White Noise Analysis: Part I. Theory in Progress

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Abstract. In the past quarter century white noise theory has made great progress. We are further given many interesting and important questions which should be discussed in line with white noise analysis. On the other hand, some of the new directions are before our eyes and they require new analytic methods to be investigated, by generalizing the method established so far. It is now the time to remind the original idea of our stochastic analysis and to think of further directions of this theory which are going to appear in front of us.

While we are reviewing the results, we have recognized the foundation of the theory should be refined. This will be done also in Part I.

The plan of Part II is as follows. From what have been established, there naturally arise several new topics to be discussed. This is, in a sense, a continuation of Part I. On the other hand, having suggested by many significant examples and problems in applications, we come to a framework of the new white noise theory which is going to be constructed. A significant development is to grow out of L^2-theory and to establish an analysis on a wider class of suitable functionals of sample paths. This approach will be proposed.

Part I. White Noise Analysis: Theory in Progress

I.1. INTRODUCTION

First, a very quick review of the analysis of (Gaussian) white noise functionals is presented. Such a review serves to summarize the established theory, as well as to give an interpretation to our basic idea behind the known results, and it helps to prepare some background towards the proposal *A Frontier of White Noise Analysis*.

When the white noise theory is discussed, functionals in question are defined on the measure space (E^*, μ) of white noise, where E^* is a space of generalized

functions on R (more precisely, E^* is the dual space of some nuclear space E which is dense in $L^2(R)$) and μ is the standard Gaussian measure on E^*, called the *white noise measure*, which is uniquely determined by the characteristic functional $C(\xi)$ on E:

$$C(\xi) = \exp[-\frac{1}{2}\|\xi\|^2].$$

The Hilbert space $(L^2) = L^2(E^*, \mu)$ is used to be taken as the start line of the white noise theory. To see the idea of such an approach, some naive explanation would be necessary. It is known that many interesting random functions and stochastic processes are expressed as functions of Gaussian random variables. The most elemental (or atomic) system of Gaussian random variables is to be independent and identically distributed. If one is allowed to take a relaxed view of mathematical rigor, then one would consider the time derivative $\dot{B}(t)$ of a Brownian motion $B(t)$ as a realization of such an idealized system, since it has stationary independent increments. The $\dot{B}(t)$ is a *white noise*, often called Gaussian white noise to distinguish from other noises. Note that the parameter t stands for the time, so the use of $\dot{B}(t)$ is fitting for expressing the time development of random phenomena.

Unfortunately, the sample functions of white noise are not ordinary functions, but generalized functions. Hence, the probability distribution of a white noise is a Gaussian measure, indeed the standard Gaussian measure, introduced on the space of generalized functions, e.g. the space E^* defined above. Although for the intuitive or visualized forms of random phenomena we often use $\dot{B}(t)$ as the variables, we can immediately come to mathematically rigorous discussion by switching to μ, (L^2) and so on.

Our aim is now to investigate *random evolutional complex systems* in question by expressing them mathematically in terms of (L^2)-functionals and by using the modern functional analysis applied to those functionals, and to apply the results to various problems. Needless to say, we shall introduce various new concepts and methods of analysis depending on the new questions.

In short, the following steps should be in order, for our study :

Reduction $\longrightarrow$ Synthesis $\longrightarrow$ Analysis

This is exactly what we are going to follow.

The basic direction is that we first construct the *innovation* (the concept of which will be prescribed later), of the given random system, as the step of reduction, then functionals of the obtained innovation are chosen to express the original random complex system, and finally those functionals are to be analyzed.

The innovation can often be formed by the variational calculus for the given random evolutional phenomena, and sometimes, like in the communication theory, it is given in advance. Our attention will be focussed towards the construction of the innovation and on the analysis of functionals of the innovations; namely, the

cases where the innovation can actually be constructed are more attractive.

The essential part of the analysis comes from the *white noise theory*, which provides the main route of the analysis of functionals of the innovation. Naturally, the analysis is infinite dimensional. Note that our analysis has a viewpoint of harmonic analysis arising from the infinite dimensional rotation group. The topic will be passed to the references [10], [24], and others.

Further step to be followed is the application of the analysis. Many applications to quantum dynamics have been known; now application to Bio-Science would be the most fruitful area, although only some part of it will be studied in what follows. There is also an area of application in mathematical finance, however we will not go into this field.

I.2. Background

This section is devoted to some more interpretation of the three steps of our approach proposed in the last section.

Reduction.

To fix the idea let us observe the case where the random complex system is taken to be a stochastic process $X(t)$ parameterized by $t \in R$ or its sub-interval. Lévy's *stochastic infinitesimal equation* for a stochastic process $X(t)$ is expressed in the form

$$\delta X(t) = \Phi(X(s), s \leq t, Y(t), t, dt),$$

where $\delta X(t)$ stands for the variation of $X(t)$ for the infinitesimal time interval $[t, t + dt)$, the Φ is a sure functional and the $Y(t)$ is the *innovation*. Intuitively speaking, the innovation is a system such that the $Y(t)$ contains the same information as what is newly gained by the $X(t)$ during the infinitesimal time interval $[t, t+dt)$. A mathematically rigorous definition of the innovation will be given later. See [2].

Once such a stochastic infinitesimal equation is obtained, then the pair $(\Phi, Y(t))$ can completely characterize the probabilistic structure of the given process $X(t)$. Under some mild assumptions, the innovation may be considered as the time derivative of an additive process $Z(t)$. It may be assumed that the additive process has no fixed discontinuity and is continuous in probability. Further assume that it has stationary independent increments. Tacitly, it can be assumed that there is no non-random part. Then, $Z(t)$ is certainly a *Lévy process*, so that the Lévy decomposition of such a process asserts that

$$Z(t) = X_0(t) + X_1(t),$$

where $X_0(t)$ is Gaussian, in fact, a Brownian motion up to constant factor, and where $X_1(t)$ is a compound Poisson process involving mutually independent Poisson

processes with various jumps. Note that we need a trick, which is essential, to have a superposition of those component processes.

Thus, a Brownian motion and Poisson processes with various jumps are all *elemental* additive processes, and their time derivatives are white noise and Poisson noises, respectively. Those noises are all elemental, so that we may call each of them a system of *idealized elemental random variables* (abb. i.e.r.v.).

The next step is to form a function of the i.e.r.v.'s, so that the function can well represent the given random complex system mathematically. It is nothing but the step called

Synthesis

There, it is expected that the function is parameterized by the time variable t or space time variable, or even a manifold like contour, so that we can discuss propagation of the random system as time (or space time etc.) goes by. For the purpose to have mathematical expression of the system, we are required to introduce *generalized functionals* of the i.e.r.v.'s.

We now pause to explain why a generalized functional has to be introduced. To be concrete, take the $\dot{B}(t)$'s which are variables of function to be discussed. One may ask what are the most elemental functions when variables are given. Certainly, the answer is that polynomials in the given variables are most elemental. Take the simplest, say $\dot{B}(t)^2$. As was mentioned before, it is the square of a generalized function of t, this is not permitted to define in the ordinary calculus. Now the idea has come up. The quantity is infinite, still random. Formally speaking, the expectation is infinite, may be written as $\frac{1}{dt}$. Let us subtract off this expectation from $\dot{B}(t)^2$, so to speak, apply a sort of renormalization (additive renormalization). Then, we were lucky to get a reasonable quantity, which is random and not ordinary variable, namely a *generalized white noise functional*.

Here, we do not want to tell the history of renormalization, but we claim that this method can be done rigorously and generally by using the S-transform, which is, intuitively speaking, an infinite dimensional analogue of the Laplace transform. The collection of those renormalized functionals forms a space of generalized white noise functionals. The space is an extension of the Hilbert space (L^2). There are, of course, various such spaces defined according to the purposes.

Then, naturally follows the analysis of functions which have been formed in our setup. Needless to say, we are required to established an infinite dimensional stochastic analysis. Thus the goal of our study has therefore to be the

Analysis

With such wider space of generalized white noise functionals, we can deal with functionals where $\dot{B}(t)$'s appear explicitly, so that it is easier to see the time

development of the random phenomena represented by those functionals. This can be done by using the unitary operator and its generalization, and as a result we are given various applications, in particular the place where propagation and causality are involved.

Our analysis of the functionals should be established in order to identify the random complex system (dynamical system) in question. Since the functionals with which we are concerned are random, being different from ordinary (non-random) case, new tools of the analysis should be introduced. In particular, the differential operators in the variables of i.e.r.v.'s and integral operators together with the ordinary differential operator in the time variable t, are all basic tools of the analysis. Thus, our calculus is really more complicated compared with the ordinary differential and integral calculus.

Beside the analysis, some more interpretation involving a short historical note is necessary to those three steps listed above.

The first step of taking suitable system of i.e.r.v.'s has been influenced by the way how to understand the notion of a *stochastic process*. We therefore have to remind the meaning of a stochastic process starting from the idea of J. Bernoulli (Ars Conjectandi, 1713) and S. Bernstein (1933). Shortly after, P. Lévy gave the definition of a stochastic process (1947), where we are suggested to consider the *innovation* of a stochastic process, as we have seen in the stochastic infinitesimal equation in a formal expression due to him.

Taking what have been explained so far, one can see the idea of our analysis of white noise functionals. Now, many significant characteristics are listed below.

(1) Essentially infinite dimensional.

There generalized white noise functionals are naturally introduced and they are analyzed systematically by taking a white noise as a system of i.e.r.v.'s to express the given random complex systems. Indeed, the analysis is infinite dimensional and does involve important parts that can not be approximated by finite dimensional calculus, as will be seen in what follows.

(2) Infinite dimensional harmonic analysis

The white noise measure supported by the space E^* of generalized functions on the parameter space R^d is invariant under the rotations of E^*. Hence a harmonic analysis arising from the group will naturally be discussed. The group contains significant subgroups which describes essentially infinite dimensional characters.

(3) Intimate connection with functional analysis

The so-called S-transform (will be defined later) applied to white noise functionals provides a bridge connecting white noise functionals and classical functionals of ordinary functions. We can therefore appeal to the theory of functionals established in the first half of the twentieth century, of course in a modernized fashion.

(4) Application

Good applications can be found in the places as long as there is fluctuation involved, e.g. in quantum dynamics, molecular biology and mathematical finance. We shall be able to show examples to which white noise theory is efficiently applied, according as the theory successively develops.

Having had great contribution by many authors, the theory has developed in the direction mentioned above and has become the present stage:
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There is one thing that can be mentioned for the future directions. As a generalization of the stochastic infinitesimal equation for $X(t)$, one can introduce a *stochastic variational equation* for random field $X(C)$ parameterized by a smooth ovaloid C:

$$\delta X(C) = \Phi(X(C'), C' < C, Y(s), s \in C, C, \delta C),$$

where $C' < C$ means that C' is in the inside of C. The system $\{Y(s), s \in C\}$ is the innovation which is understood in the similar sense to the case of $X(t)$.

The symbol $\delta X(C)$ means the variation of $X(C)$, which is the infinitesimal change of $X(C)$ defined in a suitable manner, when the parameter C changes slightly as far as $C + \delta C$ located near C. These intuitive notions will be rigorously defined in Part II so that we can go further.

The two equations for $\delta X(t)$ and for $\delta X(C)$ above have only a formal significance, however we can give rigorous meaning to the equations with some additional assumptions and the interpretations to the notations introduced there. What have been explained so far are, of course, far from the general theory, however one is given a guideline of the probabilistic approach to those random complex evolutional systems in line with the innovation theory and hence with the white noise theory.

The details of the innovation will be discussed in Part II, however one thing to be noted here is that as in the case of $X(t)$ we can consider elemental random fields, or equivalently elemental noises with multi-dimensional parameter. It should be noted that the method of restricting the parameter to a lower dimensional space contains profound probabilistic properties, and hence it is not an easy problem.

I.3 Gaussian Systems

I.3.1. Gaussian processes

First we discuss a Gaussian process $X(t), t \in T$, T being an interval of R^1, say $T = [0, \infty)$ or the entire R. Assume that $X(t)$ is separable and has no remote past,

and that $E(X(t)) = 0$. Then, the innovation which is to be a white noise $\dot{B}(t)$, or a system of independent white noises, at most countably many white noises, that can be constructed explicitly. The original idea of such an approach to Gaussian processes was proposed by the P. Lévy's 1956 paper (Proceedings of the third Berkeley Symposium). There are many literatures (see Part II in [1] and [9]) in this direction.

Let $\mathbf{B}_t(X)$ be the sigma-fields generated by the random variables $X(s), s \leq t,$.

Theorem 3.1. *Under the assumption that the process has unit multiplicity and other assumptions noted above, the Gaussian process $X(t), t \geq 0$, has the unique innovation $\dot{B}(t)$ which is a white noise such that*

$$X(t) = \int_0^t F(t, u)\dot{B}(u)du,$$

where $F(t, u)$ is a sure (non-random) kernel function and where

$$\mathbf{B}_t(X) = \mathbf{B}_t(\dot{B})$$

hols for every t.

This is the so-called the *canonical representation* of $X(t)$, and $F(t, u)$ is the *canonical kernel*. This might seem to be rather elementary, however such an understanding is not correct. Profound structure behind this formula would lead us to a deep insight as we shall see later.

On the other hand, take a Brownian motion and a kernel function $G(t, u)$ of Volterra type. And we are given a Gaussian process $Z(t)$ expressed as a stochastic integral

$$Z(t) = \int_0^t G(t, u)\dot{B}(u)du.$$

Assume that $G(t, u)$ is smooth on the domain $0 \leq u \leq t < \infty$ with $G(t, t) \neq 0$ and that $Z(t)$ has the unit multiplicity. Then, $Z(t)$ is not differentiable and we have

Theorem 3.2. *The variation $\delta Z(t)$ of the Gaussian process $Z(t)$ is defined and is given by*

$$\delta Z(t) = G(t, t)\dot{B}(t)dt + dt \int_0^t G_t(t, u)\dot{B}(u)du,$$

where $G_t(t, u) = \frac{\partial}{\partial t}G(t, u)$. The $\dot{B}(t)$ is the innovation of $Z(t)$ if and only if $G(t, u)$ is the canonical kernel.

Proof. The formula for the variation of $Z(t)$ is obtained easily. If G is not a canonical kernel, then the sigma field $\mathbf{B}_t(X)$ is strictly smaller than $\mathbf{B}_t(\dot{B})$, in particular the $\dot{B}(t)$ is not expressed as a function of the $Z(s), s \leq t + 0$.

Note that for a construction of the innovation of $X(t)$ given by a non-canonical representation, we give a method of actual construction of the innovation which has been established in [2] by using a martingale theory.

Also, note that if, in particular, a canonical kernel $G(t, u)$ is of the form $f(t)g(u)$, then $Z(t)$ is a Markov process and the converse is true, and hence, $\dot{B}(t)$ is the innovation.

Significance of Multiple Markov Property

The multiple Markov property, which expresses the way of dependence of a stochastic process when t varies, is one of the most important properties of stochastic processes to be investigated. The simple Markov property is well known and the definition has been confirmed without any doubts. A notion which will come naturally right after the simple Markov property would be a multiple Markov property, which describes how the observed values give influence to predict the future values of $X(t)$. For a Gaussian process we are able to give a plausible definition of multiple Markov property as a generalization of the simple Markov property. Before our definition is given, there is a short remark.

It is known that a Gaussian process, which is a solution to an ordinary differential equation involving $\dot{B}(t)$, is called a multiple Markov Gaussian process in the restricted sense. It is, however, not quite welcome to a class of general multiple Markov processes, since we think that the dependency should not depend mainly on the local regularity or differentiability. Still, like in the case of Langevin equation, the use of a differential operator may suggest us some idea.

By restricting our attention to a Gaussian process $X(t)$ we have given a definition of multiple Markov process by using the conditional expectations. This has been proposed by the author in 1960 (see Part II [8] in Ref. [1]).

Definition *Let $X(t)$ be a separable Gaussian process with $E(X(t)) = 0$ for every t. Then, if $X(t)$ satisfies the conditions:*

(i) *For any fixed t_0 and for any choice of t_j's such that $t_0 \leq t_1 < t_2 < \cdots < t_N$, the conditional expectations $E(X(t_j)|\mathbf{B}_{t_0}(X))$ are linearly independent.*

(ii) *Let t_0 be fixed. For any choice of t_j's such that $t_0 \leq t_1 < t_2 < \cdots < t_{N+1}$, the conditional expectations $E(X(t_j)|\mathbf{B}_{t_0}(X))$ are linearly dependent.*

Then, $X(t)$ is called an N-ple Markov Gaussian process.
An interesting characteristic property is

Theorem 3.3. *Let*

$$X(t) = \int_0^t F(t, u)\dot{B}(u)du$$

be the canonical representation of $X(t)$. If it is N-ple (N $\geq$ 1) Markov, then

$F(t, u)$ *is expressed in the form*

$$F(t, u) = \sum_1^N f_i(t) g_i(u),$$

where

$$det(F_i(t_j)) \neq 0,$$

for any different t_j's, and where $g_i(u), 1 \leq i \leq N$, are linearly independent in $L^2([0, t])$ for every t.

Definition *A kernel $F(t, u), u \leq t$, satisfying the conditions in the above theorem is called a Goursat kernel.*

Corollary *If $X(t)$ in the above theorem is stationary, then the Fourier transform $\hat{F}(\lambda)$ is expressed in the form*

$$\hat{F}(\lambda) = \frac{Q(i\lambda)}{P(i\lambda)},$$

where Q and P are polynomials with the property that degree of $Q <$ degree of P, and N, being the degree of P.

These properties of the canonical kernel of multiple Markov Gaussian process has many applications.

Several remarks concerning the Gaussian Markov property are now in order.

Remark 1. The multiple Markov property is often misunderstood. For example, the projection of any future value of $X(t + h), h \geq 0$, down to the past, (that is, the past is now taken to be the linear manifold spanned by $X(s), s \leq t,$) is always N-dimensional. This is a necessary condition, but it is not sufficient.

Remark 2. The canonical kernel may be understood, roughly speaking, to be a kernel associated to an invertible linear operator. Formally, the inverse operator of the integral operator F exists, let it be denoted by $G(= F^{-1})$, then, we have

$$(GX)(t) = \dot{B}(t).$$

Hence, $\dot{B}(t)$ obtained above is nothing but the innovation of $X(t)$. These are understood in the Hilbert space $L^2(\Omega, P)$ spanned by the random variables with finite variance. It is not a sample function-wise story.

Remark 3. Since we are concerned only with Gaussian processes, the conditional expectation $E(X(t)/\mathbf{B}_s(X))$ is a linear functional of the $X(u), u \leq s$, and

further $X(t) - E(X(t)/\mathbf{B}_s(X))$ is not only orthogonal, but also independent of all $\mathbf{B}_s(X)$-measurable random variables.

Remark 4. Although assuming the differentiability of $X(t)$ is not preferable for us to define Markov property, however let us do so to see an old attempt of defining a multiple Markov property. Under the assumption that $X(t)$ is differentiable, with respect to the $L^2(\Omega, P)$-norm, as many times as $N-1$, we consider an ordinary differential equation of the form

$$L_t X(t) = \sum_{j=0}^{N} a_{N-j} \frac{d^{N-j}}{dt^{N-j}} X(t) = \dot{B}(t),$$

which is a generalization of the Langevin equation (i.e. $N = 1$). The N-th derivative $X^{(N)}(t)$ has only a formal significance like $\dot{B}(t)$. The solution to the above differential equation with initial data $X(0) = 0$ is given by

$$X(t) = \int_0^t R(t, u) \dot{B}(u) du,$$

where $R(t, u)$ is the Riemann function for a linear differential operator L_t of order N. Certainly $R(t, u)$ is a canonical kernel.

Remark 5. As was seen in the Theorem above, the canonical kernel $F(t, u)$ of an N-ple Markov Gaussian process enjoys a particular functional property, and it is not possible to guarantee any analytic property of F straightforward. However, we can take conditional expectation and covariance function, and assume differentiability in t in addition to the property of a Goursat kernel. Then, we can find an analytic method to form the innovation. This is a generalization of what is discussed in the above remark.

I.3.2. Gaussian random fields

As a particular case, we first introduce a white noise (of course, Gaussian) with a multi-dimensional parameter, say $R^d, d > 1$, parameter. It is determined in the following manner. If it is denoted by $W(u), u \in R^d$, then the characteristic functional has to be of the form: for $\xi \in E$

$$E(\exp[i\langle W, \xi \rangle]) = \exp[-\frac{1}{2}\|\xi\|^2],$$

which is exactly the same expression as in the case $d = 1$.

To fix the idea we shall be concerned with the case $d = 2$. What are interested for us is Gaussian Random fields $X(C)$ defined by the stochastic integral

$$X(C) = \int_{(C)} F(C, u) W(u) du,$$

namely a field expressed in the causal representation in terms of white noise. For the purpose of the analysis, C is assumed to be a smooth ovaloid (a convex contour). The notation (C) means the domain enclosed by C.

Such a field has been discussed so far to some extent by analogy with the case of a Gaussian process. The canonical representation and the multiple Markov property can be discussed in the similar manner to the case of a Gaussian process. As for the innovation, we can discuss by using the stochastic variational equation.

Because of the significance and importance of such a system as a random complex system, it will be discussed in more details towards the future development. We shall come back to this problem in Part II.

I.4. NONLINEAR FUNCTIONALS OF WHITE NOISE

General setup

(1) The Fock space

Following the reductionism, we take white noise as an elemental stochastic process. For this purpose we start out with the Hilbert space (L^2). It has a direct sum decomposition:

$$(L^2) = \oplus_{n=0}^{\infty} H_n,$$

which is called a *Fock space*. The subspace H_n is spanned by all the n-th degree Hermite polynomials in smeared variables of $x(t)$'s. The subspace H_n is called the space of *multiple Wiener integrals* of degree n. A member of H_n is often called a *homogeneous chaos* of degree n after N. Wiener.

One may think of a finite dimensional analogue of the direct sum decomposition of $L^2(S^d) = \oplus_{n=0}^{\infty} H_n$, where H_n is spanned by the spherical harmonics of degree n. We may immediately remind the relationship between the rotation group $SO(d+1)$ and the spherical Laplacian acting on the function space over S^d which is isomorphic to $SO(d+1)/SO(d)$. The H_n is the eigenspace of the Laplacian, and so forth.

A generalization of these beautiful relationships to infinite dimensional case will be seen easily.

In order to carry on the analysis further on (L^2), we need to extend the basic space. There are two main reasons.

1. Polynomials are most fundamental functions when variables are given. Now take $x(t), t \in R$, with $x \in E^*$, which are not assumed to be smeared. This

leads us to introduce generalized functionals of $x(t)$'s like $x(t)^n$ for which we need a kind of renormalization (see I.2. **Synthesis**), since $x(t)$ is a generalized function of t. There t appears explicitly, so that it is fitting for expressing the time development explicitly in terms of the variable t.

2. We are interested in a random fields, not only Gaussian fields, but also general random fields including the Poisson case. If the variational calculus is applied, then we meet singular functionals with respect to the variable. Our theory should be set up so as to manage those singular functionals.

We then come to introduce a visualized representation of nonlinear functionals of a generalized function $x(t)$. To this end the S-transform is introduced.

(2) The S-transform of a functional φ in (L^2) is defined by

$$(S\varphi)(\xi) = \exp[-\frac{1}{2}\|\xi\|^2] \int \exp[\langle x, \xi \rangle]\varphi(x)d\mu(x).$$

The integral representation of $\varphi \in H_n$ is expressed in the form

$$(S\varphi)(\xi) = \int_{R^n} F(u)\xi^{\otimes n}(u)du^n, \quad u \in R^n,$$

where F is a symmetric $L^2(R^n)$-function such that

$$\|\varphi\|^2 = n!\|F\|_{L^2(R^n)}^2.$$

We may consider that the φ is represented by F which is an L^2-function over a *finite* dimensional, in fact n-dimensional, Euclidean space. Thus, we have a good representation of white noise functionals.

There is another transformation called T-transform. It is like a Fourier transform, but not quite. It is given by

$$(T\varphi)(\xi) = \int \exp[i\langle x, \xi \rangle]\varphi(x)d\mu(x).$$

The T-transform plays similar roles to S-transform, but sometimes the former is more convenient.

Even generalizations of these transforms are introduced and used in our analysis.

The next notion useful for our analysis is a transformation group under which the white noise measure μ is kept invariant.

(3) The infinite dimensional rotation group $O(E)$.

The collection of the rotations g of the basic nuclear space E forms a group. The adjoint g^* of g acts on E^* and keeps the white noise measure μ on E^* invariant. We

may have a view point that the white noise analysis is like an infinite dimensional harmonic analysis. This is one of the great advantages of our white noise analysis. The infinite dimensional harmonic analysis in this sense is a fruitful area to be investigated, in particular the subgroup of $O(E)$ involving one-parameter subgroups (called whiskers) that come from the group of differmorphisms of the parameter space R^d. Some collection of whiskers which are significant in probability theory forms a group, in fact a Lie group. The structure of such a Lie group is known partly. Further investigation will certainly important.

(4) Laplacians are naturally introduced. In our case, where the dimension of the basic space is infinite, Laplacian is not unique. Three Lapalcians are often used, among others the Lévy Laplacian Δ_L is most interesting (see, e.g. Saito [32]). It would be successful if one discusses Δ_L on its domain, members of which are functionals with singularity on the diagonal (see next section). Space of eigen-functionals has special significant meaning in analysis and shall be investigated.

I.5. GENERALIZED WHITE NOISE FUNCTIONALS

In order to carry on the so-called *causal calculus* we are led to introduce a bigger space than the basic Hilbert space (L^2), as was explained before. The reason was also partly mentioned in (1) in the last section. There is a big advantage to introduce a space of generalized white noise functionals. Having been suggested by employing polynomials in $x(t)$ or in $\dot{B}(t)$'s of degree n, we are led to have an extension H_n^- of H_n. Also, a Gauss kernel $g(x, c)$ which is to be an exponential of the squares of $x(t)$'s or $\dot{B}(t)$'s

$$g(x, c) = N \exp[c \int x(t)^2 dt], c \neq \frac{1}{2},$$

N being the renormalizing constant, and the delta function of $B(t)$ will be found in a weighted sum of the H_n^-. Finally we have come to a space $(L^2)^-$ of general class of generalized functionals:

$$(L^2)^- = \bigoplus c_n H_n^-,$$

where c_n is a positive decreasing sequence that tends to 0. The sequence c_n is subject to the optional choice.

Another method of defining the space of generalized white noise functionals is an infinite dimensional analogue of introducing the Schwartz space of distributions. Take the operator

$$L = -\Delta_d + |u|^2 + 1$$

with d-dimensional Laplacian Δ_d, and use the second quantization technique to define the space (S) of test functionals and its dual space $(S)^*$. Now a Gel'fand

triple

$$(S) \subset (L^2) \subset (S)^*$$

is obtained. Thus obtained $(S)^*$ is another space of generalized white noise functionals.

A significant advantage of taking the space $(S)^*$ is that there is a characterization theorem for a member of $(S)^*$ due to Potthoff and Streit [31].
Before stating the theorem, a definition is provided.

Definition *A functional F on E is called ray entire at $\eta \in E$, if and only if for every $\eta \in E$, the mapping $\lambda \to F(\eta + \lambda\xi), \lambda \in R$, has an entire analytic extension, denoted by $F(\eta + z\xi), z \in \mathbf{C}$.*

For a ray entire function at 0, and for $R > 0$, set

$$M(R, \xi) = \sup_{z=R} |F(z\xi)|.$$

If there exists $C > 0$ such that for all $R > 0$, the inequality

$$M(R, \xi) \leq C \exp[\tau R^2 |\xi|_p^2]$$

holds, then F is said to be of growth $(2, \tau)$, where $|\xi|_p$ is the p-th norm defining the topology of the nuclear space E.

Definition *A complex-valued function F on E is called a U-functional if and only if F is ray entire on E and of growth $(2, \tau)$ on E_p for some non-negative τ.*

Theorem *(Potthoff-Streit) Let $\varphi \in (S)^*$. Then its S-transform $S(\varphi)(\xi)$ is a U-functional. Conversely, if F is a U-functional, then there exists unique φ such that $S\varphi = F$.*

See also Y. J. Lee [19] and Cochran-Kuo-Sengupta's result [4] for further study.

The two spaces $(L^2)^-$ and $(S)^*$ of generalized functionals are used depending on the purpose, and of course both of them are useful and equally important.

One significant remark is that the so-called Lévy Laplacian acts effectively on those spaces. There one can see essentially infinite dimensional analysis. Needless to say, applications are important.

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