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ON P_4-DECOMPOSITION OF GRAPHS

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Abstract. A graph G is decomposable into subgraphs G_1, G_2, $\ldots$, G_n of G if no G_i $(i = 1, 2, \ldots, n)$ has isolated vertices and the edge set $E(G)$ can be partitioned into the subsets $E(G_1)$, $E(G_2)$, $\ldots$, $E(G_n)$. If $G_i \cong P_4$ for all i, then G is called P_4-decomposable. In this paper, we show the P_4-decomposability of some classes of graphs, and prove in particular that a complete r-partite graph is P_4-decomposable if and only if its size is a multiple of 3. We also give an example of a 2-connected graph of size $3k$ which is not P_4-decomposable, disproving a conjecture of Chartrand.

1. Introduction

In this paper we only consider simple graphs. A graph G is said to be *H-decomposable*, denoted by $H|G$, if $E(G)$ can be partitioned into subgraphs such that each subgraph is isomorphic to H. Such a factorization is called an *isomorphic factorization*. The concept of isomorphic factorization was studied by F. Harary et al. [4]. In this paper we consider a conjecture of Chartrand et al. [3] that a 2-connected graph of order $p \geq 4$ and size $q \equiv 0$ (mod 3) is P_4-decomposable. We prove the conjecture for certain 2-connected graphs. We also show by an example that it is not true in general.

We follow standard notations in graph theory. The cardinality of the vertex set of a graph G, the *order* of G is denoted by $p(G)$; and the cardinality of the edge set of G, the size of G is denoted by $q(G)$.

Theorem 1. *$K_{m,n}$ is P_4-decomposable if and only if $m \geq 2$, $n \geq 2$ and $mn \equiv 0$ (mod 3).*

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Proof. As the conditions are clearly necessary, we only need to prove the sufficiency. Without loss of generality, we may assume that $m = 3r$. Write $n = 2s+3t$ with $s \geq 0$ and $t \geq 0$. Then $K_{m,n}$ can be decomposed into rs copies of $K_{2,3}$ and rt copies of $K_{3,3}$. It is easily verified that $K_{2,3}$ and $K_{3,3}$ are P_4-decomposable. Hence $K_{m,n}$ is P_4-decomposable.

Theorem 2. *If G_1, G_2 and K_{m_1,m_2} are H-decomposable, where $m_1 = p(G_1)$ and $m_2 = p(G_2)$, then $G_1 + G_2$ is H-decomposable.*

Proof. As $E(G_1 + G_2)$ is equal to the edge-disjoint union $E(G_1) \bigcup E(G_2) \bigcup E(K_{m_1,m_2})$, we have that $G_1 + G_2$ is H-decomposable.

Theorem 3. *If G_1 and G_2 are P_4-decomposable graphs and $p(G_1)$ or $p(G_2)$ is a multiple of 3, then $G_1 + G_2$ is P_4-decomposable.*

Proof. Let $p(G_1) = m$ and $p(G_2) = n$. Then $K_{m,n}$ is P_4-decomposable by Theorem 1. By Theorem 2, $G_1 + G_2$ is P_4-decomposable.

Theorem 4. *If $G_1,\ G_2, \ldots, G_n$ are P_4-decomposable graphs and $p(G_i) \equiv 0$ (mod 3) for $i = 1, 2, \ldots, n$, then $G_1 + G_2 + \ldots + G_n$ is P_4-decomposable.*

Proof. The theorem holds for the case $n = 2$ by Theorem 3. The general case follows from an induction on n, as $G_1 + G_2 + \ldots + G_n \cong (G_1 + G_2) + \ldots + G_n$.

Lemma 5. *If K_r and $K_{r,r}$ are H-decomposable, then K_{nr} is H-decomposable for any positive integer n.*

Proof. We prove the lemma by induction on n. When $n = 1$, K_r is H-decomposable by the assumption. Assume the lemma is true for $n = m - 1 \geq 1$. We prove that the lemma is true for $n = m$. Notice that $K_{mr} = K_{(m-1)r+r}$ and $E(K_{(m-1)r+r}) = E(K_{(m-1)r}) \bigcup E(K_r) \bigcup E(K_{(m-1)r,r})$. By the induction hypothesis, $K_{(m-1)r}$ is H-decomposable. As $K_{(m-1)r,r}$ can be decomposed into $m - 1$ copies of $K_{r,r}$, we have that $K_{(m-1)r,r}$ is H-decomposable. Thus K_{mr} is H-decomposable. These prove the lemma.

Using Theorems 1 and 2 and Lemma 5, we have the following propositions.

Proposition 6. *When $n \geq 2$ and $m \geq 1$, $K_{3n} + P_{3m+1}$ is P_4-decomposable.*

Proposition 7. *When $n \geq 2$ and $m \geq 2$, $K_{3n} + C_{3m}$ is P_4-decomposable.*

Proposition 8. *When $n \geq 2$ and $m \geq 2$, $C_{3n} + C_{3m}$ is P_4-decomposable.*

Theorem 9. *K_n is P_4-decomposable if and only if $n > 3$ and $n \not\equiv 2$ (mod 3).*

Proof. Clearly K_n is not P_4-decomposable for $n \le 3$. Also, if $n \equiv 2$ (mod 3), then $q(K_n) = \frac{n(n-1)}{2}$ is not divisible by 3 and hence K_n is not P_4-decomposable.

It can be easily verified that K_4 is P_4-decomposable. So, let n be an integer such that $n \not\equiv 2$ (mod 3) and $n \ge 6$.

Case 1. $n \equiv 0$ (mod 3).

When n is odd, K_n is decomposable into $\frac{n-1}{2}$ Hamiltonian cycles each of which is P_4-decomposable. It is also easily verified that K_6 is P_4-decomposable. Notice that $E(K_{6r}) = E(K_{6(r-1)}) \bigcup E(K_6) \bigcup E(K_{6(r-1),6})$. It follows from an induction on r that K_{6r} is P_4-decomposable.

Case 2. $n \equiv 1$ (mod 3), say $n = 3k + 1$.

We first show that K_7 is P_4-decomposable. Let the vertices of K_7 be v_0, v_1, v_2, v_3, v_4, v_5, v_6. Then K_7 can be decomposed into 3 Hamiltonian cycles as follows:

$$C^1 : v_0, v_1, v_2, v_6, v_3, v_5, v_4, v_0;$$

$$C^2 : v_0, v_2, v_3, v_1, v_4, v_6, v_5, v_0;$$

$$C^3 : v_0, v_3, v_4, v_2, v_5, v_1, v_6, v_0.$$

The edges $\{v_4, v_0\}$ from C^1, $\{v_0, v_2\}$ from C^2 and $\{v_2, v_5\}$ from C^3 form a path P_4. The other edges of C^1, C^2, C^3 form 2 paths P_4 each. Hence K_7 is P_4-decomposable. Notice that $E(K_{3k+1}) = E(K_{3(k-1)}) \bigcup E(K_4) \bigcup E(K_{3(k-1),4})$ and each of the graphs on the right is P_4-decomposable if $k \ge 3$. Hence K_{3k+1} is P_4-decomposable for all integers $k \ge 1$.

These complete the proof of the theorem.

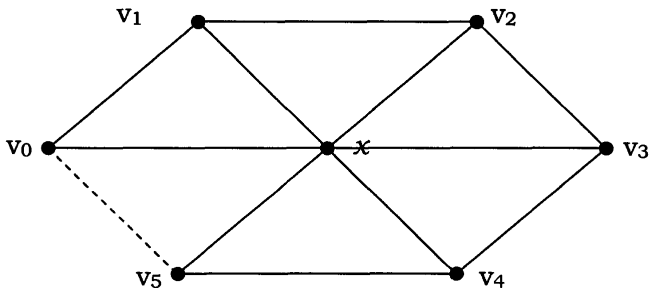
Theorem 10. *If $n \equiv 2$ (mod 3) and $n > 4$, then $K_n - e$ is P_4-decomposable.*

Proof. It can be easily verified that $K_5 - e$ is P_4-decomposable. For $n > 5$, we have $E(K_n - e) = E(K_{2,n-2}) \bigcup E(K_{n-2})$. Since $n - 2 \equiv 0$ (mod 3), $K_{2,n-2}$ is P_4-decomposable by Theorem 1, and K_{n-2} is P_4-decomposable by Theorem 9. The theorem then follows.

Proposition 11. *If $n \equiv 0$ (mod 3), then $K_{2n} - F$ is P_4-decomposable, where F is a 1-factor of K_{2n}.*

Proof. The proposition follows from the fact that $K_{2n} - F$ can be decomposed into $n - 1$ Hamiltonian cycles, each of which is P_4-decomposable.

Proposition 12. *A wheel W_n is P_4-decomposable if and only if $n \equiv 0$ (mod 3).*



Proof. The condition is clearly necessary.

Conversely, suppose that $n \equiv 0$ (mod 3). It is clear that $W_n = C_n + K_1$. Let C_n be the cycle $v_0, v_1, v_2, \ldots, v_{n-1}, v_0$ and $K_1 = x$. Notice that $q(W_n) = 2n \equiv 0$ (mod 3). It is a routine to check that $E(W_n)$ can be decomposed into $\frac{2n}{3}$ P_4's of the form $x\ v_{0+3i}, v_{1+3i}, v_{2+3i}$, and $v_{1+3i}, xv_{2+3i}, v_{3+3i}$, where $0 \leq i = \leq \frac{n}{3} - 1$ and $v_n = v_0$.

Theorem 13. *Let G be a complete tripartite graph K_{m_1,m_2,m_3}. Then G is P_4-decomposable if and only if $q(G) \equiv 0$ (mod 3) and $q(G) > 3$.*

Proof. We only need to prove the sufficiency. Since $q(G) \equiv 0$ (mod 3), there are three possibilities:

1. $m_i \equiv 1$ (mod 3) for all i;
2. $m_i \equiv 2$ (mod 3) for all i;
3. $m_i \equiv 0$ (mod 3) for at least two i.

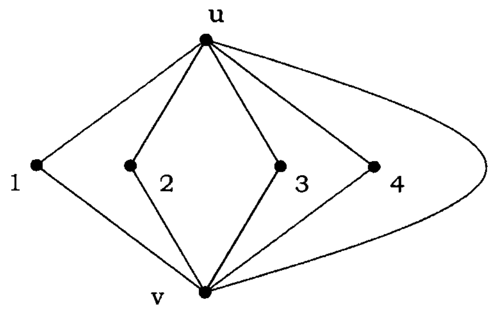
Case 1. $m_i \equiv 1$ (mod 3) for i = 1, 2, 3.

Let $m_1 = 3a + 1$, $m_2 = 3b + 1$ and $m_3 = 3c + 1$. Notice that

$$E(K_{3a+1,3b+1,3c+1}) = E(K_{3a,3b+1+3c+1}) \bigcup E(K_{1,3b+1,3c+1}),$$
$$E(K_{1,3b+1,3c+1}) = E(K_{3b,1+3c+1}) \bigcup E(K_{1,1,3c+1}),$$
$$E(K_{1,1,3c+1}) = E(K_{3(c-1),2}) \bigcup E(K_{1,1,4}).$$

By Theorem 1, $K_{3a,3b+1+3c+1}$, $K_{3b,1+3c+1}$ and $K_{3(c-1),2}$ are P_4-decomposable. [If $c = 1$, $K_{3(c-1),2}$ is a null graph.]

A P_4-decomposition of $K_{1,1,4}$ is shown below.



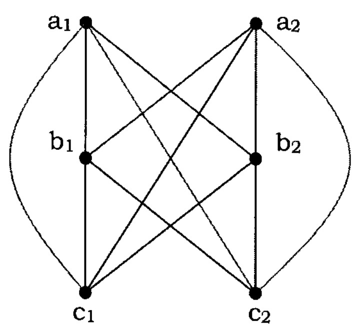
$(1,u,v,4)$, $(1,v,2,u)$, $(v,3,u,4)$ is a P_4-decomposition of $K_{1,1,4}$. Thus K_{m_1,m_2,m_3} is P_4-decomposable when $m_i \equiv 1$ (mod 3) for i = 1, 2, 3.

Case 2. $m_i \equiv 2$ (mod 3) for i = 1, 2, 3.

Let $m_1 = 3a+2$, $m_2 = 3b+2$ and $m_3 = 3c+2$. Notice that

$$E(K_{3a+2,3b+2,3c+2}) = E(K_{3a,3b+2+3c+2}) \bigcup E(K_{2,3b+2,3c+2}),$$
$$E(K_{2,3b+2,3c+2}) = E(K_{3b,2+3c+2}) \bigcup E(K_{2,2,3c+2}),$$
$$E(K_{2,2,3c+2}) = E(K_{3c,2+2}) \bigcup E(K_{2,2,2}).$$

By Theorem 1, $K_{3a,3b+2+3c+2}$, $K_{3b,2+3c+2}$ and $K_{3c,4}$ are P_4-decomposable. A P_4-decomposition of $K_{2,2,2}$ is shown below.



$a_1\ b_2\ a_2\ b_1$; $b_1\ c_2\ b_2\ c_1$; $a_2\ c_2\ a_1\ c_1$; $a_1\ b_1\ c_1\ a_2$ is a P_4-decomposition of $K_{2,2,2}$.

Thus K_{m_1,m_2,m_3} is P_4-decomposable when $m_i \equiv 2$ (mod 3) for i = 1, 2, 3.

Case 3. $m_i \equiv 0$ (mod 3) for at least two i.

Subcase 3.1. $m_1,\ m_2 \equiv 0$ (mod 3) and $m_3 \neq 1$.

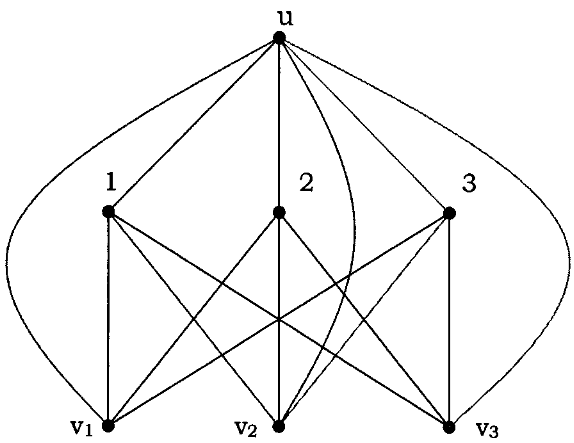
Notice that $E(K_{m_1,m_2,m_3}) = E(K_{m_1,m_2+m_3}) \bigcup E(K_{m_2,m_3})$. K_{m_1,m_2+m_3} and K_{m_2,m_3} are complete bipartite graphs of size a multiple of 3 and hence they are P_4-decomposable. Thus K_{m_1,m_2,m_3} is P_4-decomposable.

Subcase 3.2. $m_1, m_2 \equiv 0$ (*mod* 3) *and* $m_3 = 1$.

Let $m_1 = 3a$ and $m_2 = 3b$. Notice that

$$E(K_{1,3a,3b}) = E(K_{3(a-1),1+3b}) \bigcup E(K_{1,3,3b}),$$
$$E(K_{1,3,3b}) = E(K_{3(b-1),1+3}) \bigcup E(K_{1,3,3}).$$

By Theorem1, $K_{3(a-1),1+3b}$ and $K_{3(b-1),1+3}$ are P_4-decomposable. A P_4-decomposition of $K_{1,3,3}$ is shown below.



$K_{1,3,3}$ can be decomposed into 3 paths P_4: v_1, u, v_2, 1; 1, u, 2, v_2; v_2, 3, u, v_3; and the remaining edges form $K_{3,2}$ which is P_4-decomposable.

Thus K_{m_1,m_2,m_3} is P_4-decomposable.

Observation 14. Let $G = K_{3n_1+r_1,3n_2+r_2,\dots,3n_k+r_k}$, where $r_1, r_2, \dots, r_k$ are positive integers and $n_1, n_2, \dots, n_k$ are non-negative integers. Then $E(G) = E(K_{3n_1,3(n_2+\dots+n_k)+(r_2+\dots+r_k)}) \bigcup E(K_{r_1,3n_2+r_2,\dots,3n_k+r_k})$. A similar argument as in Case 2 of Theorem 13 shows that G is P_4-decomposable if $K_{r_1,r_2,\dots,r_k}$ is P_4-decomposable. This observation is repeatedly used in the proof of the next theorem.

Theorem 15. *Let G be the graph $K_{m_1,m_2,\dots,m_r}$ with $r \geq 4$. Then G is P_4-decomposable if and only if $q(G) \equiv 0$ (mod 3).*

Proof. It is enough to prove the sufficiency of the condition. We prove the result by induction on $q = q(G)$. When $q = 6$, $G = K_4$ is P_4-decomposable. When $q = 9$, $G = K_{2,1,1,1}$ is also P_4-decomposable.

Case 1. At least one $m_i \equiv 0$ (mod 3).

Let $m_1 \equiv 0$ (mod 3). Then $E(G) = E(K_{m_1, m_2+m_3+\ldots+m_r}) \bigcup E(K_{m_2, m_3, \ldots, m_r})$. Let G_1 be the graph $K_{m_1, m_2+m_3+\ldots+m_r}$ and G_2 be the graph $K_{m_2, m_3, \ldots, m_r}$. G_1 is P_4-decomposable by Theorem 1. If $r > 4$, G_2 is P_4-decomposable by the induction hypothesis. If $r = 4$, except in the case $m_2 = m_3 = m_4 = 1$, it follows by Theorem 13 that G_2 is P_4-decomposable. So, let us assume $r = 4$, $m_2 = m_3 = m_4 = 1$. It is easily verified that $K_{3,1,1,1}$ is P_4-decomposable. If $n > 1$, $E(K_{3n,1,1,1}) = E(K_{3,1,1,1}) \bigcup E(K_{3(n-1),3})$ and it can be proved by induction on n that $K_{3n,1,1,1}$ is P_4-decomposable for all $n \geq 4$. Thus G_2 and hence G is P_4-decomposable in this case.

Case 2. At least three of the m_i's $\equiv 2$ (mod 3).

Let $m_1 \equiv m_2 \equiv m_3 = 2$ (mod 3). Hence $m_1 + m_2 + m_3 \equiv 0$ (mod 3) and $E(G) = E(K_{m_1, m_2, m_3}) \bigcup E(K_{m_1+m_2+m_3, m_4, m_5, \ldots, m_r})$. By Theorem 13, the first graph on the right side is P_4-decomposable. Again, as in Case 1, if either $r \geq 5$, or if $r = 4, m_4 > 1$, the second graph on the right side is P_4-decomposable. Hence we may assume $r = 4$ and $m_4 = 1$. $K_{2,2,2,1}$ is easily verified to be P_4-decomposable. Hence, by Observation 14, G is P_4-decomposable.

Case 3. Exactly two m_i's = 2 (mod 3).

In this case, it can be verified that $q(G) \not\equiv 0$ (mod 3).

Case 4. Exactly one $m_i \equiv 2$ (mod 3).

In this case $r \equiv 1$ (mod 3). $K_{2,1,1,\ldots,1} \cong K_{r+1} - e$ is P_4-decomposable when $r + 1 \equiv 2$ (mod 3), by Theorem 10. Hence by Observation 14, G is P_4-decomposable.

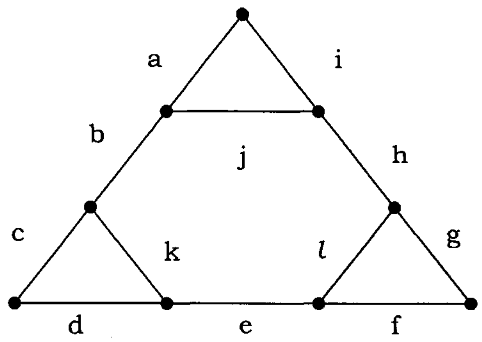
Case 5. No $m_i \equiv 2$ (mod 3). That is, all m_i'$s \equiv 1$ (mod 3).

In this case $r \equiv 0$ or 1 (mod 3). $K_{1,1,1,\ldots,1}$ is P_4-decomposable, by Theorem 9. Hence by Observation 14, G is P_4-decomposable.

These complete the proof of the theorem.

Conjecture (Chartrand et al. [3]). If G is a 2-connected graph of order $p \geq 4$ and size $q \equiv 0$ (mod 3), then G is P_4-decomposable.

The following example shows that this conjecture is not true.



It is easy to see that every P_4 must contain at least one of the edges b, e and h. Since the graph is of size 12, it cannot be decomposed into 4 edge-disjoint paths P_4.

Conjecture. Every 3-connected graph of size $q \equiv 0 \pmod 3$, is P_4-decomposable.

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