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HOW MANY THEOREMS CAN BE DERIVED FROM A VECTOR FUNCTION – ON UNIQUENESS THEOREMS FOR THE MINIMAL SURFACE EQUATION

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Abstract. In this survey article we consider equations related to the minimal surface equation div $Tu = 0$, where $Tu = \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$, ∇u is the gradient of u, and derive some structural inequalities related to the vector function Tu. These structural inequalities give rise to striking uniqueness properties of the solutions.

1. Existence and Uniqueness Theorems for Bounded Domains

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, consider the functional

$$I(u) = \int_\Omega F(x, u, \nabla u), \tag{1}$$

where ∇u is the gradient of u. If u is an extremal function then the Euler-Lagrange equation is

$$\sum_i \frac{\partial}{\partial x_i} \frac{\partial F}{\partial u_{x_i}} = \frac{\partial F}{\partial u}, \tag{2}$$

or

$$\text{div } \langle F_{u_{x_1}}, \cdots, F_{u_{x_n}} \rangle = \frac{\partial F}{\partial u}.$$

More generally, consider the equation in divergence form

$$\text{div} A = B,$$

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where

$$A = \langle A_1, \cdots, A_n \rangle (x, u, \nabla u), \quad B = B(x, u, \nabla u).$$

If $F = |\nabla u|^2 = u_{x_1}^2 + \cdots + u_{x_n}^2$, then (2) is the Laplace equation

$$\mathrm{div} \nabla u = \Delta u = 0;$$

if $F = \sqrt{1 + |\nabla u|^2}$, then (1) is the area functional and (2) is the minimal surface equation (MSE)

$$\mathrm{div} Tu = 0, \tag{3}$$

where

$$Tu = \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}. \tag{4}$$

For $\Delta u = f$, mainly two kinds of boundary value problems have been considered: the Dirichlet problem

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}; \tag{5}$$

and the Neumann problem

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \nabla u \bullet \vec{\nu} = h & \text{on } \partial\Omega \end{cases}. \tag{6}$$

where f is a function defined in Ω, g and h functions defined on $\partial\Omega$, and $\vec{\nu}$ is the outward unit normal vector on $\partial\Omega$.

Similarly, for div $Tu = f$ there are also two kinds of boundary value problems: the Dirichlet problem

$$\begin{cases} \mathrm{div} Tu = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}; \tag{7}$$

and the capillary problem

$$\begin{cases} \mathrm{div} Tu = f & \text{in } \Omega \\ Tu \bullet \vec{\nu} = \cos\gamma & \text{on } \partial\Omega \end{cases}. \tag{8}$$

where $0 \leq \gamma \leq \pi$ is a function defined on $\partial\Omega$, in capillary theory it is the contact angel between the liquid surface (the graph generated by a solution u) and the fixed boundary, see pictures in [22].

For the existence, it is well known that if the boundary $\partial\Omega$ is not too pathological bad, then (5) always has a solution if $f = 0$ and $g \in C^0(\partial\Omega)$. Roughly speaking, if $F \cong |\nabla u|^\beta$ as $|\nabla u| \to \infty$, where $\beta > 1$ is a constant, then for smooth boundary $\partial\Omega$, the Dirichlet problem

$$\begin{cases} \text{div}\langle F_{u_{x_1}}, \cdots F_{u_{x_n}} \rangle = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{9}$$

always has a solution. The assumption that $\beta > 1$ guarantees that (9) is a uniformly elliptic equation. See [25], Theorem 15.11 on page 381 for detail.

But if $F = \sqrt{1 + |\nabla u|^2}$, $\beta = 1$, then (7) is no longer uniformly elliptic. Consider the Dirichlet problem

$$\begin{cases} \text{div}Tu = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{10}$$

An interesting problem is: does there exist a sufficiently general existence theorem for the Dirichlet problem (10)? Here sufficiently general means that for any $g \in C^0(\partial\Omega)$, there is a solution for (10).

It turns out that the geometric property of the domain Ω itself holds the key. In 1965, Finn (see [18] and [21]) proved that for the case $n = 2$, (10) is solvable for any $g \in C^0(\partial\Omega)$ if and only if Ω is convex.

For the case $n \geq 3$, (10) is solvable for any $g \in C^0(\partial\Omega)$ if and only if $\partial\Omega$ has non-negative mean curvature (respect to the inner normal direction), see Theorem 16.8 on page 407 of [25]. Many contributed to the solvability of (10), among them are Jenkins, Serrin, and Bakel'man, see [39], [58], [2], and [3].

The study for non-existence of solutions of (10) for non-convex domains was initiated by Bernstein in 1912 [5]. Since then many mathematicians made contributions to this study (for a detailed discussion of this question, see [51]), until Finn proved that for $\Omega \subset \mathbb{R}^2$, convex is the necessary and sufficient condition for (10) to have a solution for any $g \in C^0(\partial\Omega)$.

To study the non-existence phenomenon, we first consider a uniqueness theorem.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\partial\Omega = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, Γ_2 smooth. Suppose that u, $v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ and Tu, $Tv \in C^0(\Omega \cup \Gamma_2)$. If*

$$\begin{cases} \text{div}Tu \geq \text{div}Tu & \text{in } \Omega \\ u \leq v & \text{on } \partial\Gamma_1 \\ Tu \bullet \vec{\nu} \leq Tu \bullet \vec{\nu} & \text{on } \partial\Gamma_2 \end{cases} \tag{11}$$

then $u \leq v$ in Ω unless $\Gamma_1 = \emptyset$, in that case $u \equiv v +$ constant.

Proof. Let $\Omega' = \{x \in \Omega |\ u - v > 0\} \neq \emptyset$, then

$$\begin{aligned}\int_{\partial\Omega'} (u-v)(Tu-Tv)\bullet\vec{\nu} &= \int_{\Omega'} \text{div}[(u-v)(Tu-Tv)] \\ &= \int_{\Omega'} (\nabla u - \nabla v)\bullet(Tu-Tv) + (u-v)(\text{div}Tu - \text{div}Tv).\end{aligned} \tag{12}$$

On $\partial\Omega' \cap (\Omega \cup \Gamma_1)$, $u = v$, on $\partial\Omega' \cap \Gamma_2$, $(u-v)(Tu-Tv)\bullet\vec{\nu} \leq 0$, so

$$\int_{\partial\Omega'} (u-v)(Tu-Tv)\bullet\vec{\nu} \leq 0.$$

In Ω', $(u-v)(\text{div}Tu - \text{div}Tv) \geq 0$. By (16), $(\nabla u - \nabla v)\bullet(Tu-Tv) \geq 0$, and $(\nabla u - \nabla v)\bullet(Tu-Tv) = 0$ if and only if $\nabla u = \nabla v$. Therefore, (12) shows that

$$\int_{\partial\Omega'} (u-v)(Tu-Tv)\bullet\vec{\nu} \geq 0,$$

hence

$$(\nabla u - \nabla v)\bullet(Tu-Tv) \equiv (u-v)(\text{div}Tu - \text{div}Tv) \equiv 0, \quad \text{and} \quad \nabla u \equiv \nabla v$$

in Ω'. We obtain that either $\Omega' = \emptyset$ and $u \leq v$ or $\Omega' = \Omega$ and $u - v \equiv$ constant. ■

For the general divergence equation $\text{div}A = B$, for simplicity, let us write $Au = A(x, u, \nabla u)$ and $Av = A(x, v, \nabla v)$, we can generalize Theorem 1.1 if either

$$\begin{aligned}(\nabla u - \nabla v)\bullet(Au - Av) &\geq 0, \\ \text{equality holds if and only if} &\quad \nabla u = \nabla v; \\ (u-v)(\text{div}Au - \text{div}Av) &\geq 0\end{aligned} \tag{13}$$

or

$$\begin{aligned}(\nabla u - \nabla v)\bullet(Au - Av) &\geq 0; \\ (u-v)(\text{div}Au - \text{div}Av) &\geq 0, \\ \text{equality holds if and only if} &\quad u = v.\end{aligned} \tag{14}$$

For

$$A = A(x_1, \cdots, x_n, z, p_1, \cdots, p_n), \quad B = B(x_1, \cdots, x_n, z, p_1, \cdots, p_n),$$

on page 429 of [58], Serrin pointed out that if the $(n+1)\times(n+1)$ matrix

$$\begin{bmatrix} D_{p_j}A^i(x,z,p) & D_{p_j}B(x,z,p) \\ D_zA^i(x,z,p) & D_zB(x,z,p) \end{bmatrix}$$

is non-negatively definite, then

$$(\nabla u - \nabla v) \bullet (Au - Av) + (u - v)(B(x, u, \nabla u) - B(x, v, \nabla v)) \geq 0.$$

There are similar conditions that can guarantee the uniqueness, see for example [62].

Minimal surface equation has the basic structural inequality:

$$\begin{aligned} &(\nabla u - \nabla v) \bullet (Tu - Tv) \ \geq \ 0, \\ &\text{equality holds if and only if} \qquad \nabla u = \nabla v. \end{aligned} \tag{15}$$

This inequality is equivalently related to ellipticity. The structural inequality (15) comes from another structural inequality (see page 542 of [51]):

$$(\nabla u - \nabla v) \bullet (Tu - Tv) \geq \frac{|\nabla u - \nabla v|^2}{\max\left((1 + |\nabla u|^2)^{\frac{3}{2}},\ (1 + |\nabla v|^2)^{\frac{3}{2}}\right)}. \tag{16}$$

Now we can give an example of non-existence. We first define $B_r = \{x^2 + y^2 < r^2\} \subset \mathbb{R}^2$ for $r > 0$.

Theorem 1.2. (Theorem 1 of [21]). *Let* $\Omega = B_2 - \overline{B_1} \subset \mathbb{R}^2$, if $u \in C^2(\Omega)$ and $Tu \in C^0(\bar{\Omega})$

$$\begin{cases} \text{div} Tu = 0 & \text{in } \Omega \\ u \geq \cosh^{-1} 2 & \text{on } \partial B_2, \end{cases}$$

then $u \geq 0$ *on* ∂B_1.

Proof. Take $v = \cosh^{-1} r$ as a comparison function, note that the graph of v is a part of a cantenoid. We have $\text{div} Tv = 0$, $v = \cosh^{-1} 2$ on ∂B_2, $v = \cosh^{-1} 1 = 0$ on ∂B_1, and $Tv \bullet \vec{\nu} = -1$ on ∂B_1.

Since $|Tu| \leq 1$, we have $Tu \bullet \vec{\nu} \geq -1$. Thus $Tu \bullet \vec{\nu} \geq Tv \bullet \vec{\nu}$ on ∂B_1. By Theorem 1.1, $u \geq v$ in Ω, therefore $u \geq v \geq 0$ on ∂B_1. ■

In Theorem 1.2, we only need to know the behavior of u on ∂B_2 ($\geq \cosh^{-1} 2$), then somehow the behavior of u on ∂B_1 is strictly restricted (≥ 0). Hence the following Dirichlet problem is unsolvable:

$$\begin{cases} \text{div}\, Tu = 0 & \text{in } B_2 - \overline{B_1}, \\ u \geq \cosh^{-1} 2 & \text{on } \partial B_2, \\ u < 0 & \text{on } \partial B_1. \end{cases}$$

Because of the non-existence of solutions of some Dirichlet problems for the minimal surface equation, the study of existence in general is very complicated.

But if we take a deeper look at the Theorem 1.2, it is really a uniqueness theorem, we only need know the behavior of u on one component of the boundary, then we know the behavior of u on another component of the boundary. In other words, the minimal surface equation has a very strong uniqueness property, therefore, some Dirichlet problem may has no solution at all.

For uniformly elliptic equation, since any Dirichlet problem has a solution, it is impossible to find such a strong uniqueness property.

So we may conclude that although uniformly elliptic equations have better existence property, the minimal surface equation has better uniqueness property. In the following sections, we will see some uniqueness examples that have no counterparts for Laplace equation. This is one of the attractiveness of the minimal surface equation.

2. Removable Singularity Theorem

The first mathematician noticed that the minimal surface equation has much better uniqueness property than the Laplace equation is Bernstein. In 1915 Bernstein [6] proved that if u is defined over the whole $\mathbb{R}^2$ and satisfies the MSE (3), then u must be an affine function. On the other hand, Re z^n, $n > 1$, Re e^z, are easy examples of global solutions to the Laplace equation.

It has long been an open problem whether or not Bernstein's theorem generalizes to higher dimensions, that is, if $u : \mathbb{R}^n \to \mathbb{R}$ satisfies the MSE, must u be an affine functions?

In 1962, applying methods of geometric measure theory, Fleming [24] gave a new proof for the original 2-dimensional Bernstein theorem. Following Fleming's idea, Bernstein theorem was proved in the case of $n = 3$ by de Giorgi [14], for $n = 4$ by Almgren [1], and for $n = 5, 6, 7$ by Simons [59]. Then Bombieri-de Giorgi-Giusti [8] supplied a non-linear global solution for $n = 8$.

The next influence of strong uniqueness is the removable singularity theorem of Bers, [7].

Theorem 2.1. *Let* $u \in C^0(\bar{B}_R - \{0\}) \cap C^2(B_R - \{0\})$, *div*$Tu = 0$ *in* $B_R - \{0\}$, *then* u *has no isolated singularity at the point* $\{0\}$.

Bers was the first to prove Theorem 2.1 for 2-dimensional minimal surface equation. It was also obtained independently by Finn. A result of more general type became Finn's dissertation in 1951 and was published in [17]. See the Zentralblatt Math-Review Zbl # 896 35001 on Nirenberg's article "Lipman Bers and partial differential equations" in [16].

The original proof of Bers is in complex analysis. Finn in [17] gave a proof of partial differential equation method. Using ideas of Concus and Finn in [12] and Hwang in [37], we can simplify Finn's proof in [17]. We first establish a comparison theorem.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, $p \in \Omega$, $u, v \in C^0(\overline{\Omega} - \{p\}) \cap C^2(\Omega - \{p\})$, and*

$$\begin{cases} \mathrm{div} Tu = \mathrm{div} Tv & \text{in } \Omega - \{p\}, \\ u = v & \text{on } \partial\Omega, \end{cases}$$

then $u \equiv v$ in $\Omega - \{p\}$.

Proof. To avoid that u, v tend to infinity at $\{p\}$, let us consider $\Omega_\epsilon = \Omega - B_\epsilon(p)$, where $B_\epsilon(p)$ is the ball of small radius $\epsilon > 0$ and centered at p. By Green's formula

$$\begin{aligned} &\int_{\partial\Omega_\epsilon} \tan^{-1}(u-v)(Tu - Tv) \bullet \vec{\nu} \\ &\qquad = \int_{\Omega_\epsilon} \nabla \tan^{-1}(u-v) \bullet (Tu - Tv) + \tan^{-1}(u-v)(\mathrm{div} Tu - \mathrm{div} Tv) \\ &\qquad = \int_{\Omega_\epsilon} \frac{\nabla u - \nabla v}{1 + (u-v)^2} \bullet (Tu - Tv) \geq 0. \end{aligned}$$

On $\partial\Omega_\epsilon = \partial\Omega \cup \partial B_\epsilon(p)$, $\tan^{-1}(u-v) = 0$ on $\partial\Omega$, and

$$\left| \int_{\partial B_\epsilon(p)} \tan^{-1}(u-v)(Tu - Tv) \bullet \vec{\nu} \right| \leq 2\pi\epsilon \frac{\pi}{2} \max_{\partial B_\epsilon(p)} |Tu - Tv| \to 0$$

as $\epsilon \to 0$, because of $|Tu - Tv| \leq 2$. Letting $\epsilon \to 0$, we obtain that

$$\int_{\Omega - \{p\}} \frac{\nabla u - \nabla v}{1 + (u-v)^2} \bullet (Tu - Tv) = 0.$$

By (15), $\nabla u \equiv \nabla v$ in $\Omega - \{p\}$. Since $u = v$ on $\partial\Omega$, we have $u \equiv v$ in $\Omega - \{p\}$. ■

Now it is easy to prove Theorem 2.1. Since B_R is convex, the Dirichlet problem

$$\begin{cases} \mathrm{div} Tv = 0 & \text{in } B_R, \\ v = u & \text{on } \partial B_R \end{cases}$$

has a unique solution v. By Theorem 2.2, $u \equiv v$ in $B_R - \{0\}$, but $v \in C^2(B_R)$, so is u.

Theorem 2.1 can be easily generalized to the case of singular set $A \subset \bar{\Omega} \subset \mathbb{R}^n$ such that $\bar{A}$ has vanishing $n-1$ Hausdorff measure.

The first extension for the removability theorem to higher dimensions appeared in Finn's 1961 paper [19]. In 1965, De Giorgi and Stampacchia [15] did not notice the result of [19]. They proved that the singular set has vanishing $n-1$ Hausdorff measure which was not proved in [19]. But as stated in Theorem 2.2, one can easily generalize Finn's method to prove the vanishing $n-1$ Hausdorff measure phenomenon.

Also in 1965, Nitsche [53] proved the De Giorgi-Stampacchia theorem in the case of $n=2$ independently.

Note that we only used the structural inequalities (15),

$$|Tu| \leq 1 \tag{17}$$

and the help of bounded function $\tan^{-1}(u-v)$, these are the three key points of the proof of the removable singularity theorem. Note that Bernstein in [6] had used the boundedness of the functions $\tan^{-1} u_x$ and $\tan^{-1} u_y$ to prove his well-known entire minimal surface theorem. Here Hwang [37] used the boundedness of $\tan^{-1}(u-v)$ to simplify the proofs in Finn [17] and Concus and Finn [12].

C. C. Chen studies removable singularity theorems for general equations in his Ph.D thesis, [9]. The bounded function $\tan^{-1}$ also significantly simplifies the proof, the author suggested the idea to Chen, see page 18 of [9].

Using the structural inequalities (15) and (17), Finn-Hwang [23] and Kurta [43] independently proved the uniqueness of capillary equation in a gravitational field.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^n$ be an unbounded smooth domain, k be a positive constant and $0 \leq \gamma \leq \pi$. Let $\partial\Omega = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, Γ_2 smooth. If*

$$\begin{cases} \operatorname{div} Tu = ku & \text{in } \Omega, \\ u = g \in C^0(\Gamma_1) & \\ Tu \bullet \vec{\nu} = \cos\gamma & \text{on } \Gamma_2 \end{cases}$$

has a solution, then it is unique.

Proof. We only give proof for the case $n=2$, $n \geq 3$ is similar. Let v be another solution. Define $\Omega_r = \Omega \cap B_r$, $\Gamma_r = \partial\Omega_r \cap \partial B_r$,

$$\begin{aligned} g(r) \; :&= \int_{\Gamma_r} (u-v)(Tu-Tv) \bullet \vec{\nu} \\ &= \int_{\partial\Omega_r} (u-v)(Tu-Tv) \bullet \vec{\nu} \\ &= \int_{\Omega_r} (\nabla u - \nabla v) \bullet (Tu - Tv) + (u-v)(\operatorname{div} Tu - \operatorname{div} Tv) \\ &= \int_{\Omega_r} (\nabla u - \nabla v) \bullet (Tu - Tv) + k(u-v)^2. \end{aligned}$$

Then

$$\begin{aligned} g'(r) &= \int_{\Gamma_r} (\nabla u - \nabla v) \bullet (Tu - Tv) + k(u-v)^2 \\ &\geq k \int_{\Gamma_r} (u-v)^2 \geq \frac{k}{2\pi r} (\int_{\Gamma_r} |u-v|)^2 \\ &\geq \frac{k}{8\pi r} (\int_{\Gamma_r} |u-v||Tu-Tv|)^2 \\ &\geq \frac{k}{8\pi r} g^2(r). \end{aligned}$$

Either $g \equiv 0$, therefore $u \equiv v$, or there is an r_0 such that $g(r) \geq g(r_0) > 0$ for $r \geq r_0$. Then for $r > r_0$

$$\frac{g'}{g^2} \geq \frac{k}{8\pi r},$$

integrate the above we obtain

$$\frac{1}{g(r_0)} - \frac{1}{g(r)} = \int_{r_0}^{r} \frac{g'}{g^2} \geq \int_{r_0}^{r} \frac{k}{8\pi r} = \frac{k}{8\pi} \log \frac{r}{r_0} \to \infty$$

as $r \to \infty$, a contradiction. ∎

For general divergence equation $\mathrm{div} A(x, u, \nabla u) = ku$, assuming the following two structural inequalities,

(18) $(Au - Av) \bullet (\nabla u - \nabla v) \geq 0$ equality holds if and only if $\nabla u = \nabla v$,

(19) $$|Au| \leq \text{constant}.$$

Then we will have the similar theorems, and it is not hard to generalize the above.

3. Buddha's Holy Palm: Compare with Infinite Boundary Value

Consider the Scherk's surface, it is a graph generated by

$$F(x, y) = \log \frac{\cos y}{\cos x}, \quad (x, y) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2}).$$

Then F approaches $\pm\infty$ on the boundary. This is a special example of solutions of MSE have infinite boundary values. Finn [20] in 1963 proved that

Theorem 3.1. Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain and $\partial\Omega$ contains a straight line segment Γ, then the Dirichlet problem

$$\begin{cases} \text{div}Tu = 0 & \text{in } \Omega, \\ u = +\infty & \text{on } \Gamma \\ u \in C^0(\partial\Omega - \Gamma). \end{cases}$$

always has a solution.

For the generalization of this theorem, see [38].

In [52] of 1965, Nitsche pointed out the phenomenon of infinite boundary value can be used as a good comparison function to prove some uniqueness theorems. Recall that in the Chinese classical novel "Journey to the West", Buddha uses only one single hand to bury the Monkey under the Five Finger Mountain for over 500 years. Here just like Buddha's holy palm the infinite boundary value controls everything near Γ. For example, let

$$\Omega_\alpha = \{y > |x| \cot \frac{\alpha}{2}\} \subset \mathbb{R}^2$$

be a sector domain. Nitsche showed that

Theorem 3.2. *Let* $0 < \alpha < \pi$ *and* $u \in C^2(\Omega_\alpha) \cap C^0(\overline{\Omega_\alpha})$ *satisfies*

$$\begin{cases} \text{div}Tu = 0 & \text{in } \Omega_\alpha, \\ u = 0 & \text{on } \partial\Omega_\alpha. \end{cases}$$

Then $u \equiv 0$.

Proof. For any $a > 0$, define $\Omega_{\alpha,a} = \Omega_\alpha \cap \{y < a\}$. Let $\Gamma = \partial\Omega_{\alpha,a} \cap \{y = a\}$ and v_α be a solution of the Dirichlet problem

$$\begin{cases} \text{div}Tu = 0 & \text{in } \Omega_{\alpha,a} \\ u = +\infty & \text{on } \Gamma \\ u = 0 & \text{on } \partial\Omega_{\alpha,a} - \Gamma. \end{cases}$$

Since $v_a \geq u$ on $\partial\Omega_{\alpha,a}$, by Theorem 1.1, $u \leq v_a$ in $\Omega_{\alpha,a}$.

But $\lim_{a\to\infty} v_a = 0$, so $u \leq 0$. Similarly, $u \geq 0$, hence $u \equiv 0$. ∎

Note that the use of solutions that are infinite or have infinite normal derivatives on the boundary goes back to Bernstein [4], Heinz [28], and Finn [21]. See paragraphs after Theorem 5.1.

Nitsche's 1965 article [52] is a survey article, in which $\lim_{a\to\infty} v_a = 0$ is stated but without a proof.

Langevin, Levitt, and Rosenberg [45] proved that for $0 < \alpha < 2\pi$, $\alpha \neq \pi$, Theorem 3.2 is also true.

C. C. Lee gave a basic proof of Theorem 3.2 in [46].

Note that when $\alpha = \pi$, $u = ay$ is a solution, for any $a \in \mathbb{R}$.

The following theorem is an easy corollary:

Theorem 3.3. Let $\Omega \subset \Omega_\alpha$, $0 < \alpha < \pi$, if

$$\begin{cases} \mathrm{div}Tu = 0 & \text{in } \Omega, \\ u \leq 0 & \text{on } \partial\Omega, \end{cases}$$

then $u \leq 0$ *in* Ω.

Nitsche conjectured that

Conjecture 3.1. (Nitsche's Conjecture, [52], page 256) *Let* $\Omega \subset \Omega_\alpha$, $0 < \alpha < \pi$, *if*

$$\begin{cases} \mathrm{div}Tu = 0 & \text{in } \Omega \\ u = f \in C^0(\partial\Omega) \end{cases}$$

has a solution, then it is unique.

Since the zero boundary value implies uniqueness, is it true that continuous boundary values also imply uniqueness? Such a question comes from the non-linearity of MSE and has no significance in the linear case.

The same problem of higher dimensional was raised by Massari-Miranda [47].

Nitsche's conjecture is true if the boundary value f is bounded. This can be proved by the combination of Theorem 3.3 and the following Theorem 3.4, proved independently by Miklyukov [49] and Hwang [31].

Theorem 3.4. Let $\Omega \subset \mathbb{R}^2$ be an unbounded domain, $\Omega_r = \Omega \cap B_r$, $\Gamma_r = \partial\Omega_r \cap \partial B_r$, $r > 0$. Let $|\Gamma_r|$ be the length of Γ_r. If

$$\begin{cases} \mathrm{div}Tu = \mathrm{div}Tv & \text{in } \Omega, \\ u = v & \text{on } \partial\Omega, \\ \max_{\Omega_r} |u - v| = O\left(\sqrt{\int_{R_0}^{R} \frac{dr}{|\Gamma_r|}}\right) \end{cases}$$

as $R \to \infty$, where R_0 is a positive constant. Then $u \equiv v$ in Ω.

In 1991 Collin and Krust found a better result independently [11]

Theorem 3.5. *Let Ω and Γ_r be as in Theorem 3.4, if*

$$\begin{cases} (i)\mathrm{div}Tu = \mathrm{div}Tv & \text{in } \Omega, \\ (ii)u = v & \text{on } \partial\Omega, \\ (iii)\max_{\Omega_r} |u-v| = o\left(\int_{R_0}^{R} \frac{dr}{|\Gamma_r|}\right) & \text{as } \; R \to \infty \end{cases}$$

then $u \equiv v$ in Ω.

For general unbounded domain in $\mathbb{R}^2$, since $|\Gamma_r| \le 2\pi r$, condition (iii) of Theorem 3.5 becomes

$$\max_{\Omega_r} |u-v| = o(\log r) \quad \text{as } r \to \infty.$$

For the strip since $|\Gamma_r| \le$ constant, (iii) becomes

$$\max_{\Omega_R} |u-v| = o(R) \quad \text{as } R \to \infty.$$

The proofs of Theorem 3.4 and 3.5 are based on the structural inequalities (15) and the following (20) which is independently due to Miklyukov [49], page 265, Hwang [31], page 342, and Collin and Krust [11], page 452.

$$\begin{aligned} (Tu - Tv) \bullet (\nabla u - \nabla v) \; &\ge \frac{\sqrt{1+|\nabla u|^2} + \sqrt{1+|\nabla v|^2}}{2} |Tu - Tv|^2 \\ &\ge |Tu - Tv|^2. \end{aligned} \tag{20}$$

In 1990, Collin [10] gave a counter-example to show that condition (iii) of Theorem 3.5 is best possible.

Theorem 3.6. *Let $\Omega = (0,1) \times \mathbb{R}$, then there are u, $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $u \not\equiv v$, but divTu = divTv in Ω, and $u = v$ on $\partial\Omega$. Furthermore, u, $v = O(|y|)$ as $|y| \to \infty$.*

Since a modification of Theorem 3.6 can show that for the case $\Omega = (0,1) \times (0,\infty)$ it is still true, Nitsche's conjecture in general is not true.

Collin and Krust [11] also proved the following theorem:

Theorem 3.7. *Let $\Omega = (0,1) \times \mathbb{R}$, if*

$$\begin{cases} \mathrm{div}Tu = 0 & \text{in } \Omega, \\ u(0,y) = ay + b \\ u(1,y) = cy + d \end{cases}$$

where a, b, c, and d are constants, then u is a helicoid or a plane. (Note: The helicoid is the only ruled minimal surface, see for example, [55].)

By Collin's counter-example, Theorem 3.5 is not applicable in the proof of Theorem 3.7, since $u = O(|y|)$. No wonder Collin and Krust used a geometric method, the study of Gauss map, to prove Theorem 3.7. Using the slightly more accurate structural inequality (20), Hwang [35] gave a proof of Theorem 3.7 by the PDE method.

Theorem 3.8. *Let $\Omega \subset \mathbb{R}^2$, u, $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and*

$$\begin{cases} \mathrm{div}Tu = \mathrm{div}Tv & \text{in } \Omega, \\ u = v & \text{on } \partial\Omega, \\ \max_{\Omega_R} |u - v| = o\left(\int_{R_0}^{R} \dfrac{\min_{\Gamma_r} \sqrt{1 + |\nabla u|^2} dr}{|\Gamma_r|} \right) & \text{as } R \to \infty \end{cases}$$

where R_0 is a positive constant. Then $u \equiv v$ in Ω.

Applying Theorem 3.8 we can prove Theorem 3.7.

This is a very interesting part of the non-linear problem, that uniqueness of zero solution does not guarantee that general solution is unique. But, even some counter-examples show that some kind of methods will not work, somehow it works anyway. The surprise is everywhere.

We would like to pose an open problem:

Problem 3.1. *Let $\Omega = (0, 1) \times (0, \infty)$ and*

$$\begin{cases} \mathrm{div}Tu = \mathrm{div}Tv = 0 & \text{in } \Omega, \\ u = v & \text{on } \partial\Omega, \\ \lim_{y\to\infty} \int_{\Gamma_y} (Tu - Tv) \bullet \vec{\nu}_1 = 0 \end{cases}$$

where $\vec{\nu}_1 = (0, 1)$ and $\Gamma_y = (0, 1) \times \{y\}$. Is $u \equiv v$ in Ω?

Modify Collin's counter-example (Theorem 3.6) and we can construct two solutions $u \not\equiv v$, satisfying $\mathrm{div}Tu = \mathrm{div}Tv = 0$ in $\Omega = (0, 1) \times (0, \infty)$, $u = v$ on $\partial\Omega$. But u, v do not satisfy $\lim_{y\to\infty} \int_{\Gamma_y} (Tu - Tv) \bullet \vec{\nu}_1 = 0$. Can we get uniqueness by adding this condition?

4. From $\mathbb{R}^n$ to Manifold

Let $(M, \langle , \rangle)$ be a complete (non-compact), m-dimensional, $m \geq 2$, Riemannian manifold and, for a fixed reference point $o \in M$, set $r(x) = \mathrm{dist}_{(M,\langle,\rangle)}(o, x)$. Let

B_R and ∂B_R denote, respectively, the geodesic ball and sphere of radius R, centered at o.

We associate to a smooth function $u : M \to \mathbb{R}$, its graph $\mathcal{G}_u : M \to M \times \mathbb{R}$, defined by

$$\mathcal{G}_u : x \to (x, u(x)).$$

We note that with (,) the product metric on $M \times \mathbb{R}$,

$$\mathcal{G}_u : (M, \mathcal{G}_u^*(\, , \,)) \to (M \times \mathbb{R}, (\, , \,))$$

becomes an isometric embedding. Let ∇, div, $| \, |$ denote the gradient, the divergence operators and the norm with respect to $\langle , \rangle$. Then $\mathcal{G}_u$ has mean curvature $\frac{1}{m}a(x)$ for some function $a \in C^\infty(M)$ if and only if

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)(x) = a(x), \quad x \in M.$$

Since graphs in $\mathbb{R}^{n+1}$ and $(M, \langle , \rangle) \times \mathbb{R}$ has similar formula for mean curvature, Rigoli, Savatori, and Vignati in [57] and Pigola, Rigoli, and Setti in [56] pointed out that the structural inequality (20) can be applied to manifolds as well and get some uniqueness theorems.

First, apply the same method as in Theorem 3.8, we can prove

Theorem 4.1. (Theorem 1.7 of [56]). *Let $\Omega \subset M$ be an unbounded domain and let u, $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy*

$$\begin{cases} \operatorname{div}Tu \geq \operatorname{div}Tv & \text{in } \Omega \\ u \leq v & \text{on } \partial\Omega \\ \max_{B_r \cap \Omega}(u-v) = o\left(\int_{R_0}^{R} \frac{\min_{\partial B_R \cap \Omega}\left(\sqrt{1+|\nabla u|^2}+\sqrt{1+|\nabla v|^2}\right)}{|\partial B_r \cap \Omega|}\, dr\right) & \text{as } R \to \infty \end{cases}$$

where R_0 is a positive constant. If $\partial\Omega \neq \emptyset$, we have $u \leq v$ on Ω. If $\Omega = M$ and $u - v$ is not a constant, then $u \leq v$.

Recall that the proofs for Theorem 3.8 and Theorem 4.1 are based on the structural inequality

$$(20) \qquad \begin{aligned} (Tu - Tv) \bullet (\nabla u - \nabla v) &\geq \frac{\sqrt{1+|\nabla u|^2} + \sqrt{1+|\nabla v|^2}}{2}|Tu - Tv|^2 \\ &\geq |Tu - Tv|^2. \end{aligned}$$

Now for the type equations $\operatorname{div}Au \geq \operatorname{div}Av$, if Au satisfies the following structural inequalities

$$(21) \qquad (Au - Av) \bullet (\nabla u - \nabla v) \geq \text{constant}|Au - Av|^2,$$

and

$$(Au - Av) \bullet (\nabla u - \nabla v) = 0 \text{ if and only if } \nabla u = \nabla v, \tag{22}$$

then we can get the same result. See Theorem 4 in [35].

Furthermore, for the type equations div$Au \geq 0$, if Au satisfies (22) and

$$Au \bullet \nabla u \geq \text{constant}|Au|^p, \tag{23}$$

where $p > 1$ is a constant, then with a little modification of the proof of Theorem 4.1, one can get similar results. Such is the idea in [57].

Remark 4.1. In Theorem 3 of [57], the authors consider the type of equations div$(|\nabla u|^{-1}\Phi(|\nabla u|)\nabla u) \geq 0$, and assume that $0 \leq \Phi(t) \leq Ct^\delta$ for some constants C, $\delta > 0$. Let $Au = |\nabla u|^{-1}\Phi(|\nabla u|)\nabla u$, then

$$Au \bullet \nabla u = \Phi(|\nabla u|)|\nabla u|.$$

Since $t \geq \text{constant}\Phi(t)^{1/\delta}$, we have

$$Au \bullet \nabla u \geq \text{constant}(\Phi(|\nabla u|))^{1+1/\delta} = \text{constant}|Au|^{1+1/\delta}.$$

Hence $p = 1 + 1/\delta$ in (23).

The following L^q-comparison result was obtained in [56].

Theorem 4.2. *Let* $\Omega \subset M$ *be an unbounded domain and let* u, $v \in C^2(\Omega) \cap C_0(\overline{\Omega})$ *satisfy*

$$\begin{cases} \text{div}Tu \geq \text{div}Tv & \text{in } \Omega \\ u \leq v & \text{on } \partial\Omega. \end{cases}$$

Assume that for some $q > 1$,

$$\int_{B_r \cap \Omega} |u - v|^q = O(r^2 \log r) \text{ as } r \to \infty.$$

If $\partial\Omega \neq \emptyset$, *then* $u = v$ in Ω, otherwise $u = v+$ constant in M.

Theorem 4.2 can be proved by applying (20). Hence it can also be applied to the case that $\Delta u \geq 0$ in Ω. In particular, we can obtain Yau's well-known L^q theorem:

Theorem 4.3. ([64]). *Let* M *be a complete Riemannian manifold. Suppose that* u *is a positive subharmonic function such that* $u \in L^p(M)$ *with* $p > 1$. *Then* u *must be a constant.*

The following theorem is a result of [56].

Theorem 4.4. *Let $\Omega \subset M$ be an unbounded domain with boundary $\partial\Omega$. Let $p : (0, +\infty) \to (0, +\infty)$ be a continuous function such that for some $\overline{R} > 0$ and for each $R \geq \overline{R}$, either one of the following conditions is satisfied:*

$$(i) \quad \frac{\exp(D(\int_0^R \sqrt{p(s)}\, ds)^2)}{|\partial B_R \cap \Omega|} \notin L^1(0, +\infty)$$

for some constant $D > 0$;

$$(ii) \quad \frac{\left(\int_R^{\frac{3R}{2}} \sqrt{p(s)}\, ds\right)^2}{R \log |\partial B_{2R} \cap \Omega|} \geq h(R) \notin L^1(\overline{R}, +\infty),$$

where $h : (\overline{R}, +\infty) \to (0, +\infty)$ is continuous and monotonically non-increasing.

Let u, $v \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ satisfies

$$\begin{cases} \text{div}Tu - \text{div}Tv \geq p(r(x)) & \text{in } \Omega \\ u \leq v & \text{on } \partial\Omega. \end{cases}$$

if $\sup_{\overline{\Omega}}(u - v) < +\infty$, then $u \leq v$ in Ω.

The proof of Theorem 4.4 is the application of (20) and a variation of some ideas from Grigor'yan [26] and [27]. Originally, Grigor'yan proved the non-existence of the non-trivial bounded solutions of the Schrödinger equation $\Delta u - b(x)u = 0$, $b(x) \geq 0$, $b(x) \not\equiv 0$. For the proof of Theorem 4.4, Pigola, Rigoli, and Setti applied some variations of Grigor'yan's argument. Furthermore, there still are similar theorems without the assumption $\sup_{\overline{\Omega}}(u - v) < +\infty$ in Theorem 4.4, see [13].

5. From the 21 Point Principle to the Half-bound Principle

In fact, there are two kinds of infinite control for a bounded domain Ω, the first one is u takes $+\infty$ on a line segment $\Gamma \subset \partial\Omega$, the second is for a subset $\gamma \subset \partial\Omega$, $Tu \bullet \vec{\nu} = 1$ on γ. In any case, if div$Tv \geq$ divTu in Ω and $v \leq u$ on $\partial\Omega - \Gamma$, or on $\partial\Omega - \gamma$, we have $v \leq u$ in Ω. Finn's non-existence theorem (Theorem 1.2) is based on the second principle.

One can ask that, is there a bounded domain Ω, $Tu \bullet \vec{\nu} = 1$ on $\partial\Omega$? If exists, what is the property of the solution u?

It turns out that there are no such domains and solutions for the MSE, but for any constant $H > 0$, it is possible to have such domains and solutions. For example,

$$\begin{cases} \text{div}Tu = 2 & \text{in } B_1, \\ Tu \bullet \vec{\nu} = 1 & \text{on } \partial B_1 = S^1 \end{cases}$$

has a solution $u(x,y) = -\sqrt{1-x^2-y^2}$, which graph is the lower hemisphere.

In fact, such a solution must be unique up to a constant, we have

Theorem 5.1. (Finn [21]). *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, $H > 0$. If the capillary problem*

$$\begin{cases} \mathrm{div} Tu = H & \text{in } \Omega, \\ Tu \bullet \vec{\nu} = 1 & \text{on } \partial\Omega \end{cases}$$

has a solution u, then for any $v : \Omega \to \mathbb{R}$ such that div$Tv = H$, it must be that $v \equiv u + C$, where C is a constant.

Note that we do not need boundary value hypothesis on v.

Proof.

$$\int_{\partial\Omega} (Tu - Tv) \bullet \vec{\nu} = \int_{\Omega} (\mathrm{div} Tu - \mathrm{div} Tv) = 0.$$

Since $(Tu - Tv) \bullet \vec{\nu} = 1 - Tv \bullet \vec{\nu} \geq 0$ on $\partial\Omega$, then we know that $Tv \bullet \vec{\nu} = 1$ on $\partial\Omega$. Then by Theorem 1.1, $v \equiv u + C$. ■

If we check the proof a little more carefully, we will see that since u has infinite normal derivatives on the boundary, hence $Tu \bullet \vec{\nu} = 1$ on the whole $\partial\Omega$. It is so full just like achieving 21 points in poker game, any more will be blow up. Hence in Ω div$Tu = H$ can have only one solution (up to a constant). Bernstein [4] and Heinz [28] used the same idea to derive similar results.

Based on this 21 points principle, Finn raised a problem in [21],

Problem 5.1. ([Finn]). *Let $\Omega = (-1, 1) \times \mathbb{R}$ and div$Tu = 1$ in Ω. Is the graph generated by u a regular cylinder?*

Note that for such u over a regular cylinder, $Tu \bullet \vec{\nu} = 1$ on $\partial\Omega$.

A.N.Wang [63] and Collin [10] independently gave counter-examples to show that the answer to this problem is negative.

L. F. Tam in [60] and [61] considered the related problem

Theorem 5.2. *Let $\Omega = (-1, 1) \times \mathbb{R}$, $H > 0$ and $0 \leq \gamma < \frac{\pi}{2}$ be two constants, and u satisfies*

$$\begin{cases} \mathrm{div} Tu = H & \text{in } \Omega, \\ Tu \bullet \vec{\nu} = \cos\gamma & \text{on } \partial\Omega. \end{cases}$$

Then

$$u = -\frac{1}{\sqrt{1-\beta^2}} \sqrt{\left(\frac{1}{2\cos\gamma}\right)^2 - x^2} + \frac{\beta}{\sqrt{1-\beta^2}} y + \text{constant},$$

where $|\beta| < 1$ is a constant.

The original proof of of Theorem 5.2 is based on the geometric measure theory. Hwang proved a more general theorem in [34],

Theorem 5.3. *Let $\Omega = (-1, 1) \times \mathbb{R}$, $\Gamma_y = (-1, 1) \times \{y\}$ and $\vec{\nu}_1 = (0, 1)$. If u and v satisfy*

$$\begin{cases} \text{div}Tu = \text{div}Tv & \text{in } \Omega, \\ (Tu - Tv) \bullet \vec{\nu} = 0 & \text{on } \partial\Omega, \\ \text{there is a } y_0 \text{ such that} & \\ \displaystyle\int_{\Gamma_{y_0}} Tu \bullet \vec{\nu}_1 = \int_{\Gamma_{y_0}} Tv \bullet \vec{\nu}_1, & \end{cases}$$

and for any $0 < \epsilon < 1$, $|\nabla u|$ is uniformly bounded in $[-1 + \epsilon, 1 - \epsilon] \times \mathbb{R}$, then $v \equiv u + C$, where C is a constant.

Theorem 5.3 implies Theorem 5.2.

The proof of Theorem 5.3 is based on the structural inequalities (15), (17), and

$$(24) \quad (Tu - Tv) \bullet (\nabla u - \nabla v) \geq \frac{|\nabla u - \nabla v|^2}{\sqrt{1 + (|\nabla u| + |\nabla u - \nabla v|)^2}} \left(1 - \frac{|\nabla u|}{\sqrt{1 + |\nabla u|^2}}\right).$$

The estimate in (24) essentially only involves $|\nabla u|$. It has nothing to do with $|\nabla v|$, thus if $|\nabla u|$ is bounded, then

$$(Tu - Tv) \bullet (\nabla u - \nabla v) \geq C(|\nabla u|) \frac{|\nabla u - \nabla v|^2}{\sqrt{1 + |\nabla u - \nabla v|^2}}.$$

This kind of asymmetry also is reflected in (20).

Problem 5.2. *Is the hypothesis that for any $0 < \epsilon < 1$, $|\nabla u|$ is uniformly bounded in $[-1 + \epsilon, 1 - \epsilon] \times \mathbb{R}$ in Theorem 5.3 necessary?*

6. Two Directions of Generalizations of Phragmèn-lindelöf Theorem for Firm and Soft Domains

Nitsche's theorem, Theorem 3.2, has various extensions. Besides the generalization of Langevin-Levitt-Rosenberg in [45], there are two other directions to generalize, i.e., for the firm and soft domains.

In 1988, Hwang [32] used the infinite boundary values as comparison function to estimate the interior growth property of solutions of MSE. Let $\Omega \subset \{-f(y) < x < f(y) : y > 0\}$, where $f \in C^1([0, \infty))$ is nondecreasing and $f(0) \geq 0$. If div$Tu = 0$ in Ω, $u \leq g(y)$ on $\partial\Omega$ with g nondecreasing, then the growth property of u only depends on the values of f and g. This kinds of domain are properly contained in the halfplane hence we can use the infinite boundary value to control the growth property, so this kind of estimates are called P-L (Phragmèn-Lindelöf) theorems for solid domains. Hwang then generalized his result in [33] and [36].

Theorem 6.1. *Let $\Omega \subset \{|x| < f(y);\ y > 0\} \subset \mathbb{R}^2$ be an unbounded domain and*

$$\begin{cases} \text{div}Tu = 0 & \text{in } \Omega, \\ u \leq 0 & \text{on } \partial\Omega. \end{cases}$$

If

(1) $f = ay^m$, $a > 0$, $m > 1$, then $u \leq f(y)h_m\left(\dfrac{x}{f(y)}\right)$;

(2) $f = ay$, $a > 0$, then $u \leq 0$;

(3) $f = ae^{cy}$, $a > 0$, $c > 0$, then $u \leq \sqrt{a^2e^{2cy} - x^2}$.

Here h_m satisfies

$$\left(1 - \frac{1}{m}\right)(h_m - th'_m)(1 + (h'_m)^2) + h''_m(h_m^2 + t^2) = 0$$

and $h_m(\pm 1) = 0$, $h_m(t) > 0$ for $t \in (-1, 1)$.

If the f increases faster than e^y, then the formula will be more complicated, see Hwang, [33]. Insert

$$F(x,y) = f(y)h_m\left(\frac{x}{f(y)}\right)$$

we can get div$TF \leq 0$. In [36] Hwang gave an example to show that the estimate in Theorem 6.1 is optimal.

Instead using infinite boundary value as comparison function, we can also use the geometric boundary condition $Tu \bullet \nu = +1$ to consider comparison.

We give the proof of (3) of Theorem 6.1. Without loss of generality, assume that $a = c = 1$. Let $\Omega' = \{(x, y) \in \Omega :\ u - F > 0\}$, where $F = \sqrt{e^{2y} - x^2}$. It is easy to check that div$F \leq 0$ in Ω. Let $\Omega_{y_0} = \Omega' \cap \{y < y_0\}$, $\Gamma_{y_0} = \partial\Omega_{y_0} \cap \{y = y_0\}$, then

$$\begin{aligned} \int_{\Gamma_y} \tan^{-1}(u - F)(Tu - TF) \bullet \vec{\nu} &= \int_{\partial\Omega_y} \tan^{-1}(u - F)(Tu - TF) \bullet \vec{\nu} \\ = \quad \int_{\Omega_y} \frac{(\nabla u - \nabla v)}{1 + (u - F)^2} &\bullet (Tu - TF) + (u - F)(\text{div}Tu - \text{div}TF) \geq 0. \end{aligned}$$

Direct calculation gives $(Tu - TF) \bullet \vec{\nu} \leq 2/e^{2y}$ on Γ_y, hence

$$\int_{\Gamma_y} \tan^{-1}(u-F)(Tu-TF)\bullet\vec{\nu} \to 0$$

as $y \to \infty$.

The method works because of $Tu \bullet \vec{\nu} \to 1$ on Γ_y as $y \to \infty$. This is also an application of Buddha's holy palm of second type.

There are two generalizations of Theorem 6.1, one is for higher dimension:

Theorem 6.2. (Hsieh-Hwang-Liang [30]). Let $\Omega \subset \{|x| < f(y);\ y > 0\} \times \mathbb{R}^{n-2} \subset \mathbb{R}^n$ be an unbounded domain,

$$\begin{cases} \text{div}Tu = 0 & \text{in } \Omega, \\ u \leq 0 & \text{on } \partial\Omega. \end{cases}$$

Then

(1) $f = ay^m$, $m > 0$, $a > 0$, then
$u \leq f(y)h_m\left(\frac{x}{f(y)}\right)$;
(2) $f = ay$, $a > 0$, then $u \leq 0$;
(3) $f = ae^{cy}$, $a > 0$, $c > 0$, then $u \leq \sqrt{a^2e^{2cy} - x^2}$.

Here h_m is the same as in Theorem 6.1.

When f grows faster than e^y, we have the same results as stated after Theorem 6.1. Note that the statement and result of Theorem 6.2 is the same as in Theorem 6.1, but since $\Omega_{y_0} := \Omega \cap \{y < y_0\}$ is no longer compact, the proof is more complicated.

Another generalization is for symmetric domain:

Theorem 6.3. (Hsieh [29]) *Let $f(y) = y^m$, $m > 1$ be a constant or $f(y) = e^y$. Let*

$$\Omega \subset \{(x_1, \cdots, x_{d+1}, y);\ |x| = \sqrt{x_1^2 + \cdots + x_{d+1}^2} < f(y);\ y > 0\} \subset \mathbb{R}^{d+2}$$

and

$$\begin{cases} \text{div}Tu = 0 & \text{in } \Omega, \\ u \leq 0 & \text{on } \partial\Omega. \end{cases}$$

Then

$$u(x_1, \cdots, x_{d+1}, y) \leq \frac{1}{t_0} f(y) h\left(\frac{|x|}{f(y)} t_0\right),$$

where h satisfies

$$(h - th')(1 + (h')^2) + \mu h''(h^2 + t^2) + \frac{\lambda}{t} h'(h - th') = 0,$$

$h(0) = 1$, $h'(0) = 1$,

$$\lambda = \frac{dm}{m-1}, \quad \mu = \frac{m}{m-1},$$

for $f(y) = y^m$*;* $\lambda = d, \quad \mu = 1$*, for* $f(y) = e^y$*. Finally,* $t_0 > 0$ *is the first root of* $h(t)$*.*

We can also generalize Theorem 6.3 to the domains

$$\Omega \subset \{(x_1, \cdots, x_{d+1}, y, z_1, \cdots, z_n);\ |x| = \sqrt{x_1^2 + \cdots + x_{d+1}^2} < f(y);\ y > 0\} \times \mathbb{R}^n \subset \mathbb{R}^{n+d+2},$$

just like in Theorem [6.2].

The above P-L theorems are about domains properly contained in a halfspace $\{y > 0\}$, we call them firm domain. There is another kind of P-L theorems in which the domains are called soft domains which need not be properly contained in a halfspace.

Let $\Omega \subset \mathbb{R}^2$, we define the generalized angle of Ω as follows,

$$\beta(\Omega) = \liminf_{r \to \infty} \frac{1}{\log r} \int_{\Omega \cap (B_r - \overline{B_1})} \frac{1}{r^2}. \tag{25}$$

It is easy to calculate that $\beta(\Omega_\alpha) = \alpha$.

Definition 6.1. *We call a simply connected, unbounded domain $\Omega \subset \mathbb{R}^2$ a m-domain, $m > 0$, if and only if $\partial\Omega$ has m connected components.*

In 1981, Miklyukov [50] proved that

Theorem 6.4. *Suppose that $\Omega \subset \mathbb{R}^2$ is a m-domain, $m \geq 1$ and*

$$\begin{cases} \text{div}Tu = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

If $\beta(\Omega) < 2$, then $u \equiv 0$ in Ω.

Problem 6.1. *Suppose that $\Omega \subset \mathbb{R}^2$, $\beta(\Omega) < \pi$ and*

$$\begin{cases} \text{div}Tu = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Is it true that $u \equiv 0$ *in* Ω*?*

Combining the uniqueness theorems in this survey, we would like to make a conjecture:

Conjecture 6.1. *Let* $\Omega \subset \mathbb{R}^2$ *be a simply connected, unbounded domain,* $\partial\Omega$ *be a smooth, proper, non-compact, complete curve. Suppose that*

$$\begin{cases} \mathrm{div}Tu = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

and $\lim_{|x|\to\infty} \vec{\nu}(x)$ *does not exist,* $x \in \partial\Omega$*,* $\vec{\nu}(x)$ *is the unit outward normal vector of* $\partial\Omega$*. Then* $u \equiv 0$ *in* Ω*.*

7. Postscript

The study of minimal surfaces involves many mathematical research areas.

PDE $\mathrm{div}Tu = 0$

$\Uparrow$

Complex Analysis		Geometry
	$\Longleftarrow$ Minimal Surfaces $\Longrightarrow$	
$\Delta_M X = \vec{0}$		$H = 0$

$\Downarrow$

Geometric Measure Theory
Higher Dimensional Bernstein Theorem
Capillary Surface Theory

It is well-known that the structural inequality method is a powerful tool in the investigation of existence problem, see [25]. During the 50's to 70's of the last century, Robert Finn applied the structural inequality method to the study of uniequeness. He obtained non-existence theorems and removable singularity theorems by applying (15) and (17). Later development shows that this is a very nice method.

Roughly speaking, the way of applying structural inequalities to the uniqueness study falls in the following three categories:

(1) Find new structural inequalities to discover and prove new theorems.
(2) Find new structural inequalities to give new proofs of known theorems.
(3) Generalize the known structural inequalities to apply them to more general cases.

These are challenging problems.

But there is no such thing as a universal approach to any certain problem, Finn himself shifted to the geometric measure theory method in the studying of capillary surfaces, see [22].

Even for uniqueness problem, the structural inequality method sometimes does not work well.In 1995 W. H. Huang [34] proved the following:

Let $\Omega \subset \mathbb{R}^2$ be an unbounded domain, let $H(x) \geq H_0 > 0$ and $f = o(\log r)$, H_0 a constant. If the Dirichlet problem

$$\begin{cases} \mathrm{div}Tu = H(x) & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$

has a solution, then it must be unique.

The statement of this result seems similar to Theorem 3.5, in fact the proof used Theorem 3.5, but the key of the proof is a method of Meeks, see [48].

Z.Jin, K. Lancaster, and J. Stanley established some uniqueness theorems by applying the strongly singularity elliptic idea of Serrin in [58], see [40], [41], [42], and [44].

Every method has its own advantages and drawbacks.

Are there more structural inequalities for divTu so we can get more uniqueness theorems? The answer is yes but so far there are no better results to report. We should stop here.

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References

1. I. J. Almgren, Jr. Some interior regularity theorems for minimal surfaces and an extension of Berstein's theorem. *Ann. of Math.* **84** (1966), 277-292.

2. I. Y. A. Bakel'man. Mean curvature and quasilinear elliptic equations. *Sibirsk. Mat. Z.* **9** (1968), 1014-1040, [Russian]. English Translation in *Siberian Math. J.* **9** (1968), 752-771.

3. I. Y. A. Bakel'man. Geometric problems in quasilinear elliptic equations. *Uspehi Mat. Nauk.* **25(3)** (1970), 49-112, [Russian]. English Translation in *Russian Math. Surveys* **25(3)** (1970), 45-109.

4. S. Bernstein. Sur les surfaces definies au moyen de leur courbure moyenne ou total. *Ann. Sci. Ecole Norm. Sup. (3)* **27** (1910), 233-256.

5. S. Bernstein. Sur les éuqations du calcul des variations. *Ann. Sci. Ecole Norm. Sup. (3)* **29** (1912), 431-485.

6. S. Bernstein. Sur un théorème de géométrie et son application aux équations aus dérivées partielles du type elliptique. *Comm. Soc. Math. de Kharkov* 2éme sér, **15** (1915-1917), 38-45. German trans: Uber ein geometrisches Theorem und seine Anwendung auf die partiellen Differentialgleichungen vom elliptischen Typus, *Math. Zeit.*, **26** (1927), 551-558.
7. L. Bers. Isolated singularities of minimal surfaces. *Ann. of Math.* (2), **53** (1951), 364-386.
8. E. Bombieri, E. de Giorgi, and E.Giusti. Minimal areas and Berstein problem. *Invent. Math.*, **7** (1969), 243-268.
9. C. C. Chen. Removability of singularities for minimizers; Uniqueness of positive solutions of $\Delta u + f(u) = 0$. Ph.D. Thesis, National Taiwan University, 1991.
10. P.Collin. Deux exemples de graphes de courbure moyenne constante sur une bande de R^2. (French) [Two examples of constant mean curvature surfaces on a strip in $\mathbf{R}^2$] *C. R. Acad. Sci. Paris S. I Math.*, **311(9)** (1990), 539-542.
11. P. Collin and R. Krust. Le problème de Dirichlet pour l'équation des surfaces minimales sur des domaines non bornés. (French) [The Dirichlet problem for the minimal surface equation in unbounded domains] *Bull. Soc. Math. France*, **119(4)** (1991), 443-462.
12. P. Concus and R. Finn. On capillary free surfaces in the absence of gravity. *Acta Math.*, **132** (1974), 177-198.
13. Y. Fang and J. F. Hwang. On uniequeness of solutions to the prescribed mean curvature equation on non-compact complete Riemannian minifolds. In preparation.
14. R. de Giorgi. Vna extensione del theorena du Bernstein. *Ann. Scuola Norm. Sup. Pisa*, **19** (1965), 79-85.
15. R. de Giorgi and G. Stampacchia. Sulle singolarità eliminabili delle ipersuperficie minimali. *Atti. Accad. Naz. Lincei. Rand. Cl. Sci. Fis. Mat. Natur. (Ser 8)*, **38** (1965), 352-357.
16. Dodziuk, Józef (ed.) et al., Lipa's legacy. Proceedings of the 1st Bers colloquium, New York, NY, October 19-20, 1995. Providence, RI: American Mathematical Society. Contemp. Math. **211** (1997), 455-461.
17. R. Finn. Isolated singularities of solutions of non-linear partial differential equations. *Trans. Amer. Math. Soc.*, **75** (1953), 385-404.
18. R. Finn. On equations of minimal surface type. *Ann. of Math. (2)*, **60** (1954), 397-416.
19. R. Finn. On partial differential equations (whose solutions admit no isolated singularities). *Scripta Math.*, **26** (1961), 107-115.
20. R. Finn. New estimates for equations of minimal surface type. *Arch. Rational Mech. Anal.*, **14** (1963), 337-375.

21. R. Finn. Remarks relevant to minimal surfaces, and to surfaces of prescribed mean curvature. *J. Analyse Math.*, **14** (1965), 139-160.

22. R. Finn. Equilibrium Capillary Surfaces. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **284.** Springer-Verlag, New York, (1986).

23. R. Finn and J. F. Hwang. On the comparison principle for capillary surfaces. *J. Fac. Sci. Univ. Tokyo Sect. I.A. Math.*, **36** (1989), 132-134.

24. W. H. Fleming. On the oriented Plateau problem. *Rend Circ. Ma. Palemo Serie II*, **11** (1965), 69-90.

25. D. Gilbarg and N. Trudinger. Elliptic Partial Differential Equations of Second Order. Grundlehren math. Wiss. Vol. **224.** Springer-Verlag. Berlin, Heidelberg, New York. Second edition (1983).

26. A. Grigor'yan. On stochastically complete manifolds. (In Russian) *PANSSSR*, **290** (1986), 534-537. English tranlation: *Sovie. Math. Dokl.*, **34** (1987), 310-313.

27. A. Grigor'yan. Bounded solutions of the Schrödinger equation on non-compact Riemannian manifolds. (In Russian) *Trudy Seminara, I.G. Petrovskogi*, **14** (1989), 66-77. English tranlation: *Sovie. Math.* , **51** (1990), 2340-2349.

28. E. Heinz. Uber Flächen mit eineindeutiger Projektion auf eine Ebene, deren Krümmugen durch Ungleichungen eingeschränkt sind. *Math. Ann.*, **129** (1955), 451-454.

29. C. C. Hsieh. Phragmèn-Lindelöf theorem for minimal surface equation in domains with symmetry. *Geom. Dedicata*, **71** (1998), 97-109.

30. C. C. Hsieh, J. F. Hwang, and F. T. Liang. Phragmèn-Lindelöf theorem for minimal surface equation in higher dimensions. *Pacific J. Math.*, **207(1)** (2002), 183-198.

31. J. F. Hwang. Comparison principles and Liouville theorems for prescribed mean curvature equations in unbounded domains. *Annali. Scuola Norm. Sup. Pisa*, IV **15** (1988), 341-355.

32. J. F. Hwang. Phramén-Lindelöf theorem for the minimal surface equation. *Proc. Amer. Math. Soc.*, **104** (1988), 825-828.

33. J. F. Hwang. Growth property for the minimal surface equation in unbounded domains. *Proc. Amer. Math. Soc.*, **121(4)** (1994), 1027-1037.

34. J. F. Hwang. On the uniqueness of capillary surfaces over an infinite strip. *Pacific J. Math.*, **169** (1995), 95-105.

35. J. F. Hwang. A uniqueness theorem for the minimal surface equation. *Pacific J. Math.*, **176(2)** (1996), 357-364.

36. J. F. Hwang. Catenoid-like solutions for the minimal surface equation. *Pacific J. Math.*, **183** (1998), 91-102.

37. J. F. Hwang. Structural inequalities mathod for uniequeness theorems for the minimal surface equation. Joint Australia-Taiwan Workshop on Analysis and Applications, Brisbane, March 1997. Proceedings of CMA, Australian National University, Vol. **37** (1999), 47-52.

38. H. Jenkins and J.Serrin. Variational problems of minimal surface type II. Boundary value problems for the minimal surface equation. *Arch. Rat. Mech. Analysis*, **21** (1966), 321-342.

39. H. Jenkins and J. Serrin. The Dirichlet problem for the minimal surface equation in higher dimensions. *J. Reine Angew. Math.*, **229** (1968), 170-187.

40. Z. Jin and K. Lancaster. A comparison principle for quasilinear elliptic equations and its application. *Pacific J. Math.*, **184** (1998), 279-294.

41. Z. Jin and K. Lancaster. Phragmèn-Lindelöf theorems and the asymptotic behavior of solutions of quasilinear elliptic equations in slabs. *Proc. Roy. Soc. Edinburgh*, **130A** (2000), 335-373.

42. Z. Jin and K. Lancaster. The convergence rate of solutions of quasilinear elliptic equations in slabs. *Comm. Part. Diff. Equ.*, **25** (2000), 957-985.

43. V. V. Kurta. On the uniqueness of the solutions of the Dirichlet problem for equations of mean curvature type. *Math. Zametki*, **53** (1993), 53-61. English transl. in *Math. Notes*, **53** (1993), 394-399.

44. K. Lancaster and J. Stanley. On the asymtoticbehavior of solutions of quasilinear elliptic equations. Preprint.

45. R. Langevin, G. Levitt, and H. Rosenberg. Complete minimal surfaces with long line boundaries. *Duke Math. Journ.*, **55(4)** (1987), 985-995.

46. C. C. Lee. A uniqueness theorem for the minimal surface equation on an unbounded domain in $\mathbb{R}^2$. *Pacific J. Math.*, **177** (1997), 103-107.

47. U. Massari and M. Miranda. Minimal Surfaces of Codimension One. North-Holland Math. Studies, 91, 1984.

48. W. Meeks III. Uniqueness theorems for minimal surfaces. *Illinois J. Math.*, **25(2)** (1981), 318-336.

49. V. M. Miklyukov. On a new approach to Berenstein's theorem and related questions for for equations of minimal surface type. *Mat. Sb.*, **108(150)** (1979), 268-289. English transl. in *Math. USSR Sb.*, **36** (1980), 252-271.

50. V. M. Miklyukov. Some singularities in the behavior of solutions of equations of minimal-surface type in unbounded domains. *Mat. Sb.*, **1116(158)** (1981), 72-86. English transl. in *Math. USSR Sb.*, **44** (1983), 61-73.

51. J. C. C. Nitsche. On the non-solvability of Dirichlet's problem for the minimal surface equation. *J. Math. Mech.*, **14** (1965), 779-788.

52. J. C. C. Nitsche. On new results in the theory of minimal surfaces. *Bull. Amer. Math. Soc.*, **71** (1965), 195-270.

53. J. C. C. Nitsche. Über ein verallgemeinertes Dirichletsches Problem für die Minimal flächengleichung und hebbare Unstetigkeiten ihrer Lösungen. *Math. Ann.*, **158** (1965), 203-214.

54. J. C. C. Nitsche. Vorlesungen über Minimälflaschen. Springer-Verlag, Berlin-Heidelberg-New York, 1975.

55. R. Osserman. A Survey of Minimal Surfaces. Dover Publications, Inc. New York, 1986.

56. S. Pigola, M. Rigoli, and A. Setti. Some remarks on the prescribed mean curvature equation on complete manifolds. To appear in *Pacific J. Math.*.

57. M. Rigoli, M. Salvatori, and M. Vignati. A Liouville type theorem for a general class of operators on complete manifolds. *Pacific J. Math.*, **194** (2000), 439-453.

58. J. Serrin. The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables. *Philos. Trans. Roy. Soc. London Ser. A*, **264** (1969), 413-496.

59. J. Simons. Minimal varieties in Riemannian manifolds. *Ann. of Math.*, **88** (1968), 62-105.

60. L. F. Tam. On the uniqueness of capillary surfaces without gravity over an infinite strip. *Indiana Univ. Math. J.*, **36** (1987), 79-89.

61. L. F. Tam. On the uniqueness of capillary surfaces. In *Variations Methods for Free Surfaces Interfaces.*, Edited by P. Concus and R. Finn, Springer-Verlag, Berlin-Heidelberg-New York, 1987.

62. N. S. Trudinger. On problems of Dirichlet for quasilinear elliptic differential equations with many independent variables. *Arch. Rational Mech. Anal.*, **57** (1974), 128-133.

63. A. N. Wang. Constant mean curvature surfaces on a strip. *Pacific J. Math.*, **145** (1990), 395-396.

64. S. T. Yau. Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry. *Indiana Univ. Math. J.*, **25** (1976), 659-670.

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