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ON SOME SPECIAL GENERALIZED FUNCTIONAL IDENTITIES

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Abstract. We consider some special generalized functional identities on a prime ring R, such that the existence of nonstandard solutions of these identities yields a more refined conclusion than just that R satisfies a generalized polynomial identity.

1. Introduction

Speaking in a loose manner we can say that a *generalized functional identity* (GFI) on a ring R is an identity holding for all elements in R which besides maps on R also involves some fixed elements. The usual goal of the study of (generalized) functional identities is to find the form of the maps involved, or, when this is not possible, to determine the structure of the ring. Usually we first search for the solutions of GFI's that do not depend on structual properties of the ring but are merely consequences of a formal calculation. We call them standard solutions. In case there exists a nonstandard solution, it turns out that the ring has a special structure. The study of GFI's was initiated in 1995 by M. Brešar [4]. The basic result about GFI's is a theorem due to M.A. Chebotar [10, Theorem 1.3]. It states that a GFI on a prime ring R consisting of expressions of the form

$$E(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)x_ia \quad \text{and} \quad bx_iF(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$$

(here E,F are arbitrary maps from $R\times\ldots\times R$ to R and a,b are fixed elements) has only standard solutions, unless R is a GPI ring (i.e. it satisfies a generalized polynomial identity). For more details we refer the reader to [10] and also to the survey [5].

While the theorem of Chebotar deals with a rather general class of GFI's, it is the purpose of this paper to study certain quite special GFI's, which, however, enable

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a stronger conclusion than the one obtained by Chebotar. We were motivated by the study of some special GPI's (see [3, Section 6.6] and also the more recent paper [2] and by resuts on some special functional identities [7,8].

The result of Brešar [4, Proposition 8] shows that the GFI

$$\sum_{i=1} F_i(x)ya_i + \sum_{j=1} b_j xG_j(y) = 0$$

has only standard solutions in arbitrary (GPI or non-GPI) prime rings. Note that here the indeterminates x and y always appear in the same order. Let us mention that this result has turned out to be applicable to the study of the so-called generalized derivations [11]. In Section 3 we shall give a shorter proof and a generalization of Brešar's result, using similar methods as developed in [1, 10].

By Chebotar's theorem a prime ring admitting a GFI with a nonstandard solution is GPI, and so, by a well-known Martindale's theorem, its central closure has nonzero socle. In Section 4 we shall see that in some special GFI's one can say more, namely, that the coefficients of these GFI's actually lie in the socle. In this context we also mention the recent paper [9] which contains some results in a similar direction.

There are various results on some special GPI's and functional identities (see [3, Theorem 6.6.2] and main theorems in [2], [7] and [8] where the conclusion is that the considered prime ring is GPI and its associated division algebra is a field, i.e. is of dimension 1 over the extended centroid. Given any positive integer n, we shall in Section 5 characterize via some special GFI those GPI prime rings that their associated division algebra is of dimension $\leq n^2$.

2. Preliminaries

In this section we introduce some notation and recall some basic facts from the theory of rings with generalized identities that will be used in the sequel without specific mention. For a complete account on this theory we refer the reader to the book of Beidar, Martindale and Mikhalev [3].

Throughout the paper R denotes a *prime ring*. As usually in the theory of (generalized) functional identities we will need some rings of quotients of R and other related notions. By C, $A = RC$, and Q_{ml} we denote the extended centroid, the central closure, and the maximal left ring of quotients of R, respectively. We refer to [3] for definitions and basic properties; let us just point out here that $R \subseteq A \subseteq Q_{ml}$, that C is a field defined as the center of Q_{ml}, and that A is the subring of Q_{ml} generated by R and C. One of the most useful properties of the extended centroid is the following well-known fact: If $a_i; b_i \in Q_{ml}$, $i = 1; \ldots; n$, are nonzero elements such that $\sum_{i=1}^{n} a_i x b_i = 0$ for all $x \in R$, then the a_i's are linearly dependent over C, and the b_i's are linearly dependent over C. This is essentially a result of Martindale [12, Theorem 2], but the version we stated follows for example from [3, Proposition 6.3.13].

Given $x \in Q_{ml}$ we denote by $\deg_C(x)$ the degree of x over C (if x is algebraic over C) or ∞ (if x is not algebraic over C). For any nonempty subset $S \subseteq Q_{ml}$ we define $\deg_C(S) = \sup\{\deg_C(x) \mid x \in S\}$. Using standard facts of the PI theory we see that $\deg_C(R) = \sqrt{\dim_C(A)}$ holds in case R is a PI ring.

The fundamental result of the GPI theory is due to Martindale [12] (see also [3, Theorem 6.1.6]. It states that a prime ring R is GPI if and only if A is a primitive ring with nonzero socle and for each minimal idempotent $e \in A$ the associated division algebra eAe is finite dimensional over C.

For a positive integer n and elements $x_1, x_2, \ldots, x_n \in R$ we write

$$\overline{x}_n = (x_1, x_2, \ldots, x_n) \in R^n,$$

where R^n denotes the Cartesian product of n copies of R. Further, for any $1 \le i \le n$ we define

$$\overline{x}_n^i = (x_1, \ldots, x_n)^i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in R^{n-1}.$$

3. GFI's with only Standard Solutions in all Prime Rings

Lemma 3.1. *Let $F : R \to Q_{ml}$ be an additive map and let $a \in Q_{ml}$ be a nonzero element. Assume that there is a dense left ideal L of R such that $La \subseteq R$ and $F(yax) = ayF(x)$ for all $x \in R$, $y \in L$. Then there exists a unique $q \in Q_{ml}$ such that $F(x) = axq$ for all $x \in R$.*

Lemma 3.1 is a slight modification of [1, Lemma 2.9]. More precisely, instead of taking the element a from R (as in [1, Lemma 2.9]), we take it from Q_{ml}. However, almost the same proof works in this more general situation, so we omit it.

Theorem 3.2. *Let R be a prime ring and let k, l, m, n be positive integers. Suppose that*

$$\sum_{i=1}^{k} F_i(\overline{x}_n) y_1 a_i^1 y_2 a_i^2 \ldots y_m a_i^m + \sum_{j=1}^{l} b_j^1 x_1 b_j^2 x_2 \ldots b_j^n x_n G_j(\overline{y}_m) = 0 \tag{1}$$

for all $\overline{x}_n \in R^n$, $\overline{y}_m \in R^m$, where $F_i : R^n \to Q_{ml}$, $G_j : R^m \to Q_{ml}$ are any maps and $a_i^\alpha, b_j^\beta \in Q_{ml}$ are nonzero elements, $i = 1, \ldots, k$, $j = 1, \ldots, l$, $\alpha = 1, \ldots, m$, $\beta = 1, \ldots, n$. If $\{a_1^m, \ldots, a_k^m\}$ and $\{b_1^1, \ldots, b_l^1\}$ are C-independent sets, then there exist elements $q_{ij} \in Q_{ml}$, $i = 1, \ldots, k$, $j = 1, \ldots, l$, such that

$$F_i(\overline{x}_n) = \sum_{j=1}^{l} b_j^1 x_1 b_j^2 x_2 \ldots b_j^n x_n q_{ij}, \tag{2}$$

$$G_j(\overline{y}_m) = -\sum_{i=1}^{k} q_{ij}\, y_1 a_i^1 y_2 a_i^2 \dots y_m a_i^m \tag{3}$$

for all $\overline{x}_n \in R^n$, $\overline{y}_m \in R^m$, $i = 1, \dots, k$, $j = 1, \dots, l$.

Proof. It is enough to show that only the maps G_j have the desired form. Namely, if G_j's are of the form (3), then a standard argument based on the C-independence of $\{a_1^m, \dots, a_k^m\}$ shows that F_i's must satisfy (2). Moreover, we fix some $s \in \{1, \dots, l\}$ and note that it suffices to show that G_s has the desired form.

By [3, Theorem 2.3.3] there exist elements $u_\gamma, v_\gamma \in R$, $\gamma = 1, \dots, N$, such that

$$\sum_{\gamma=1}^{N} u_\gamma b_s^1 v_\gamma = b \neq 0 \quad \text{and} \quad \sum_{\gamma=1}^{N} u_\gamma b_j^1 v_\gamma = 0 \quad \text{for all } j \neq s.$$

Let $E_i(\overline{x}_n) = \sum_{\gamma=1}^{N} u_\gamma F_i(v_\gamma x_1, x_2, \dots, x_n)$ for each i. Then

$$\begin{aligned}
&\sum_{i=1}^{k} E_i(\overline{x}_n) y_1 a_i^1 y_2 a_i^2 \dots y_m a_i^m \\
&= \sum_{i=1}^{k} \sum_{\gamma=1}^{N} u_\gamma F_i(v_\gamma x_1, x_2, \dots, x_n) y_1 a_i^1 y_2 a_i^2 \dots y_m a_i^m \\
&= \sum_{\gamma=1}^{N} u_\gamma \left(\sum_{i=1}^{k} F_i(v_\gamma x_1, x_2, \dots, x_n) y_1 a_i^1 y_2 a_i^2 \dots y_m a_i^m \right) \\
&= -\sum_{\gamma=1}^{N} u_\gamma \left(\sum_{j=1}^{l} b_j^1 v_\gamma x_1 b_j^2 x_2 \dots b_j^n x_n G_j(\overline{y}_m) \right) \\
&= -\sum_{j=1}^{l} \left(\sum_{\gamma=1}^{N} u_\gamma b_j^1 v_\gamma \right) x_1 b_j^2 x_2 \dots b_j^n x_n G_j(\overline{y}_m) \\
&= -b x_1 b_s^2 x_2 \dots b_s^n x_n G_s(\overline{y}_m).
\end{aligned}$$

So we have

$$\begin{aligned}
&H(x_1, \dots, x_n, y_1, \dots, y_m) \\
&\sum_{i=1}^{k} E_i(\overline{x}_n) y_1 a_i^1 y_2 a_i^2 \dots y_m a_i^m + b x_1 b_s^2 x_2 \dots b_s^n x_n G_s(\overline{y}_m) = 0
\end{aligned} \tag{4}$$

for all $x_i, y_i \in R$.

We proceed by induction on n. If $n = 1$, we have

$$(5)\qquad H(x_1; y_1; \ldots; y_m) = \sum_{i=1}^{k} E_i(x_1)y_1a_i^1y_2a_i^2\ldots y_ma_i^m + bx_1G_s(\overline{y}_m) = 0$$

for all $x_1; y_1; \ldots; y_m \in R$: Let L be a dense left ideal of R such that $Lb \subseteq R$. Pick any $z \in L$. Then

$$\begin{aligned} 0 &= H(zbx_1; y_1; \ldots; y_m) - bzH(x_1; y_1; \ldots; y_m) \\ &= \sum_{i=1}^{k} (E_i(zbx_1) - bzE_i(x_1))\, y_1a_i^1y_2a_i^2\ldots y_ma_i^m \end{aligned}$$

for all $x_1; y_1; \ldots; y_m \in R$, and so, by the same standard argument as at the beginning of the proof, it follows that $E_i(zbx_1) = bzE_i(x_1)$ for all $x_1 \in R$, $z \in L$, $i = 1; \ldots; k$. According to Lemma 3.1 there exist uniquely determined elements $q_{is} \in Q_{ml}$, $i = 1; \ldots; k$; such that $E_i(x_1) = bx_1q_{is}$. Now (5) readily implies that G_s is of the form (3). So the proof is complete in the case when $n = 1$.

Now let $n > 1$. Fix $x_2; \ldots; x_n \in R$ in (4). Then

$$\sum_{i=1}^{k} \tilde{E}_i(x_1)y_1a_i^1y_2a_i^2\ldots y_ma_i^m + bx_1\tilde{G}_s(\overline{y}_m) = 0$$

for all $x_1; y_1; \ldots; y_m \in R$, where $\tilde{E}_i(x_1) = E_i(\overline{x}_n)$ and $\tilde{G}_s(\overline{y}_m) = b_s^2x_2\ldots b_s^nx_nG_s(\overline{y}_m)$. By the argument above there exist $p_{is} = p_{is}(x_2; \ldots; x_n) \in Q_{ml}$, $i = 1; \ldots; k$, such that $\tilde{E}_i(x_1) = bx_1p_{is}$. Thus

$$\sum_{i=1}^{k} p_{is}(x_2; \ldots; x_n)y_1a_i^1y_2a_i^2\ldots y_ma_i^m + b_s^2x_2\ldots b_s^nx_nG_s(\overline{y}_m) = 0$$

for all $x_2; \ldots; x_n \in R$, $\overline{y}_m \in R^m$. Now, the induction hypothesis yields $G_s(\overline{y}_m) = -\sum_{i=1}^{k} q_{is}y_1a_i^1y_2a_i^2\ldots y_ma_i^m$ for some $q_{is} \in Q_{ml}$, $i = 1; \ldots; k$. ■

4. GFI's and the Socle

Lemma 4.1. *Suppose there exist nonzero maps $f; g : R \to A$ and nonzero elements $a; b \in A$ such that*

$$(6)\qquad f(x)ya = g(y)xb$$

for all $x; y \in R$. Then R is GPI, the associated division algebra of A is a field, and both elements $a; b$ belong to the same minimal left ideal of A.

Proof. Pick a nonzero ideal I of R such that $bI \subseteq R$, and let $z \in I$. Replacing x by xbz in (6) and using (6) once again we obtain

$$f(xbz)ya = g(y)xbzb = f(x)yazb$$

for all $x, y \in R$. Since $f \neq 0$ it follows that a and azb are linearly dependent over C for each $z \in I$. Take $u \in I$ such that $ub \neq 0$. Then $0 \neq aRub \subseteq aIb \subseteq Ca$. Consequently, $aAub = Ca$ and so $a \in aAub$. Whence we see that $aAa = Ca$. We claim that Aa is a minimal left ideal. Namely, if there was a nonzero left ideal L of A such that $L \subseteq Aa$, there would exist $s, t \in A$ such that $sa, ta \in L$ and $sata \neq 0$. Therefore, $ata \neq 0$ and hence $ata = \lambda a$ for some nonzero $\lambda \in C$. This would yield

$$Aa = A\left(\lambda^{-1}ata\right) \subseteq Ata \subseteq L \subseteq Aa,$$

showing that $L = Aa$. Thus our claim is true and so there exists a minimal idempotent $e \in A$ such that $Aa = Ae$ [3, Proposition 4.3.3]. In particular, $e = va$ for some $v \in A$ which readily implies $eAe = Ce$. Therefore, R is a GPI ring and the associated division algebra is a field. Clearly, $a \in aAa \subseteq Aa = Ae$. Multiplying (6) by $1 - e$ we get $g(y)x(b - be) = 0$ for all $x, y \in R$. Accordingly, $b = be \in Ae$. ■

Lemma 4.1 in particular shows that a and b lie in $\mathrm{soc}(A)$, the socle of A. We now extend this conclusion to any number of variables.

Theorem 4.2. *Let R be a prime ring and let $n \geq 2$ be an integer. Suppose that $E_1, \dots, E_n : R^{n-1} \to A$ and $a_1, \dots, a_n \in A$ are such that*

$$E_1(\overline{x}_n^1)x_1a_1 + E_2(\overline{x}_n^2)x_2a_2 + \dots + E_n(\overline{x}_n^n)x_na_n = 0 \tag{7}$$

for all $\overline{x}_n \in R^n$. If $E_1 \neq 0$ and $a_1 \neq 0$, then R is a GPI ring and $a_1 \in \mathrm{soc}(A)$.

Proof. By Chebotar's result [10, Theorem 2.6] R is a GPI ring. We use the induction on n to prove that $a_1 \in \mathrm{soc}(A)$. Since $E_1 \neq 0$, $a_1 \neq 0$ and R is prime, there is no loss of generality in assuming that also $E_n \neq 0$ and $a_n \neq 0$. If $n = 2$ the result follows from Lemma 4.1, so let $n > 2$. Since A is a primitive ring with nonzero socle, by [3, Theorem 4.3.7(ii)] there exists a minimal idempotent $e \in Aa_n$. Let $u \in A$ be such that $e = ua_n$. There is a nonzero ideal I of R such that $Iu \subseteq R$. Since $Ie \neq 0$, we can pick $b \in I$ such that $be \neq 0$ and hence $bu \neq 0$. Replacing x_n by x_nbu in (7) we get

$$\sum_{i=1}^{n-1} E_i\left(x_1, \dots, x_{n-1}, x_nbu\right)^i x_ia_i + E_n(\overline{x}_n^n)x_nbe = 0 \tag{8}$$

for all $\overline{x}_n \in R^n$. Since $E_n \neq 0$ and $be \neq 0$, for some $1 \leq s \leq n-1$ the map $\overline{x}_n \mapsto E_s(x_1, \dots, x_{n-1}, x_n bu)^s$ is nonzero. Fix $c_n \in R$ such that the map $\overline{x}_{n-1} \mapsto E_s(x_1, \dots, x_{n-1}, c_n bu)^s$ is also nonzero. Multiplying (8) from the right by $1-e$ and setting $x_n = c_n$ gives us

$$\sum_{i=1}^{n-1} E_i\,(x_1, \dots, x_{n-1}, c_n bu)^i\, x_i\,(a_i - a_i e) = 0 \tag{9}$$

for all $\overline{x}_{n-1} \in R^{n-1}$. By the induction assumption it follows that $a_s - a_s e \in \mathrm{soc}(A)$, and hence $a_s \in \mathrm{soc}(A)$. We may therefore assume that $s \neq 1$, say $s = 2$ with no loss of generality. According to Litoff's theorem [3, Theorem 4.3.11] there exists an idempotent $f \in \mathrm{soc}(A)$ such that $a_2 = f a_2 f$. Multiplying (7) from the right by $1-f$ gives us

$$E_1(\overline{x}_n^1)x_1\,(a_1 - a_1 f) + E_3(\overline{x}_n^3)x_3\,(a_3 - a_3 f) + \dots + E_n(\overline{x}_n^n)x_n\,(a_n - a_n f) = 0$$

for all $x_1, x_3, \dots, x_n$. Since E_1 is a nonzero map, there is $c \in R$ such that the map $(x_3, \dots, x_n) \mapsto E_1(c, x_3, \dots, x_n)$ is still nonzero. Setting $x_2 = c$ in the last identity and using the induction hypothesis it follows that $a_1 - a_1 f \in \mathrm{soc}(A)$, and hence also $a_1 \in \mathrm{soc}(A)$. The proof is thereby complete. ∎

A simple example illustrating Lemma 4.1 is given by the identity $(exbe)ye = (eye)xbe$ which holds true for any minimal idempotent e such that $eRe = Ce$ and any $b \in R$. In Theorem 4.2 one certainly cannot conclude in general neither that the associated division algebra is a field nor that a_1 lies in some minimal left ideal. Indeed, if R is any PI ring, then it satisfies some multilinear polynomial identity of the smallest possible degree, which can be interpreted as the functional identity

$$E_1(\overline{x}_n^1)x_1 + E_2(\overline{x}_n^2)x_2 + \dots + E_n(\overline{x}_n^n)x_n = 0$$

for all $\overline{x}_n \in R^n$, where E_1 (a "polynomial" of degree $n-1$) is a nonzero map. Note that this is just a very special case of (7) with $a_i = 1$ for each i. Now take for example $R = M_n(D)$, the ring of $n \times n$ matrices (with $n \geq 2$) over a noncommutative division algebra finite dimensional over its center. Then the associated division algebra of R is D and is not a field, and 1 of course does not lie in any minimal left ideal of $A = R$.

5. A Result on Traces

Let F be an n-additive map on R^n, i.e. the map that is additive in each argument. The map q on R defined by $q(x) = F(x, \dots, x)$ is called the *trace* of F.

Theorem 5.1. *Let R be a prime ring, let n be a positive integer and assume that char$(R) = 0$ or char$(R) > n$. Further, let $e \neq 0, 1$ be an idempotent in A. Then the following two conditions are equivalent:*

(i) There exists a nonzero trace $q : R \to A$ of an n-additive map such that $q(x)xe = 0$ for all $x \in R$;

(ii) $\dim_C(eAe) \leq n^2$.

Proof. Suppose that q is a nonzero trace of an n-additive map $F : R^n \to A$ such that $q(x)xe = 0$ for all $x \in R$. The complete linearization of this identity gives us

$$\sum_{\sigma\in S_{n+1}} F\left(x_{\sigma(1)},\ldots,x_{\sigma(n)}\right)x_{\sigma(n+1)}e = 0$$

for all $\bar{x}_{n+1} \in R^{n+1}$. Here $\mathcal{S}_n$ denotes the symmetric group of order n. Let $E(\bar{x}_n) = \sum_{\sigma\in S_n} F\left(x_{\sigma(1)},\ldots,x_{\sigma(n)}\right)$. Clearly, E is a symmetric n-additive mapping satisfying

$$\sum_{i=1}^{n+1} E\left(\bar{x}^i_{n+1}\right)x_ie = 0 \tag{10}$$

for all $\bar{x}_{n+1} \in R^{n+1}$. Moreover, $E \neq 0$ in view of the characteristic assumption. Let us pick a nonzero ideal I of R such that $eI, Ie \subseteq R$. Fix $y \in I$. Replacing x_1 by x_1ey in (10) we get

$$E\left(\bar{x}^1_{n+1}\right)x_1eye + \sum_{i=2}^{n+1} E(x_1ey, x_2,\ldots,x_{n+1})^i x_ie = 0.$$

Using (10) once again we obtain

$$-\sum_{i=2}^{n+1} E\left(\bar{x}^i_{n+1}\right)x_ieye + \sum_{i=2}^{n+1} E(x_1ey, x_2,\ldots,x_{n+1})^i x_ie = 0. \tag{11}$$

Next, replacing x_2 by x_2ey in (11) gives us

$$\begin{aligned}
&- E\left(\bar{x}^2_{n+1}\right)x_2eyeye + E(x_1ey, x_3,\ldots,x_{n+1})\,x_2eye\\
&- \sum_{i=3}^{n+1} E(x_1, x_2ey, x_3,\ldots,x_{n+1})^i x_ieye\\
&+ \sum_{i=3}^{n+1} E(x_1ey, x_2ey, x_3,\ldots,x_{n+1})^i x_ie = 0;
\end{aligned}$$

which can be according to (11) written as

$$\begin{aligned}
&\sum_{i=3}^{n+1} E\left(\overline{x}^i_{n+1}\right) x_i eyeye - \sum_{i=3}^{n+1} E\,(x_1ey, x_2, \ldots, x_{n+1})^i\, x_i eye \\
-&\sum_{i=3}^{n+1} E\,(x_1, x_2ey, x_3, \ldots, x_{n+1})^i\, x_i eye \\
+&\sum_{i=3}^{n+1} E\,(x_1ey, x_2ey, x_3, \ldots, x_{n+1})^i\, x_i e = 0.
\end{aligned}$$

After further $n-2$ repetitions of this procedure we obtain

$$\begin{aligned}
&(-1)^n E\,(\overline{x}_n)\, x_{n+1}(eye)^n \\
+\ &(-1)^{n-1} \Big(E(x_1ey, x_2, \ldots, x_n) + \cdots + E\,(x_1, \ldots, x_{n-1}, x_n ey) \Big)\, x_{n+1}(eye)^{n-1} \\
+\ &\cdots - \Big(E\,(x_1, x_2ey, \ldots, x_n ey) + \cdots + E\,(x_1ey, \ldots, x_{n-1}ey, x_n) \Big)\, x_{n+1}eye \\
+\ &E\,(x_1ey, x_2ey, \ldots, x_n ey)\, x_{n+1}e = 0
\end{aligned}$$

for all $\overline{x}_{n+1} \in R^{n+1}$. Since we can choose $\overline{x}_n \in R^n$ such that $E\,(\overline{x}_n) \neq 0$, it follows that $e, eye, \ldots, (eye)^n$ are C-dependent for each $y \in I$. Consequently, $C_{2n+1}\,(e, eye, \ldots, (eye)^n, x_1, \ldots, x_n) = 0$ for all $y \in I$ and $x_1, \ldots, x_n \in I$, where C_{2n+1} is the so-called Capelli polynomial defined by

$$C_{2n+1}(x_1, \ldots, x_{2n+1}) = \sum_{\sigma \in S_{n+1}} \varepsilon(\sigma) x_{\sigma(1)} x_{n+2} x_{\sigma(2)} x_{n+3} \cdots x_{\sigma(n)} x_{2n+1} x_{\sigma(n+1)},$$

here $\varepsilon(\sigma)$ is the sign of the permutation $\sigma \in \mathcal{S}_{n+1}$ (see e.g. [3, Theorem 2.3.7]). Recall that I and A satisfy the same generalized polynomial identities (this follows from [3, Proposition 2.1.10 and Theorem 6.4.1]. Thus, $C_{2n+1}(e, eye, \ldots, (eye)^n, x_1, \ldots, x_n) = 0$ for all $y \in A$ and $x_1, \ldots, x_n \in R$. Again using [3, Theorem 2.3.7] we see that $e, eye, \ldots, (eye)^n$ are C-dependent for each $y \in A$. This means that $\deg_C(eAe) \leq n$, and hence $\dim_C(eAe) \leq n^2$.

It remains to prove that **(ii)** implies **(i)**. Suppose $\dim_C(eAe) = k^2$, where $1 \leq k \leq n$. Similarly as above, by making use of the Capelli polynomial, we see that $\deg_C(eAe) = \deg_C(eRe) = k$. By [6, Lemma 3.3] there exist maps $\tau_i : eAe \to C$, $i = 1, \ldots, k-1$, such that

$$(exe)^k + \tau_1(exe)\,(exe)^{k-1} + \tau_2\,(exe)\,(exe)^{k-2} + \cdots + \tau_{k-1}\,(exe)\, exe \in Ce$$

for all $x \in A$, and each τ_i is the trace of an i-additive map.

We claim that there is $x_0 \in R$ such that $(ex_0e)^k + \lambda_1(ex_0e)(ex_0e)^{k-1} + \dots + \lambda_{k-1}(ex_0e)ex_0e = \lambda e$ for some nonzero $\lambda \in C$ (in particular, ex_0e is therefore invertible in eAe). Indeed, if this was not true, then we would have $q'(x)xe = 0$ for all $x \in R$, where $q'(x) = (exe)^{k-1} + \lambda_1(exe)(exe)^{k-2} + \dots + \lambda_{k-1}(exe)e$ is the trace of a $(k-1)$-additive map and so, by the first part of the proof either $q' = 0$ or $\dim_C(eAe) \le (k-1)^2$. However, $q' = 0$ implies $\deg_C(eAe) \le k-1$ which in turn yields $\dim_C(eAe) \le (k-1)^2$, so we arrive at a contradiction in any case.

Further, pick $b \in R$ such that $eb(1-e) \neq 0$, and define $q'' : R \to A$ by

$$\begin{aligned} q''(x) &= \left((exe)^k + \lambda_1(exe)(exe)^{k-1} + \dots + \lambda_{k-1}(exe)exe\right) eb \\ &\quad - ebxe\left((exe)^{k-1} + \lambda_1(exe)(exe)^{k-2} + \dots + \lambda_{k-1}(exe)e\right). \end{aligned}$$

Clearly, $q''(x)xe = 0$ for all $x \in R$. Defining $q : R \to A$ by

$$q(x) = (exe)^{n-k} q''(x)$$

we see that q is the trace of an n-additive map, $q(x)xe = 0$ for all $x \in R$, and $q \neq 0$. Namely, if q was a zero map, then we would have $(ex_0e)^{n-k}q''(x_0) = 0$, and so $0 = (ex_0e)^{n-k}q''(x_0)(1-e) = (ex_0e)^{n-k}\lambda eb(1-e)$, which contradicts the invertibility of ex_0e. ∎

Theorem 5.2. *Let R be a prime ring, let n be a positive integer and assume that char$(R) = 0$ or char$(R) > n$. Then the following two conditions are equivalent:*

(i) *There exist a nonzero $a \in A$ and a nonzero trace $q : R \to A$ of an n-additive map such that $q(x)xa = 0$ for all $x \in R$;*

(ii) *R is a GPI ring, A is not a division algebra, and the associated division algebra of A is at most n^2-dimensional over C.*

Proof. Suppose that q is a nonzero trace of an n-additive map $F : R^n \to A$ such that $q(x)xa = 0$ for all $x \in R$. Similarly as in the proof of Theorem 5.1 we obtain

$$\sum_{i=1}^{n+1} E\left(\bar{x}^i_{n+1}\right) x_i a = 0$$

for all $\bar{x}_{n+1} \in R^{n+1}$, where $E : \bar{x}_n \mapsto \sum_{\pi \in S_n} F\left(x_{\pi(1)}, \dots, x_{\pi(n)}\right)$ is a nonzero symmetric n-additive map. Applying [10, Theorem 2.6] it follows that R is a GPI ring. Consequently, A is a primitive ring with nonzero socle, which further implies that the right ideal aA contains a minimal idempotent e [3, Theorem 4.3.7]. So, we have $q(x)xe = 0$ for all $x \in R$. Note that $e \neq 0; 1$, since e is minimal and q is nonzero. Thus, using Theorem 5.1 we see that $\dim_C(eAe) \leq n^2$. Obviously, A can not be a division algebra.

Let us prove that **(ii)** implies **(i)**. Since R is GPI there exists a minimal idempotent $e \in A$ such that eAe is a finite dimensional division algebra over C. Note that $e \neq 1$ since A is not a division algebra. Now apply Theorem 5.1. ∎

We conclude the paper by remarking that according to Theorem 4.2 the element a from Theorem 5.2 lies in $\mathrm{soc}(A)$.

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