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RATE OF CONVERGENCE BY THE BEZIER VARIANT OF PHILLIPS OPERATORS FOR BOUNDED VARIATION FUNCTIONS

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Abstract. In the present paper, we introduce the Bezier variant of Phillips operators and study the rate of convergence for the Phillips-Bezier operators for bounded variation functions.

1. Introduction

The Phillips operators [6] are defined by

$$
\begin{aligned}
S_\lambda(f;t) &= \int_0^\infty e^{-\lambda(t+u)} \left(\sum_{m=1}^\infty \frac{(\lambda^2 t)^m u^{m-1}}{m!(m-1)!} \right) f(u)du + e^{-\lambda t} f(0) \\
&= \int_0^\infty W(\lambda;t;u) f(u)du;
\end{aligned}
\tag{1.1}
$$

where

$$
W(\lambda;t;u) = e^{-\lambda(t+u)} \left(\sum_{m=1}^\infty \frac{(\lambda^2 t)^m u^{m-1}}{m!(m-1)!} + \delta(u) \right)
$$

with $\delta(u)$ being the Dirac delta function. Alternatively, the operators (1.1) may be written as follows:

$$
S_\lambda(f;t) = \lambda \sum_{m=1}^\infty p_{\lambda;m}(t) \int_0^\infty p_{\lambda;m-1}(u) f(u)du + e^{-\lambda t} f(0) \qquad t \in [0;\infty)
$$

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where $p_{\lambda,m}(t) = e^{-\lambda t}\frac{(\lambda t)^m}{m!}$.

The Phillips operators defined by (1.1) are similar to the modified Szasz- Mirakyan operators studied in [3,4]. Some approximation properties of the Phillips operators were discussed in [1], [2] and [5]. Now we introduce the Bezier variant of these Phillips operators. For $\alpha \geq 1$ and for a function f defined on $[0,\infty)$, the Bezier variant of the Phillips operator is defined as

$$S_{\lambda,\alpha}(f;t) = \lambda \sum_{m=1}^{\infty} Q^{(\alpha)}_{\lambda,m}(t) \int_0^{\infty} p_{\lambda,m-1}(u) f(u)du + Q^{(\alpha)}_{\lambda,0}(t) f(0); \tag{1.2}$$

where $Q^{(\alpha)}_{\lambda,m}(t) = J^{\alpha}_{\lambda,m}(t) - J^{\alpha}_{\lambda,m+1}(t)$; $J_{\lambda,m}(t) = \sum_{j=m}^{\infty} p_{\lambda,j}(t)$. These generalized operators (1.2) may also be written in the alternative form as follows:

$$S_{\lambda,\alpha}(f;t) = \int_0^{\infty} W_{\alpha}(\lambda;t;u) f(u)du; \tag{1.3}$$

where

$$W_{\alpha}(\lambda;t;u) = \lambda \sum_{m=1}^{\infty} Q^{(\alpha)}_{\lambda,m}(t) p_{\lambda,m-1}(u) + Q^{(\alpha)}_{\lambda,0}(t) Q^{(\alpha)}_{\lambda,0}(u)\delta(u)$$

$\delta(u)$ being the Dirac delta function.

Obviously, the operators $S_{\lambda,\alpha}(f;t)$ defined by (1.2) are linear positive operators. Particularly when $\alpha = 1$, the operators (1.2) reduce to the Phillips operators $S_{\lambda,1} \equiv S_{\lambda}$. Also $S_{\lambda,\alpha}(1;t) = 1$. Some basic properties of $J_{\lambda,m}$ are as follows:

(i.) $J_{\lambda,m}(t) - J_{\lambda,m+1}(t) = p_{\lambda,m}(t)$; $m = 0,1,2,\ldots$

(ii). $J'_{\lambda,m}(t) = \lambda p_{\lambda,m-1}(t)$; $m = 1,2,3,\ldots$

(iii). $J_{\lambda,m}(t) = \lambda \int_0^t p_{\lambda,m-1}(u)du$; $m = 1,2,3\ldots$

(iv). $\sum_{m=1}^{\infty} J_{\lambda,m}(t) = \lambda \int_0^t \sum_{m=1}^{\infty} p_{\lambda,m-1}(u)du = \lambda t$

(v). for every $m \in N$, $0 \leq J_{\lambda,m}(t) < 1$ and $J_{\lambda,m}(t)$ increases strictly on $[0,\infty)$.

In the present paper, we study the rate of convergence of the Bezier variant of Phillips operators, defined by (1.2), for bounded variation functions.

2. Auxiliary Results

In this section we give certain lemmas which are needed by us to prove our main result (Theorem 1 below).

Lemma 1. [7, p. 159]. *If* $\{\xi_m\}(m \geq 1)$ *are independent random variables with the same distribution and* $0 < D\xi_m < \infty; \beta_3 = E|\xi_1 - a_1|^3 < \infty;$ *then*

$$\max\left|P\left(\frac{1}{b_1\sqrt{\lambda}}\sum_{m=1}(\xi_m - a_1) \leq y\right) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-u^2/2}du\right| < \frac{C\beta_3}{b_1^3\sqrt{\lambda}};$$

where $a_1 = E(\xi_1)$ (*expectation of* ξ_1), $b_1^2 = D\xi_1 = E(\xi_1 - a_1)^2$ (*variance of* ξ_1) *and* $1/\sqrt{2\pi} \leq C < 0.82.$

It is well known that the basis function $p_{\lambda,m}(t)$ corresponds with the Poisson distribution in the probability theory. Using Lemma 1, Gupta and Pant [3] obtained the inequality

$$p_{\lambda,m}(t) \leq \frac{32t^2 + 2t + 5}{2\sqrt{\lambda t}}; t \in (0;\infty).$$

In Lemma 3 of [9] it is proved that $p_{\lambda,m}(t) \leq \frac{1}{2\sqrt{\pi\lambda t}}$. Recently, Zeng and Zhao [10] improved these results and obtained the exact bound as follows:

Lemma 2. [10]. *Let j be a fixed non negative integer and* $H(j) = \frac{(j+1/2)^{j+1/2}}{j!}$ $e^{-(j+1/2)}$ *Then, for all* m, t *such that* $m \geq j$ *and* $t \in (0;\infty)$, *there holds*

$$p_{\lambda,m}(t) \leq \frac{H(j)}{\sqrt{t\lambda}}.$$

Moreover, the coefficient $H(j)$ *and the asymptotic order* $\lambda^{-1/2}$ (*for* $\lambda \to \infty$) *are the best possible.*

Lemma 3. [2]. *Let the function* $\mu_{\lambda,r}(t); r \in N^0$ (*the set of non negative integers*) *be defined by* $\mu_{\lambda,r}(t) = S_\lambda((u-t)^r; t)$: *Then* $\mu_{\lambda,0}(t) = 1; \mu_{\lambda,1}(t) = 0$ *and* $\mu_{\lambda,2}(t) = \frac{2t}{\lambda}$; *and the following recurrence relation holds:*

$$\mu_{\lambda,r+1}(t) = \frac{2t}{\lambda}\mu_{\lambda,r}(t) + \frac{t}{\lambda^2}\mu^{(2)}_{\lambda,r}(t) + \frac{2tr}{\lambda}\mu_{\lambda,r-1}(t) + \frac{tr(r-1)}{\lambda^2}\mu_{\lambda,r-2}(t) + \frac{2tr}{\lambda^2}\mu^{(2)}_{\lambda,r-1}(t)$$

Furthermore

(i:) $\mu_{\lambda,r}(t)$ *is a polynomial in* t *and* $1/\lambda$ *for every* $t \in [0,\infty)$.

(ii:) $\mu_{\lambda,r}(t) = O(\lambda^{-[(r+1)/2]})$, *for every* $t \in [0,\infty)$.

Lemma 4. *For the kernel* $W_\alpha(\lambda,t,u)$ *of the operators* $S_{\lambda,\alpha}$, *we have*

$$\int_0^y W_\alpha(\lambda,t,u)du \le \frac{2\alpha t}{\lambda(t-y)^2} \quad 0 \le y < t; \tag{2.1}$$

$$\int_z^\infty W_\alpha(\lambda,t,u)du \le \frac{2\alpha t}{\lambda(z-t)^2} \quad t < z < \infty. \tag{2.2}$$

Proof. We first prove (2.1). Clearly

$$\begin{aligned}\int_0^y W_\alpha(\lambda,t,u)du &\le \int_0^y \frac{(t-u)^2}{(t-y)^2} W_\alpha(\lambda,t,u)du \\ &= \frac{1}{(t-y)^2} S_{\lambda,\alpha}((u-t)^2,t) \le \alpha(t-y)^{-2}\mu_{\lambda,2}(t).\end{aligned}$$

Using Lemma 3, (2.1) follows. The proof of (2.2) is similar.

Lemma 5. *For all* $t \in (0,\infty)$ *and* $\lambda, m \in N$, *there holds*

$$J^\alpha_{\lambda,m}(y)p_{\lambda,m}(t) \le Q^{(\alpha)}_{\lambda,m}(t) \le \alpha p_{\lambda,m}(t) \le \frac{\alpha H(j)}{\sqrt{\lambda t}}.$$

3. Main Theorem

In this section we prove the following main theorem.

Theorem 1. *Let* f *be a function of bounded variation on every finite subinterval of* $[0,\infty)$ *and let* $V_a^b(g_t)$ *be the total variation of* g_t *on* $[a,b]$. *Also let* $\alpha \ge 1$ *and* $f(t) = O(t^r), t \to \infty$, *for some* $r > 0$. *Then, for every* $t \in (0,\infty)$ *and* λ *sufficiently large, we have*

$$\begin{aligned}\left|S_{\lambda,\alpha}(f,t) - \left\{\frac{1}{\alpha+1}f(t+) + \frac{\alpha}{\alpha+1}f(t-)\right\}\right| &\le |f(t+) - f(t-)|\frac{\alpha H(j)}{\sqrt{\lambda t}} \\ &+ \frac{4\alpha+t}{\lambda t}\sum_{m=1} V_{t-t/\sqrt{m}}^{t+t/\sqrt{m}}(g_t) + O(\lambda^{-r})\end{aligned} \tag{3.1}$$

where

$$g_t(u) = \begin{cases} f(u) - f(t-); & 0 \le u < t \\ 0; & u = t \\ f(u) - f(t+); & t < u < \infty \end{cases}$$

Proof. Following [8, eq. (28)], we have

$$\left| S_{\lambda,\alpha}(f;t) - \left\{ \frac{1}{\alpha+1} f(t+) + \frac{\alpha}{\alpha+1} f(t-) \right\} \right| \le |S_{\lambda,\alpha}(g_t;t)| + \frac{1}{2} \left| S_{\lambda,\alpha}(\text{sign}(u-t);t) + \frac{\alpha-1}{\alpha+1} \right| |f(t+) - f(t-)| \tag{3.2}$$

First,

$$\begin{aligned} S_{\lambda,\alpha}(\text{sign}(u-t);t) &= \int_t^\infty W_\alpha(\lambda,t,u)du - \int_0^t W_\alpha(\lambda,t,u)du \\ &= -1 + 2\int_t^\infty W_\alpha(\lambda,t,u)du \end{aligned}$$

Now using the fact that $\lambda \int_t^\infty p_{\lambda,m}(u)du = \sum_{j=0}^{m} p_{\lambda,j}(t)$ for $t \in (0,\infty)$, we have

$$\begin{aligned} S_{\lambda,\alpha}(\text{sign}(u-t);t) &= -1 + 2\lambda \sum_{m=1}^{\infty} Q^{(\alpha)}_{\lambda,m}(t) \int_t^\infty p_{\lambda,m-1}(u)du \\ &\quad + \int_t^\infty Q^{(\alpha)}_{\lambda,0}(t) Q^{(\alpha)}_{\lambda,0}(u)\delta(u)du \\ &= -1 + 2\sum_{m=1}^{\infty} Q^{(\alpha)}_{\lambda,m}(t) \sum_{j=0}^{m-1} p_{\lambda,j}(t); \ \delta(u) = 0 \text{ as } t > 0 \\ &= -1 + 2\sum_{j=0}^{\infty} p_{\lambda,j}(t) \sum_{m=j}^{\infty} Q^{(\alpha)}_{\lambda,m}(t) = -1 + 2\sum_{j=0}^{\infty} p_{\lambda,j}(t) J^{\alpha}_{\lambda,j}(t) \end{aligned}$$

Thus we have

$$S_{\lambda,\alpha}(\text{sign}(u-t);t) + \frac{\alpha-1}{\alpha+1} = 2\sum_{j=0}^{\infty} p_{\lambda,j}(t) J^{\alpha}_{\lambda,j}(t) - \frac{2}{\alpha+1} \sum_{j=0}^{\infty} Q^{(\alpha+1)}_{\lambda,j}(t)$$

Since $\sum_{j=0}^{\infty} Q^{(\alpha+1)}_{\lambda,j}(t) = 1$, by mean value theorem, it follows that

$$\begin{aligned} Q^{(\alpha+1)}_{\lambda,j}(t) &= J^{\alpha+1}_{\lambda,j}(t) - J^{\alpha+1}_{\lambda,j+1}(t) \\ &= (\alpha+1)\, p_{\lambda,j}(t)\, \gamma^{\alpha}_{\lambda,j}(t) \end{aligned}$$

where $J^{\alpha}_{\lambda,j+1}(t) < \gamma^{\alpha}_{\lambda,j}(t) < J^{\alpha}_{\lambda,j}(t)$. Therefore ,

$$\begin{aligned}\left|S_{\lambda,\alpha}(\mathrm{sign}(u-t);t)+\frac{\alpha-1}{\alpha+1}\right| &= 2\sum_{j=0}^{\infty} p_{\lambda,j}(t)(J^{\alpha}_{\lambda,j}(t)-\gamma^{\alpha}_{\lambda,j}(t))\\ &\le 2\sum_{j=0}^{\infty} p_{\lambda,j}(t)(J^{\alpha}_{\lambda,j}(t)-J^{\alpha}_{\lambda,j+1}(t)) \le 2\alpha\sum_{j=0}^{\infty} p^{2}_{\lambda,j}(t);\end{aligned}$$

where we have used the inequality $b^{\alpha}-a^{\alpha} < \alpha(b-a); 0\le a;b\le 1$ and $\alpha\ge 1$. Applying Lemma 2, we get

(3.3) $$\left|S_{\lambda,\alpha}(\mathrm{sign}(u-t);t)+\frac{\alpha-1}{\alpha+1}\right| = \frac{\alpha H(j)}{\sqrt{\lambda t}};\ t\in(0;\infty):$$

Next we estimate $S_{\lambda,\alpha}(g_t;t)$ as follows:

(3.4) $$\begin{aligned}S_{\lambda,\alpha}(g_t;t) &= \int_0^{\infty} W_{\alpha}(\lambda;t;u)g_t(u)du = \left(\int_{I_1}+\int_{I_2}+\int_{I_3}\right) W_{\alpha}(\lambda;t;u)g_t(u)du\\ &= E_1+E_2+E_3 \ \text{say};\end{aligned}$$

where $I_1=[0;t-t/\sqrt{\lambda}]$, $I_2=[t-t/\sqrt{\lambda};t+t/\sqrt{\lambda}]$ and $I_3=[t+t/\sqrt{\lambda};\infty)$. We start with the estimate of E_2. For $u\in[t-t/\sqrt{\lambda};t+t/\sqrt{\lambda}]$, we have

(3.5) $$|E_2| \le V_{t-t/\sqrt{\lambda}}^{t+t/\sqrt{\lambda}}(g_t) \le \frac{1}{\lambda}\sum_{m=1} V_{t-t/\sqrt{m}}^{t+t/\sqrt{m}}(g_t):$$

Suppose $\beta_{\lambda,\alpha}(t;y)=\int_0^{y} W_{\alpha}(\lambda;t;u)du$. We now estimate E_1 , writing $y=t-t/\sqrt{\lambda}$ and using Lebesgue-Stieltjes integration by parts, we have

$$E_1=\int_0^{y} g_t(u)d_u(\beta_{\lambda}(t;y)) = g_t(y+)\beta_{\lambda,\alpha}(t;y)-\int_0^{y}\beta_{\lambda,\alpha}(t;u)d_u(g_t(u))$$

Using Eq. (2.1) of Lemma 4, we have

(3.6) $$|E_1|\le V_{y+}^{t}(g_t)\frac{2\alpha t}{\lambda(t-y)^2}+\frac{2\alpha t}{\lambda}\int_0^{y}\frac{1}{(t-u)^2}d_u(-V_u^{t}(g_t)) \le \frac{4\alpha}{\lambda t}\sum_{m=1} V_{t-t/\sqrt{m}}^{t}(g_t)$$

Finally, we estimate E_3 . We define

$$g_t(u)=\begin{cases} g_t(u); 0\le t<2t\\ g_t(2u); 2t<u<\infty\end{cases}$$

and decompose E_3 into two parts as follows:

$$
\text{(3.7)}\qquad \begin{aligned} E_3 &= \int_{t+t/\sqrt{\lambda}}^{\infty} W_{\alpha}(\lambda;t;u)\,g_t(u)\,du + \int_{2t}^{\infty} W_{\alpha}(\lambda;t;u)[g_t(u)-g_t(2t)]\,du \\ &= E_{31} + E_{32},\ \text{say}. \end{aligned}
$$

With $z = t + t/\sqrt{\lambda}$, the first integral can be written in the form

$$
E_{31} = \lim_{R\to\infty}\left\{ g_t(z)[1-\beta_{\lambda,\alpha}(t;z)] + g_t(R)[\beta_{\lambda,\alpha}(t;R)-1] + \int_z^R [1-\beta_{\lambda,\alpha}(t;u)]\,d_u g_t(u) \right\}.
$$

By (2.1) of Lemma 4, we have

$$
|E_{31}| \le \frac{2\alpha t}{\lambda}\lim_{R\to\infty}\left\{ \frac{V_t^z(g_t)}{(z-t)^2} + \frac{|g_t(R)|}{(R-t)^2} + \int_z^R \frac{1}{(u-t)^2}\,d_u(V_t^u(g_t)) \right\}.
$$

Integrating by parts the last term, we obtain

$$
\text{(3.8)}\qquad |E_{31}| \le \frac{4\alpha}{\lambda t}\sum_{m=1} V_t^{t+t/\sqrt{m}}(g_t)
$$

To estimate E_{32}, by our assumption there exists an integer r such that $f(u) = O(u^{2r}),\ u\to\infty$. Thus, for a certain constant M depending on f, t and r, we have

$$
|E_{32}| \le M\int_{2t}^{\infty} W_{\alpha}(\lambda;t;u)\,u^{2r}\,du \le \alpha M\int_{2t}^{\infty} W(\lambda;t;u)\,u^{2r}\,du,
$$

where we have used Lemma 5. Obviously, $u \ge 2t$ implies that $u \le 2(u-t)$ and it follows by Lemma 3 that

$$
\text{(3.9)}\qquad |E_{32}| \le \alpha 2^{2r} M \mu_{\lambda,2r}(t) = O(\lambda^{-r}),\ \lambda\to\infty.
$$

Collecting the estimates of (3.2) to (3.9), we get the required result.

This completes the proof of the theorem.

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