_i, $i = 1, 2$, and B_j, $j = 1, 2, 3$ are independent of λ. By the assumption of Theorem 2.1, it is easy to show that the system of algebraic equations

$$\begin{cases} \bar{r} - \bar{b}v_1 - \dfrac{\bar{c}_1 v_2}{v_1 + a_1 v_2} = 0, \\ \dfrac{\bar{m}_1 v_1}{v_1 + a_1 v_2} - \bar{d}_1 - \dfrac{\bar{c}_2 v_3}{v_2 + a_2 v_3} = 0, \\ -\bar{d}_2 + \dfrac{\bar{m}_2 v_2}{v_2 + a_2 v_3} = 0, \end{cases}$$

has a unique solution $(v_1^*, v_2^*, v_3^*)^T \in \mathrm{int}R_+^2$ with $v_i^* > 0, i = 1, 2, 3$. Denote $B = B_1 + B_2 + B_3 + B_4$, where $B_4 > 0$ is sufficiently large satisfying

$$\|(\ln\{v_1^*\}, \ln\{v_2^*\}, \ln\{v_3^*\})\| = |\ln\{v_1^*\}| + |\ln\{v_2^*\} + \ln\{v_3^*\}| < B.$$

Let

$$\Omega = \{x(t) \in X : \|x\| < B\}.$$

It is clear that Ω satisfies the condition (a) of the Lemma 2.1. of

$$x = (x_1, x_2, x_3)^T \in \partial\Omega \cap \mathrm{Ker}L = \partial\Omega \cap R^3,$$

with $\|x\| = M$, then

$$QNx = \begin{bmatrix} \bar{r} - \bar{b}\exp\{x_1\} - \dfrac{\bar{c}_1\exp\{x_2\}}{\exp\{x_1\} + a_1\exp\{x_2\}} \\ \dfrac{\bar{m}_1\exp\{x_1\}}{\exp\{x_1\} + a_1\exp\{x_2\}} - \bar{d}_1 - \dfrac{\bar{c}_2\exp\{x_3\}}{\exp\{x_2\} + a_2\exp\{x_3\}} \\ -\bar{d}_2 + \dfrac{\bar{m}_2\exp\{x_2\}}{\exp\{x_2\} + a_2\exp\{x_3\}} \end{bmatrix} \neq 0.$$

Furthermore, let J :Im$Q \to$Ker$L$, $x \to x$, and by the assumption in Theorem 2.1, it follows that

$$\deg\{JQN, \Omega \cap \mathrm{Ker}L, 0\} \neq 0.$$

Now Ω satisfies all the conditions in Lemma 2.1, hence the system (4) has at least one ω-periodic solution. By (3), we prove that the system (2) has at least one positive ω-periodic solution. The proof is complete.

Next, we consider the following predator-prey systems with distributed delays

$$(20)\quad \begin{cases} \dfrac{dx(t)}{dt} = x(t)\left(r(t) - b(t)\displaystyle\int_{-\tau_1}^{0} x(t+\theta)\,d\mu(\theta)\right) - \dfrac{c_1(t)x(t)y(t)}{a_1y(t)+x(t)}; \\ \dfrac{dy}{dt} = y(t)\left[\dfrac{m_1(t)\displaystyle\int_{-\tau_2}^{0} x(t+\theta)\,d\mu(\theta)}{a_1\displaystyle\int_{-\tau_2}^{0} y(t+\theta)\,d\eta(\theta) + \displaystyle\int_{-\tau_2}^{0} x(t+\theta)\,d\mu(\theta)} - d_1(t) \right. \\ \qquad\qquad \left. - \dfrac{c_2(t)z(t)}{a_2z(t)+y(t)}\right]; \\ \dfrac{dz}{dt} = z(t)\left[\dfrac{m_2(t)\displaystyle\int_{-\tau_3}^{0} y(t+\theta)\,d\eta(\theta)}{a_2\displaystyle\int_{-\tau_3}^{0} z(t+\theta)\,d\upsilon(\theta) + \displaystyle\int_{-\tau_3}^{0} y(t+\theta)\,d\eta(\theta)} - d_2(t)\right]; \end{cases}$$

where τ_1, τ_2 and τ_3 are positive constants and $\mu, \eta\ \upsilon$ are nondecreasing functions such that

$$\mu(0^+) - \mu(-\tau_i^-) = 1;\ \eta(0^+) - \eta(-\tau_i^-) = 1;\ \upsilon(0^+) - \upsilon(-\tau_i^-);\ i = 1; 2; 3:$$

Theorem 2.2. *If*

$$a_1\bar{r} - \bar{c}_1 > 0;\ \bar{m}_1a_2 - \bar{d}_1a_2 - \bar{c}_2 > 0 \text{ and } \bar{m}_2 - \bar{d}_2 > 0;$$

then the system (20) has at least one positive ω-periodic solution.

Proof. The proof is similar to that of Theorem 2.1. Hence we omitted the proof.

Remark 2.1. From the proofs of Theorem 2.1, one can see that in (20) , even if some of the τ_1's, τ_2's and τ_3's or all of them are ∞; the conclusion of Theorem 2.2 remains true.

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