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SPECTRA OF DIMENSIONS FOR POINCARÉ RECURRENCES FOR SPECIAL FLOWS

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Abstract. We prove the variational principle for dimensions for Poincaré recurrences, in the case of invariant sets of dynamical systems with continuous time. To achieve this goal we show that these dimensions can be expressed as roots of a non–homogeneous Bowen equation.

1. Introduction

Typical motions in dynamical systems repeat their behavior in time. Complexity or simplicity of dynamics depends on this repetition, and quantitative properties of behavior of orbits can be often expressed in terms of Poincaré recurrences. Of course, it is generally impossible to describe completely the return behavior of every orbit. Traditionally people study statistical properties of Poincaré recurrences (see, for instance [13]). In this approach one does not take into account sets of zero measure which can be very large when measured by means of Hausdorff dimension or topological entropy [3].

A new approach has been proposed recently [2, 4] that makes use of ideas and methods of dimension theory [9]. The spectra of dimensions for Poincaré recurrences has been introduced and studied [1, 2]. It was shown in [2] that for maps on conformal zero-dimensional repellers, the spectrum of dimensions is defined by roots of a non-homogeneous Bowen equation. A similar problem for systems with continuous time was unsolved.

While introducing a dimension-like characteristic of sets, one can immediately introduce a dimension for the measure as follows (see [9] for a more general definition).

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Let $\mathcal{F}$ be a collection of subsets of a metric space (X, d), such that for any $\epsilon > 0$ there exists finite or countable subcollection $\mathcal{G} \subset \mathcal{F}$, of subsets with diameter less than or equal to ϵ, which is a cover of X. Let ξ, $\eta : \mathcal{F} \to \mathbb{R}^+$ be functions such that $\xi(\emptyset) = \eta(\emptyset) = 0$, and $\eta(U) = 0$ iff $U = \emptyset$. Given $Z \subset X$, $\epsilon, \alpha > 0$, form the partition function

$$\mathcal{M}(Z, \alpha) \equiv \lim_{\epsilon \to 0} \left(\inf_{\text{diam}(\mathcal{G}) \leq \epsilon} \sum_{B_i \in \mathcal{G}} \xi(B_i)(\eta(B_i))^\alpha \right).$$

where the infimum is take over finite or countable covers from $\mathcal{F}$ of the set Z. It was shown in [9] that there exists a critical value $\alpha = \alpha_c \in [-\infty, \infty]$, such that $\mathcal{M}(Z, \alpha) = 0$ if $\alpha > \alpha_c$, and $\mathcal{M}(Z, \alpha) = \infty$ if $\alpha < \alpha_c$. The value α_c is called the Carathéodory dimension of the set Z, corresponding to the Carathéodory structure $(\mathcal{F}, \xi, \eta)$. We will say that β is a dimension-like characteristic if $\beta = \alpha_c$ for some Carathéodory structure $(\mathcal{F}, \xi, \eta)$.

When one has a dimension-like characteristic β, and a probability measure μ on X, one may introduce the dimension of measure $\dim_\beta(\mu)$ as follows

$$\dim_\beta(\mu) = \inf\{\beta(Z) \ : \ \mu(Z) = 1\}.$$

Two problems appear to be considered.

(1) Variational principle: is it true that

$$\sup\{\dim_\beta(\mu)\} = \beta(X)$$

where supremum is taken over all Borel probability measures?

(2) Existence of measures of full dimension: is there at least one Borel probability measure μ_0 such that

$$\dim_\beta(\mu_0) = \beta(X)?$$

In the present work, in the capacity of β we consider dimensions for Poincaré recurrences (see below). Let us notice that for this dimension-like characteristic the first problem was solved for maps acting on Cantor sets resulting from Moran type constructions (conformal zero-dimensional repellers) [2]. The authors made use of non-homogeneous Bowen equation.

In the present article we prove a similar result for a class of systems with continuous time, namely suspension flows on conformal repellers maps. (Let us recall that special flows over subshifts of finite type are the basic ingredient in the construction of the symbolic dynamics of hyperbolic and Anosov flows [5].) Our strategy is also the derivation of a non-homogeneous Bowen equation for special flows, and then to show that spectra for Poincaré recurrences for flows are determined by roots of this equation. This, together with known results for maps in [4], lead to the variational principle for dimensions for Poincaré recurrences, the main result of this article.

Let us notice that asymptotic properties of Poincaré recurrences are not so well studied and understood. The main classical result concerns the distribution of periodic orbits (zeta functions); see for instance [8]. Therefore our results and the ones in [4, 7] provide a new insight into the nature of recurrences for special flows.

As far as the above problem (2) is concerned, the problem is solved for Hausdorff dimension in the case of a class of maps on conformal repellers (see for instance [11]). For the conformal axiom A flows, this problem was solved in [10]. For dimensions of Poincaré recurrences, however, the question remains open.

In Section 2 we list notions and results needed for the formulation and the proof of the main result. In Section 3 we formulate and prove the variational principle for special flows over conformal zero-dimensional repellers.

2. Preliminaries

2.1. Maps under consideration

Following the definitions and notation in [4], let (X, T) be a subshift of $(\Sigma^{\mathbb{N}}, T)$. Denote ζ the partition of X into 1-cylinders, and $\zeta^n = \bigvee_{j=0}^{n-1} T^{-j}\zeta$, $n \geq 1$, the dynamical refinements of this partition. Finally, let us denote ζ^0 the trivial partition $\{X, \emptyset\}$. The length of a cylinder $\Delta \in \bigcup_{n=0}^{\infty} \zeta^n$ is the integer $|\Delta| = \max\{m \in \mathbb{Z}^+ : \Delta \in \zeta^m\}$ which coincides with the number of symbols in Σ needed to determine Δ.

We assume that (X, T) is weakly specified, i. e., there exists an integer $n_0 \in \mathbb{Z}^+$ such that for any two cylinders $\Delta \in \zeta^n$ and $\Delta' \in \zeta^m$, and for each integer $p \geq n + m + n_0$, there exists a periodic point $\tilde{x}$ of period p such that $\tilde{x} \in \Delta$ and $T^{n+n_0}\tilde{x} \in \Delta'$. Classical examples of specified subshifts are subshifts of finite type and their topological factors, namely sofic subshifts.

For each $x \in X$ let $\Delta^n(x)$ be the atom of ζ^n containing x, i. e. $\Delta^n(x)$ is the only cylinder of length n containing x. We now define a metric on X generating the product topology. Let $u : X \to (0, \infty)$ be a continuous function and define the metric

$$d_X(x, y) = e^{-S^n(u(\Delta^n(x)))} \tag{1}$$

where $n = \sup\{k \in \mathbb{N} : \Delta^k(x) = \Delta^k(y)\}$ and $S^n(u(\Delta^n(x))) = \max_{z \in \Delta^n(x)} \sum_{j=0}^{n-1} u(T^j z)$. In the sequel we will assume that u is Hölder continuous.

It is not hard to see that d_X an ultrametric. Furthermore, open balls coincides with cylinder sets, i. e., for each $x \in X$ and $\epsilon > 0$ there exists a unique $n(x, \epsilon) \in \mathbb{Z}^+$ such that $B(x, \epsilon) = \Delta^{n_{x,\epsilon}}(x)$. And vice versa, for each $x \in X$ and $n \in \mathbb{N}$ there exists $\epsilon(x,n) > 0$ such that $\Delta^n(x) = \{y \in X : d_X(x,y) < \epsilon(x,n)\}$, thought the choice of $\epsilon(x,n)$ is not unique.

All T-invariant probability measures are assumed to have a strictly positive entropy, i.e. $h_\mu(T) > 0$.

2.2. Poincaré recurrences

For any set $U \subset X$ define its Poincaré recurrence as

$$\tau_T(U) = \min\{k \in \mathbb{N} : T^k(U) \cap U \neq \emptyset\}.$$

Note that $\tau_T(U) = \tau_T(T^{-1}U)$ for any $U \subset X$, and that $\tau_T(A) \leq \tau_T(B)$ whenever $A \subset B$.

2.3. Special space and special flow

Let $\phi : X \to (0,\infty)$ be a Hölder continuous function. Consider the interval $[0, \max_{x\in X} \phi(x)]$ with the usual topology, and endow $X \times [0, \max_{x\in X} \phi(x)]$ with the product topology. The special space defined by ϕ over X is the quotient space

$$X^\phi \equiv \{(x,t) : x \in X,\ t \in [0,\phi(x))\},$$

defined by the quotient map

$$(x,t) \mapsto \begin{cases} (x,t) & \text{if } t < \phi(x) \\ (Tx,0) & \text{if } t \geq \phi(x). \end{cases}$$

To simplify the notation let $(x, t\phi(x))$ be denoted by x_t.

In the special space X^ϕ we define the special flow $\Phi : X^\phi \times \mathbb{R}^+ \to X^\phi$ by

$$\Phi(x_s, t\phi(x)) = \begin{cases} x_{s+t} & \text{if } \ s+t < 1 \\ (T^n(x))_{s'} & \text{if } \ s+t \geq 1, \end{cases}$$

where $n = \min\{k \in \mathbb{N} : \sum_{j=0}^{k} \phi(T^j(x)) \geq (s+t)\phi(x)\}$ and $s' = \frac{(s+t)\phi(x) - \sum_{j=0}^{n-1} \phi(T^j(x))}{\phi(T^n(x))}$.

2.4. Bowen-Walters' distance

For $\phi = 1$ there exists a natural metric generating the quotient topology on X^1, which was first introduced in [6]. This definition can be readily adapted for the general case.

Consider the t-horizontal sections $X_t \equiv \{x_t \in X^\phi : x \in X\}$, and define the t-horizontal distance $\rho_t(x_t, y_t) = (1-t)d_X(x,y) + t d_X(Tx, Ty)$.

A path p between x_t and $y_{t'}$ is a finite sequence $p = \{x_t = x_{t_0}^{(0)}, x_{t_1}^{(1)}, \ldots, x_{t_{n-1}}^{(n-1)}, x_{t_n}^{(n)} = y_{t'}\}$, such that for each $0 \leq i < n$, if $x^{(i+1)} \notin \{x^{(i)}, Tx^{(i)}\}$ then $t_i = t_{i+1}$. The length of the path p is given by $|p| \equiv \sum_{i=0}^{n-1} |\{x_{t_i}^{(i)}, x_{t_{i+1}}^{(i+1)}\}|$, with

$$|\{x_{t_i}^{(i)}, x_{t_{i+1}}^{(i+1)}\}| = \begin{cases} 1 - t_i + t_{i+1} & \text{if } Tx^{(i)} = x^{(i+1)} \text{ and } t_i > t_{i+1}, \\ \rho_{t_i}(x_{t_i}^{(i)}, x_{t_i}^{(i+1)}) + |t_{i+1} - t_i| & \text{otherwise.} \end{cases}$$

Finally, the distance in the special space is given by $d_{X^\phi}(x_t, y_{t'}) = \inf\{|p| : p \in [x_t \to y_{t'}]\}$, where $[x_t \to y_{t'}]$ denotes the set of all paths from x_t to $y_{t'}$.

2.5. Spectrum of dimensions

In the computation of the Poincaré spectrum for Φ, we will use covers of X^ϕ by special open sets we call rectangles.

Given $\epsilon > 0$ and $x_s \in X^\phi$, define the s-horizontal open ball of radius ϵ, $\mathcal{S}(x_s, \epsilon) = \{y_s \in X_s : \rho_s(x_s, y_s) < \epsilon\}$. The rectangle with base $\mathcal{S}(x_s, \epsilon)$ and of height $\delta > 0$ is the set

$$R(x_s, \epsilon, \delta) = \bigcup_{y_s \in \mathcal{S}(x_s,\epsilon)} \bigcup_{0<t<\delta} \Phi(y_s, \phi(y)t).$$

These set are of course open in X^ϕ.

The Poincaré recurrence $\tau_\Phi(B)$, for $B \subset X^\phi$, is the minimal time that one orbit starting in B spends in $X^\phi \setminus B$ before it reenters B. In the particular case of a rectangle $R(x_s, \epsilon, \delta)$, the Poincaré recurrence can be computed as follows. For each $y_s \in \mathcal{S}(x_s, \epsilon)$ let $\tau_\Phi(y_s, R(x_s, \epsilon, \delta)) = \inf\{t \geq \phi(y)\delta : \Phi(y_s, t) \in R(x_s, \epsilon, \delta)\}$, then $\tau_\Phi(R(x_s, \epsilon, \delta)) = \inf\{\tau_\Phi(y_s, R(x_s, \epsilon, \delta)) : y_s \in R(x_s, \epsilon, \delta)\}$.

Let $\mathcal{R}$ be a finite cover of X^ϕ by rectangles, $\alpha, q \in \mathbb{R}$, and $Z \subset X^\phi$. Define

$$\mathcal{M}(Z, \alpha, q, \mathcal{R}) = \sum_{R_i \in \mathcal{R}} \exp(-q\tau_\Phi(R_i))(\text{diam} R_i)^\alpha.$$

Monotonicity implies that the limit $\mathcal{M}(Z, \alpha, q) \equiv \lim_{\epsilon \to 0} \left(\inf_{\text{diam}(\mathcal{R}) \leq \epsilon} \mathcal{M}(\alpha, q, \mathcal{R})\right)$ exists. Here, by diam$(\mathcal{R})$ we mean the maximum of the diameters of rectangles in $\mathcal{R}$. Note that for each fixed $q \in \mathbb{R}$, $\mathcal{M}(\alpha, q) = \infty$ for α sufficiently small. We can now define the spectrum for Poincaré recurrences for $Z \subset X^\phi$ as

$$\alpha(Z, q) \equiv \sup\{\alpha \in \mathbb{R} : \mathcal{M}(\alpha, q) = \infty\}.$$

2.6. Spectrum for measures

For each T–invariant Borel probability measure μ in X, define the Φ-invariant probability measure $\bar{\mu} = \bar{\mu}_\phi$ such that

$$\int_{X^\phi} F \, d\bar{\mu} \equiv \frac{\int_X \left(\int_0^{\phi(x)} F(x,t) \, dt \right) \, d\mu(x)}{\int_X \phi(x) \, d\mu(x)},$$

for each continuous function $F : X^\phi \to \mathbb{R}$. It can be shown that every Φ-invariant probability measure on X^ϕ can be obtained in this way, by using Fubini's Theorem.

For $\alpha, q \in \mathbb{R}$, $Z \subset X^\phi$, and $\bar{\mu}$ a Φ-invariant Borel probability measure, define the spectrum for the measure $\bar{\mu}$ as $\alpha^{\bar{\mu}}(q) \equiv \inf\{\alpha(V,q) : V \subset X^\phi, \ \bar{\mu}(V) = 1\}$.

In [4] it was shown the following theorem.

Theorem 2.1. *For all $q \in \mathbb{R}$, the spectrum $\alpha^{\bar{\mu}}$ for the measure $\bar{\mu}$ satisfy the following equation*

$$h_\mu(T) + \int_X \left((1 - \alpha^{\bar{\mu}}) u(x) - q\phi(x) \right) \, d\mu(x) = 0,$$

where $\bar{\mu}$ is obtained from μ as described above, and $h_\mu(T) (> 0)$ is the entropy of the measure μ with respect to the shift map. (Recall that u is the function defining the metric, see relation (1).)

This theorem is just a reformulation of Theorem 7 in [4]. Let us stress that it is valid only for measures μ with positive entropy, that is $h_\mu(T) > 0$; see [4] for counter-examples.

2.7. Topological pressure

Let (X, T) be a subshift of $(\Sigma^{\mathbb{N}}, T)$, and ψ a real-valued continuous function on X. Let $Z_n(\psi, X) = \sum_{\Delta \in \zeta^n} \exp(S^n(\psi(\Delta)))$, where $S^n(\psi(\Delta)) = \sup_{x \in \Delta} \sum_{j=0}^{n-1} \psi(T^j x)$. It was proved in [12] that the limit $P_X(\psi|T) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\psi, X)$ exists. This limit is called "the topological pressure of the function ψ on X with respect to T".

For every constant $c \in \mathbb{R}$, the topological pressure satisfies $P_X(c + \phi|T) = c + P_X(\phi|T)$.

There is a dimension-like definition of the topological pressure [9] based on a Carathéodory construction. For a countable cover $\mathcal{C}$ of S by cylinders and $\beta \in \mathbb{R}$ let

$$\mathcal{Z}(\beta, \psi, \mathcal{C}, X) = \sum_{\Delta \in \mathcal{C}} \exp(-\beta|\Delta| + \psi(\Delta)).$$

It was proved in [9] that the topological pressure $P_S(\psi)$ coincides with the threshold value

$$P_X(\psi|T) = \sup\left\{\beta : \lim_{n\to\infty}(\inf\{\mathcal{Z}(\beta,\psi,\mathcal{C},X) : |\mathcal{C}| \geq n\}) = \infty\right\},$$

where $|\mathcal{C}| = n$ denotes the minimal length of the cylinders in $\mathcal{C}$.

3. A Variational Principle for the Poincaré Spectrum

As mentioned in the introduction, the aim of this paper is to prove a variational principle for the spectrum for Poincaré recurrences for a special flow. Our main result is the following.

Theorem 3.1. *Let $\bar{\mathcal{M}}_e$ be the set of all ergodic Φ-invariant probability measures in X^ϕ. Then, for $\alpha \geq 1$ and $q \geq 0$, $\alpha(X^\phi, q) = \sup\{\alpha^{\bar{\mu}}(q) : \bar{\mu} \in \bar{\mathcal{M}}_e\}$.*

The proof of this result is based on Theorem 2.1 and the Bowen-like equation stated in the next theorem.

Theorem 3.2. *For $\alpha \geq 1$ and $q \geq 0$, the spectrum for Poincaré recurrences $\alpha^* \equiv \alpha(X^\phi, q)$ satisfies the Bowen-like equation $P_X((1-\alpha^*)u - q\phi|T) = 0$.*

Remark 3.1. If $q = 0$, we obtain the result of Theorem 4.2 in [10].

This equation follows from two inequalities.

Claim 3.1. *If $\alpha \geq 1$ and $q \geq 0$ are such that $P_X((1-\alpha)u - q\phi|T) < 0$, then $\alpha \geq \alpha^* \equiv \alpha(X^\phi, q)$.*

Claim 3.2. *If $\alpha \geq 1$ and $q \geq 0$ are such that $P_X((1-\alpha)u - q\phi|T) > 0$, then $\alpha \leq \alpha^* \equiv \alpha(X^\phi, q)$.*

The first inequality is proved by exhibiting a particular sequence of covers by rectangles $\{\mathcal{R}_n : n \in \mathbb{N}\}$, such that $\operatorname{diam}(\mathcal{R}_n) \to 0$ when $n \to \infty$, and $\lim_{n\to\infty} \mathcal{M}(X^\phi, \alpha, q, \mathcal{R}_n) < \infty$, whenever $P_X((1-\alpha)u - q\phi|T) < 0$. For the second inequality we will require the definition of the pressure. In both cases we will need the following results.

3.1. Concerning the diameter of rectangles

Lemma 3.1. *Let $x_t, y_{t'} \in X^\phi$ be such that $\Delta^1(x) = \Delta^1(y)$, and $|t - t'| + \rho_s(x_s, y_s) \leq 1 - t$, then $d_{X^\phi}(x_t, y_{t'}) = \rho_s(x_t, y_t) + |t' - t|$, with $s = \min(t, t')$.*

Proof. Let $\delta(z, z') = d_X(Tz, Tz') - d_X(z, z')$ for each $z, z' \in X$. Since $\rho_{s'}(z_{s'}, z'_{s'}) = \rho_s(z_s, z'_s) + (s' - s)\delta(z, z')$, then $\rho_{s'}(z_{s'}, z'_{s'}) \geq \rho_s(z_s, z'_s)$ whenever $s < s'$ and $\delta(z, z') \geq 0$.

Non-traversing and positive paths. Let $q = \{x_t = z^{(0)}_{t_0}, z^{(1)}_{t_1}, \ldots, z^{(m-1)}_{t_{m-1}}, z^{(m)}_{t_m} = y_{t'}\} \in [x_t \to y_{t'}]$ be "non–traversing", i. e., $z^{(i+1)} = z^{(i)}$ whenever $t_i \neq t_{i+1}$. Suppose also that q is "positive", i. e., $\delta(z^{(i)}, z^{(i+1)}) \geq 0$ for each $0 \leq i < m$. Then we have

$$\begin{aligned}
|q| &\geq \sum_{i=0}^{m-1} \rho_{s'}(z^{(i)}_{s'}, z^{(i+1)}_{s'}) + \sum_{i=0}^{m-1} |t_{i+1} - t_i| \\
&\geq \rho_{s'}(z^{(0)}_{s'}, z^{(m)}_{s'}) + |t_m - t_0| + 2(s - s') \\
&= \rho_s(z^{(0)}_s, z^{(m)}_s) + (t_0 + t_m - 2s') + (s - s')(2 - \delta(z^{(0)}, z^{(m)})) \\
&\geq \rho_s(x_s, y_s) + |t' - t|,
\end{aligned}$$

where $s' = \min_i t_i$ and $s = \min(t_0, t_m)$.

Hence, our first partial conclusion is that

$$\inf\{|q| : \ q \in [x_t \to y_{t'}] \text{ non-traversing and positive }\} \geq \rho_s(x_s, y_s) + |t' - t|.$$

Non-traversing general paths. Let $q = \{x_t = z^{(0)}_{t_0}, z^{(1)}_{t_1}, \ldots, z^{(m-1)}_{t_{m-1}}, z^{(m)}_{t_m}\} \in [x_t \to y_{t'}]$ be non-traversing. Suppose that $\delta(z^{(j)}, z^{(j+1)}) < 0$ for each $j \in \{i, i+1, \ldots, i+k-1\} \subset \{0, 1, \ldots, m-1\}$. Then necessarily $z^{(j)} \neq z^{(j+1)}$ and $t_j = t_i$ for all $j \in \{i, i+1, \ldots, i+k\}$. Since $\rho_{t_i}(z^{(j)}_{t_i}, z^{(j+1)}_{t_i}) = (1 - t_i) d_X(z^{(j)}, z^{(j+1)}) = 1 - t_i$, then we have

$$|\{z^{(i)}_{t_i}, z^{(i+1)}_{t_i}, \ldots, z^{(i+k)}_{t_i}\}| = (k-1)(1 - t_i) \geq \rho_{t_i}(z^{(i)}_{t_i}, z^{(i+k)}_{t_i}) = 1 - t_i$$

whenever $\delta(z^{(i)}, z^{i+1}) < 0$.

Hence, to each non-traversing path q, we may associate another non-traversing path $q' = \{x_t = w^{(0)}_{t_0}, w^{(1)}_{t_1}, \ldots, w^{(m'-1)}_{t_{m'-1}}, w^{(m')}_{t_{m'}} = y_{t'}\} \in [x_t \to y_{t'}]$, with $m \leq m'$, such that if $\delta(w^{(i)}, w^{(i+1)}) < 0$ then $\delta(w^{(i-1)}, w^{(i)}) \geq 0$, for each $1 \leq i < m'$. Furthermore $|q'| \leq |q|$.

Now, for $1 \leq i < m'$ such that $\delta(w^{(i)}, w^{(i+1)}) < 0$, and for arbitrary small $\eta > 0$ we have

$$\begin{aligned}
|\{w^{(i-1)}_{t_{i-1}}, w^{(i)}_{t_i}, w^{(i+1)}_{t_i}\}| &= \rho_{t_{i-1}}(w^{(i-i)}_{t_{i-1}}, w^{(i)}_{t_{i-1}}) + |t_i - t_{i-1}| + 1 - t_i \\
&\leq \rho_{t_{i-1}}(w^{(i-i)}_{t_{i-1}}, y^{(i)}_{t_{i-1}}) + |t_i - t_{i-1}| + \rho_{t_{i-1}}(y^{(i)}_{t_i}, w^{(i+1)}_{t_i}) + \eta \\
&\leq |\{w^{(i-1)}_{t_{i-1}}, y^{(i)}_{t_{i-1}}, y^{(i)}_{t_i}, w^{(i+1)}_{t_i}\}| + \eta,
\end{aligned}$$

provided that $y^{(i)} \notin \{w^{(i)}, Tw^{(i)}\}$ satisfies $d_X(z^{(i)}, y^{(i)}) < (1+2\exp(\max u))^{-1}\eta$. Furthermore, both $\underset{\cdot}{(}w^{(i)}, y^{(i)})$ and $\underset{\cdot}{(}y^{(i)}, w^{(i+1)})$ are non-negative. By doing this replacement for each $1 \leq i < m$ such that $\underset{\cdot}{(}w^{(i)}, w^{(i+1)}) < 0$, we obtain a non-traversing chain

$$p' = \{x_t = y^{(0)}_{t_0}, y^{(1)}_{t_1}, \ldots, y^{(n-1)}_{t_{n-1}}, y^{(n)}_{t_n} = y_{t'}\} \in [x_t \to y_{t'}],$$

such that with $n < 2m' < 2m$ elements. This chain is that $|p'| \geq |q'| + m\eta$.

If in addition p' is such that $\delta(y^{(0)}, y^{(1)}) \geq 0$, then p' is non-traversing and positive, and as we proved before, it has length bounded below by $\rho_s(x_s, y_s)+|t'-t|$.

Suppose on the contrary that $\underset{\cdot}{(}y^{(0)}, y^{(1)}) = \underset{\cdot}{(}x, y^{(1)}) < 0$, then necessarily $t_1 = t$, $Tx = Ty^{(1)}$ and $\Delta^1(x) \cap \Delta^1(y^{(0)}) = \emptyset$. Since $y \in \Delta^1(x)$, then $d_X(y^{(1)}, y) = 1 > d_X(x, y)$. Using this and the fact that $\{y^{(1)}_t, \ldots, y^{(n-1)}_{t_{n-1}}, y^{(n)}_{t_n} = y_{t'}\}$ is non-traversing and positive, we obtain

$$\begin{aligned}
|p'| &= (1-t) + |\{y^{(1)}_t, \ldots, y^{(n-1)}_{t_{n-1}}, y^{(n)}_{t_n} = y_{t'}\}| \\
&\geq (1-t) + \rho_s(y^{(1)}_s, y_s) + |t'-t| \\
&= (1-t) + (1-s) + s d_X(Tx, Ty) + |t'-t| \\
&\geq (1-s) d_X(x, y) + s d_X(Tx, Ty) + |t'-t| \\
&= \rho_s(x_s, y_s) + |t'-t|.
\end{aligned}$$

In conclusion, for each non-traversing $q \in [x_t \to y_{t'}]$, and each $\eta > 0$ we have $|q| \geq |t-t'| + \rho_s(x_s, y_s) + m\eta$, where m is the number of elements in q. Since η can be taken arbitrarily small, then

$$\inf\{|q| : q \in [x_t \to y_{t'}] \text{ non-traversing}\} \geq \rho_s(x_s, y_s) + |t'-t|.$$

General paths. It remains to prove that the minimal length is reached inside the set of non-traversing paths. For this, take $p = \{x_t = x^{(0)}_{t_0}, x^{(1)}_{t_1}, \ldots, x^{(m-1)}_{t_{m-1}}, x^{(m)}_{t_m} = y_{t'}\} \in [x_t \to y_{t'}]$ such that $t_i \neq t_{i+1}$ and $x^{(i+1)} = Tx^{(i)}$ for some $i \in \{0, 1, \ldots, m-1\}$. Let

$$j = \min\{i : \ t_i \neq t_{i+1} \text{ and } x^{(i+1)} = Tx^{(i)}\},$$

so that the path $\{x_t = x^{(0)}_{t_0}, \ldots, x^{(j)}_{t_j}\}$ is non-traversing. Let $s' = \min(t, t_j)$, then, by using $|t_j - t| = t + t_j - 2s'$ we have

$$\begin{aligned}
|p| &\geq d_\Phi(x_t, x^{(j)}_{t_j}) + 1 - t_j + t_{j+1} + d_\Phi(x^{(j+1)}_{t_{j+1}}, y_{t'}) \\
&\geq |t - t_j| + \rho_{s'}(x_{s'}, x^{(j)}_{s'}) + 1 - t_j \\
&\geq t - s' + 1 - s' \\
&= 1 - t.
\end{aligned}$$

Since by hypothesis $1 - t \geq \rho_s(x_s, y_s) + |t' - t|$, we conclude

$$\inf\{|p| : p \in [x_t \to y_{t'}]\} \geq \rho_s(x_s, y_s) + |t' - t|,$$

whenever $\rho_s(x_s, y_s) + |t' - t| < 1 - t$. On the other hand $|\{x_t, x_s, y_s, y_{t'}\} = \rho_s(x_s, y_s) + |t' - t|$, the lemma follows.

Remark 3.2. Since $|t' - t| + \rho_s(x_s, y_s) = |\{x_t, x_s, y_s, y_{t'}\}|$, then $d_{X^\phi}(x_t, y_{t'}) \leq |t' - t| + \rho_s(x_s, y_s)$, where $s = \min(t, t')$. On the other hand, if $|t' - t| > 1 - |t' - t|$ then, using the path $\{x_t, \tilde{y}_t, y_{t'}\}$ with $\tilde{y}$ such that $T(\tilde{y}) = y$, we obtain $d_{X^\phi}(x_t, y_{t'}) \leq \rho_t(x_t, \tilde{y}_t) + 1 - |t - t'|$. Since the system (X, T) is assumed to have the specification property, one always has $T^{-1}(y) \neq \emptyset$.

The projection in 2^X of the rectangle $R(x_s, \epsilon, \delta)$ is the set $\{y \in X : y_s \in \mathcal{S}(x_s, \epsilon)\}$. This projection does not depend on δ, so that we can denote it by $\Delta(x_s, \epsilon)$. Furthermore, $\Delta(x_s, \epsilon)$ is a cylinder set:

Proposition 3.1. *For each $x_s \in X^\phi$ and $\epsilon > 0$ there exists $m(x_s, \epsilon) \in \mathbb{N}$ such that $\Delta(x_s, \epsilon) \in \zeta^{m(x_s, \epsilon)}$.*

Proof. For every $y \in X \setminus \{x\}$, the distances $\rho_s(x_s, y_s)$ belong to the countable set

$$D_{x,s} \equiv \left\{ d_{x,s,n} = (1 - s) e^{-S^n(u(\Delta^n(x))} + s\, e^{-u(\Delta^{n-1}(Tx))} : n \in \mathbb{N} \right\},$$

with no accumulation points other than zero. Since $u > 0$, then $d_{x,s,n} = d_{x,s,m}$ if and only if $m = n$. Clearly $d_{x,s,0} > d_{x,s,1} > \cdots$ is a decreasing sequence converging to zero. Since $y \in \Delta(x_s, \epsilon) \iff \rho_s(x_s, y_s) = d_{x,s,m} < \epsilon$, with $m = \max\{n \in \mathbb{N} : \Delta^n(x) = \Delta^n(y)\}$, then the result follows with $m(x_s, \epsilon) = \min\{n \in \mathbb{N} : d_{x,s,n} < \epsilon\}$. ■

3.2. Concerning Poincaré recurrences

Lemma 3.2. *There exists a constant $\overline{a} \geq 0$ such that for any $x_s \in X^\phi$ and $0 < \delta, \epsilon < 1$,*

$$S^\tau(\phi(\Delta(x_s, \epsilon))) - \overline{a} \leq \tau_\Phi(R(x_s, \epsilon, \delta)) \leq S^\tau(\phi(\Delta(x_s, \epsilon))) + \overline{a}$$

where $\tau := \tau_T(\Delta(x_s, \epsilon))$, and $S^\tau(\phi(\Delta(x_s, \epsilon))) = \max_{z \in \Delta(x_s, \epsilon)} \sum_{j=0}^{\tau} \phi(T^j z)$.

Proof. Let $\tau = \tau_T(\Delta^{n_{x,\epsilon}}(x))$. For each $z \in \Delta(x_s, \epsilon) \cap T^{-\tau}(\Delta(x_s, \epsilon))$, and each $\eta \in (0, \min(\delta, 1 - s))$, if

$$s(\phi(T^\tau(z)) - \phi(z)) - \eta\phi(z) < t - \sum_{j=0}^{\tau - 1} \phi(T^j(z)) < (s + \eta)(\phi(T^\tau(z)) - \phi(z)),$$

then both $z_{s+\eta}$ and $\Phi(z_{s+\eta}, t)$ belong to $R(x_s, \epsilon, \delta)$. Therefore

$$\tau_\Phi(R(x_s, \epsilon, \delta)) \leq \max_{z \in \Delta(x_s, \epsilon)} \sum_{j=0}^{\tau} \phi(T^j z) + 2 \max \phi.$$

In this way we prove one of the inequalities. Now, for each $z \in \Delta(x_s, \epsilon)$ let $\tau(z) = \min\{k \in \mathbb{N} : T^k(z) \in \Delta(x_s, \epsilon)\}$. Since $\tau = \min_{z \in \Delta(x_s,\epsilon)} \tau(z)$, then

$$\begin{aligned} \tau_\Phi(R(x_s, \epsilon, \delta)) &\geq -(s+\delta) \max \phi + \min_{z \in \Delta(x_s,\epsilon)} \sum_{j=0}^{\tau(z)-1} \phi(T^j z) \\ &\geq -(s+\delta) \max \phi + \min_{z \in \Delta(x_s,\epsilon)} \sum_{j=0}^{\tau-1} \phi(T^j z). \end{aligned}$$

Furthermore, since ϕ is a Hölder continuous function, then there exists a constant $\theta > 0$ such that $|\phi(z) - \phi(y)| \leq e^{-\theta S^n(u(\Delta^n(z)))}$, with $n = \max\{k \in \mathbb{N} : \Delta^k(y) = \Delta^k(z)\}$. Therefore

$$\tau_\Phi(R(x_s, \epsilon, \delta)) \geq \max_{z \in \Delta(x_s,\epsilon)} \sum_{j=0}^{\tau} \phi(T^j z) - \left((s + \epsilon + 1) \max \phi + \frac{1}{1 - \exp(-\theta \min \phi)} \right)$$

with $\max \phi = \max_{z \in X} \phi(z)$ and similarly for $\min \phi$. Finally, the result follows with $\overline{a} = 3 \max \phi + (1 - \exp(-\theta \min \phi))^{-1}$. ∎

3.3. Proof of Claim 3.1.

For each $n \geq 2$ let $\zeta^n = \{\Delta_1, \Delta_2, \dots, \Delta_{P(n)}\}$. For each $1 \leq i \leq P(n)$ let $\delta_i = e^{-S^n(u(\Delta_i))}$. Now, for each $i \in \{1, 2, \dots, P(n)\}$ let $N(i) = \lfloor e^{S^n(u(\Delta_i))}(1 - e^{-n \max u})^{-1} \rfloor$. For $i \in \{1, 2, \dots, P(n)\}$ fixed, and each $j \in \{0, 1, \dots, N(i)\}$ define $s(i,j) = j \times e^{-S^n(u(\Delta_i))}(1 - e^{-n \max u})$, and $\epsilon_{i,j} > 0$ be such that $y_{s(i,j)} \in \mathcal{S}(x^{(i)}_{s(i,j)}, \epsilon_{i,j}, \delta_i) \iff y \in \Delta_i$. Here $x^{(i)} \in \Delta_i$ is taken arbitrary.

The collection $\mathcal{R}_n = \{R_{i,j} = R(x^{(i)}_{s(i,j)}, \epsilon_{i,j}, \delta_i) : x^{(i)} \in \Delta_i, i \in \{1, 2, \dots, P(n)\}, j \in \{0, 1, \dots, N(i)\}\}$, is a cover of X^ϕ by (squared) rectangles. We use this particular cover to compute an upper bound for the spectra. The bound stated in Remark 3.2 implies that

$$\operatorname{diam} R_{i,j} \leq \max\{\rho_s(x_s, y_s) + |t - s| : \; x_s, y_t \in R_{i,j}\}.$$

and taking into account the definition of $R_{i,j}$, we obtain

$$\begin{aligned}\mathcal{M}(X^\phi,\alpha,q,\mathcal{R}_n) &= \sum_{i=1}^{P(n)}\sum_{j=0}^{N(i)} e^{-q\tau_\phi(R_{i,j})}(\text{diam } R_{i,j})^\alpha \\ &\leq e^{q\overline{a}}\sum_{i=1}^{P(n)}\sum_{j=0}^{M(i)} e^{-qS^{\tau_i}(\phi(\Delta_i))}\left(e^{-S^n(u(\Delta_i)))+\max u}+\delta_i\right)^\alpha \\ &\quad e^{q\overline{a}}\sum_{i=1}^{P(n)}\sum_{j=M(i)+1}^{N(i)} e^{-qS^{\tau_i}(\phi(\Delta_i))}(\text{diam } R_{i,j})^\alpha,\end{aligned}$$

with $\tau_i=\tau_T(\Delta_i)$ and $M(i)=\max\{0\leq j\leq N(i):\ s(i,M(i))+\delta_i\leq 1\}$. Now, for $j>M(i)$ and $x_t,y_{t'}\in R_{i,j}$, either $|t-t'|\leq\delta_i$ or $y=T(y')$ for $y'\in\Delta_i$. In any case $d_{X^\phi}(x_t,y_t')\leq\delta_i+d_X(T^2x,T^2y)$, and from this we have

$$\mathcal{M}(X^\phi,\alpha,q,\mathcal{R}_n)\leq\left(1+e^{2\max u}\right)^\alpha e^{q\overline{a}}\sum_{i=1}^{P(n)} N(i)\times e^{-qS^{\tau_i}(\phi(\Delta_i))-\alpha S^n(u(\Delta_i))}.$$

Furthermore, since $\delta_j=e^{-S^n(u(T(\Delta_i))}$ and $N(i)\leq e^{S^n(u(\Delta_i))}(1-e^{-n\max u})^{-1}$, then we have

$$\mathcal{M}(X^\phi,\alpha,q,\mathcal{R}_n)\leq\frac{(1+e^{2\max u})^\alpha}{1-e^{-n\max u}}e^{q\overline{a}}\sum_{i=1}^{P(n)} e^{(1-\alpha)S^n(u(\Delta_i))-qS^{\tau_i}(\phi(\Delta_i))}.$$

Now, define $P_{n,k}=\{1\leq i\leq P(n):\tau_i=k\}$. Since (X,T) is specified, then $P_{n,k}=\emptyset$ for all $k\geq n+n_0$. We have,

$$\mathcal{M}(X,\alpha,q,\mathcal{R}_n)\leq\frac{(1+e^{2\max u})^\alpha}{1-e^{-n\max u}}e^{q\overline{a}}\sum_{k=1}^{n+n_0}\sum_{i\in P_{n,k}} e^{(1-\alpha)S^k(u(\Delta^k(x^{(i)})))-qS^k(\phi(\Delta^k(x^{(i)}))},$$

where, as before, $x^{(i)}\in\Delta_i$ is taken arbitrary.

For $i\neq i'\in P_{n,k}$ there necessarily exists $1\leq j\leq k$ such that $\Delta^j(x^{(i)})\neq\Delta^j(x^{(i')})$, which means that $\left\{\Delta^k(x^{(i)}):i\in P_{n,k}\right\}\subset\zeta^k$. Then, taking $\delta>0$ such that $\delta+P_X((1-\alpha)u-q\phi|T))<0$ we obtain

$$\begin{aligned}\mathcal{M}(X,\alpha,q,\mathcal{R}_n) &\leq\frac{(1+e^{2\max u})^\alpha}{1-e^{-n\max u}}e^{q\overline{a}}\sum_{k=1}^{\infty}\sum_{\Delta\in\zeta^k} e^{(1-\alpha)S^k(u(\Delta))-qS^k(\phi(\Delta))} \\ &\leq K_\delta+\frac{(1+e^{2\max u})^\alpha}{1-e^{-n\max u}}e^{q\overline{a}}\sum_{k=1}^{\infty}e^{k(\delta+P_X((1-\alpha)u-q\phi|T))}<\infty,\end{aligned}$$

for some $K_\delta > 0$. This bound is independent on n, then $\mathcal{M}(X^\phi, \alpha, q) \le \lim_{n\to\infty} \mathcal{M}(X^\phi, \alpha,\ q, \mathcal{R}_n) < \infty$ as far as $P_X((1-\alpha)u - q\phi|T) < 0$. The function $\alpha \mapsto P_X((1-\alpha)u - q\phi|T)$ is non-increasing and the result follows. ■

3.4. Proof of Claim 3.2.

Let $\mathcal{R} = \{R(x_s, \epsilon, \delta) : (x_s, \epsilon, \delta) \in B_\mathcal{R}\}$ be a cover of X^ϕ by rectangles. The elements of this cover are indexed by a finite set $B_\mathcal{R} \subset X^\phi \times (0,1)^2$. Supose that diam $\mathcal{R} = \epsilon_0$ for some given $\epsilon_0 \in (0,1)$.

The vertical size. For each $(x_s, \epsilon, \delta) \in B_\mathcal{R}$ let $n(x_s, \epsilon, \delta) \equiv \lceil \delta \times e^{S^{m(x_s,\epsilon)}(u(\Delta(x_s,\epsilon)))} \rceil$, with $m(x_s, \epsilon)$ as in Proposition 3.1 Lemma 3.1 implies that, for each $(x_s, \epsilon, \delta) \in B_\mathcal{R}$ satisfying $1 - s \ge \epsilon_0$,

$$\text{diam } R(x_s, \epsilon, \delta) \ge \text{diam } \mathcal{S}(x_s, \epsilon) + \delta \ge e^{-S^m(u(\Delta(x_s,\epsilon)))} \times n(x_s, \epsilon, \delta).$$

For $\alpha \ge 1$ we have $(\text{diam } R(x_s, \epsilon, \delta))^\alpha \ge e^{-\alpha S^m(u(\Delta(x_s,\epsilon)))} \times n(x_s, \epsilon, \delta)$, implying

$$\mathcal{M}(X^\phi, \alpha, q, \mathcal{R}) \ge \sum_{(x_s,\epsilon,\delta)\in B^*_\mathcal{R}} n(x_s, \epsilon, \delta) e^{-q\tau_\Phi(R(x_s,\epsilon,\delta)) - \alpha S^{m(x_s,\epsilon)}(u(\Delta(x_s,\epsilon)))},$$

where $B^*_\mathcal{R} = \{(x_s, \epsilon, \delta) \in B_\mathcal{R} : 1 - s \ge \epsilon_0\}$.

According to Lemma 3.2, $\tau_\Phi(R(x_s, \epsilon, \delta)) \le \overline{a} + S^{\tau(\Delta(x_s,\epsilon))}(\phi(\Delta(x_s, \epsilon)))$. Since the (X, T) is specified, then $\tau(\Delta(x_s, \epsilon)) \le m(x_s, \epsilon) + n_0$. We have, $\tau_\Phi(R(x_s, \epsilon, \delta)) \le (\overline{a} + n_0 \max \phi) + S^{m(x_s,\epsilon)}\phi(\Delta(x_s, \epsilon))$, and then

$$\mathcal{M}(X^\phi, \alpha, q, \mathcal{R}) \ge e^{-q(\overline{a}+n_0 \max u)} \sum_{(x_s,\epsilon,\delta)\in B^*_\mathcal{R}} n(x_s, \epsilon, \delta) e^{-S^{m(x_s,\epsilon)}((q\phi+\alpha u)(\Delta(x_s,\epsilon)))}.$$

On the other hand, since $n(x_s, \epsilon, \delta) \ge \delta e^{S^{m(x_s,\epsilon)}(u(\Delta(x_s,\delta)))}$, then

$$\mathcal{M}(X^\phi, \alpha, q, \mathcal{R}) \ge e^{\overline{a}+n_0 \max u} \sum_{(x_s,\epsilon,\delta)\in B^*_\mathcal{R}} \delta e^{S^{m(x_s,\epsilon)}((1-\alpha)u - q\phi)((\Delta(x_s,\epsilon)))}.$$

The horizontal grouping. Now we will group in the 'horizontal direction' elements in $B^*_\mathcal{R}$.

For each $N \in \mathbb{N}$ fixed define the j-th "horizontal slice"

$$S(N, j) = \bigcup_{x\in X} \bigcup_{j/N<t<(j+1)/N} \Phi(x_0, \phi(x)t).$$

We say that $S(N,j)$ is properly covered if there exists a collection $B \subset B^*_{\mathcal{R}}$ such that for each $(x_s, \delta, \epsilon) \in B$ we have: a) $s \le j/N$, b) $s + \delta \ge (j+1)/N$, and c) $S(N,j) \subset \bigcup_{(x_s,\epsilon,\delta)\in B} R(x_s,\epsilon,\delta)$. Let $PC(N) = \{0 \le j < N : S(N,j)$ is properly covered$\}$.

To each $j \in PC(N)$ we associate the collection

$$B^{(j)} = \left\{(x_s,\epsilon,\delta) \in B^*_{\mathcal{R}} : s \le \frac{j}{N}, s+\delta \ge \frac{j+1}{N}\right\}.$$

For each $j \in PC(N)$, the collection

$$\mathcal{R}^{(j)} = \{R(x_s,\epsilon,\delta) : (x_s,\epsilon,\delta) \in B^*_{\mathcal{R}}\}$$

covers $S(N,j)$. Consequently, its projection in 2^X,

$$\mathcal{C}^{(j)} = \{\Delta(x_s,\epsilon) :\ R(x_s,\epsilon,\delta) \in \mathcal{R}^{(j)}\}$$

is a cover of X. It is important to remark that each $(x_s,\epsilon,\delta) \in B^*_{\mathcal{R}}$ can be associated to more than one horizontal slice. Let $C_N(x_s,\epsilon,\delta) = \#\{j \in PC(N) : (x_s,\epsilon,\delta) \in B^{(j)}\}$, be the number of horizontal silices associated to a given $(x_s,\epsilon,\delta) \in B^*_{\mathcal{R}}$. It is clear that $C_N(x_s,\epsilon,\delta) \le \delta N$.

For $N \in \mathbb{N}$ sufficiently large, almost every horizontal slice is properly covered. Indeed, if $S(N,j)$ is not properly covered by, then either $S(N,j)$ intersects at least one of the two "horizontal border" of a rectangle in $\mathcal{R}$, or $j > N(1-\epsilon_0)$. Here, by the two horizontal borders of $R(x_s,\epsilon,\delta)$ we mean $\mathcal{S}(x_s,\epsilon)$ and $\{y_{s+\delta} : y_s \in \mathcal{S}(x_s,\epsilon)\}$. Hence, for $N > \min\{\delta^{-1} : (x_s,\epsilon,\delta) \in B_{\mathcal{R}}\}$, $\#PC(N) \ge N(1-\epsilon_0) - 2\#\mathcal{R}$.

For $N > \min\{\delta^{-1} : (x_s,\epsilon,\delta) \in B_{\mathcal{R}}\}$, we have

$$\begin{aligned}\mathcal{M}(X^\phi,\alpha,q,\mathcal{R}) &\ge e^{\overline{a}+n_0\max u} \sum_{j\in PC(N)} \sum_{(x_s,\epsilon,\delta)\in B^{(j)}} \\ &\qquad \frac{\delta}{C_N(x_s,\epsilon,\delta)} e^{S^{m(x_s,\epsilon)}((1-\alpha)u-q\phi)(\Delta(x_s,\epsilon)))} \\ &\ge \frac{e^{\overline{a}+n_0\max u}}{N} \sum_{j\in PC(N)} \sum_{(x_s,\epsilon,\delta)\in B^{(j)}} e^{S^{m(x_s,\epsilon)}((1-\alpha)u-q\phi)(\Delta(x_s,\epsilon)))} \\ &\ge \frac{e^{\overline{a}+n_0\max u}}{N} \#PC(N) \times \min_{\mathcal{C}^{(j)}:j\in P(N)} \sum_{\Delta\in\mathcal{C}^{(j)}} e^{-S^{|\Delta|}(((1-\alpha)u-q\phi)(\Delta))},\end{aligned}$$

since, as we mentioned before, $C_N(x_s,\epsilon,\delta) \le N\delta$. Using the lower bound for $\#P(N)$ we obtain

$$\mathcal{M}(X^\phi,\alpha,q,\mathcal{R}) \ge e^{\overline{a}+n_0\max u}\left(1-\epsilon_0-2\frac{\#\mathcal{R}}{N}\right) \times \min_{\mathcal{C}\subset\mathcal{C}_{\mathcal{R}}} \sum_{\Delta\in\mathcal{C}} e^{S^{|\Delta|}(((1-\alpha)u-q\phi)(\Delta))},$$

where the minimum is taken over all the subcovers of the projection $\mathcal{C}_{\mathcal{R}} = \{\Pi(x_s, \epsilon, \delta) : (x_s, \epsilon, \delta) \in B_{\mathcal{R}}\}$. Thus, for $N \geq 2\#\mathcal{R}/\epsilon_0$, we have

$$\mathcal{M}(X^\phi, \alpha, q, \mathcal{R}) \geq e^{\overline{a}+n_0 \max u}(1 - 2\epsilon_0) \times \min_{\mathcal{C} \subset \mathcal{C}_{\mathcal{R}}} \sum_{\Delta \in \mathcal{C}} e^{S^{|\Delta|}(((1-\alpha)u - q\phi)(\Delta))}.$$

As the diameter ϵ_0 goes to zero, the length of the cylinders in $\mathcal{C}_{\mathcal{R}}$ increase. Indeed, if $\Delta \in \mathcal{C}_{\mathcal{R}}$, then $|\Delta| \geq 1 - \log(\epsilon_0)(\min u)^{-1}$. From all this,

$$\mathcal{M}(X^\phi, \alpha, q) \geq C \times e^{\overline{a}+n_0 \max u} \times \lim_{n\to\infty} \left(\inf\{\mathcal{Z}(0, ((1-\alpha)u - q\phi), \mathcal{C}, X) : |\mathcal{C}| \geq n\}\right),$$

with $\mathcal{Z}$ as defined in the paragraph 2.7. Hence according to what was exposed in that paragraph, $\mathcal{M}(X^\phi, \alpha, q) = \infty$ whenever $P_X((1-\alpha)u - q\phi|T) > 0$. ■

In this way, from Claims 3.1 and 3.2 it follows Theorem 3.2. Now we are in the position to prove our main result, Theorem 3.1.

3.5. Proof of Theorem 3.1.

We have that, for all $\alpha \geq 1$ and $q \geq 0$, the spectrum for Poincaré recurrences $\alpha^* \equiv \alpha(X^\phi, q)$ satisfies the Bowen-like equation $P_X((1-\alpha^*)u - q\phi|T) = 0$.

On the other hand, the spectrum $\alpha^{\bar{\mu}}$ for the measure $\bar{\mu} \in \tilde{\mathcal{M}}_e$ satisfies the following equation

$$h_\mu(T) + \int_X \left((1-\alpha^{\bar{\mu}})u(x) - q\phi(x)\right)\, d\mu(x) = 0,$$

where $\bar{\mu}$ is obtained from an T–ergodic measure μ as described before, and $h_\mu(T)$ is the entropy of the measure μ with respect to T. Now, the classical variational principle for T–invariant measures (see for instance [12]), applied to the potential $(1-\alpha)u - q\phi$, establishes that

$$P_X((1-\alpha)u - q\phi|T) = \sup_{\mu \in \mathcal{M}(T)} \left\{h_\mu(T) + \int_X ((1-\alpha)u(x) - q\phi(x))\, d\mu(x)\right\},$$

where $\mathcal{M}(T)$ is the set of all T-invariant measures. Now, for $q \geq 0$ fixed and taking $\alpha = \alpha^* := \alpha_c(X^\phi, q)$ in both sides of the previous equation, we have

$$0 = P_X((1-\alpha^*)u - q\phi|T) = \sup_{\mu \in \mathcal{M}} \left\{h_\nu(T) + \int_X ((1-\alpha^*)u(x) - q\phi(x))\, d\mu(x)\right\},$$

which implies that $h_\nu(T) + \int_X ((1-\alpha^*)u(x) - q\phi(x))\, d\mu(x) \leq 0$ for all $\mu \in \mathcal{M}(T)$. Since, for each ν fixed and $u > 0$, the function $\alpha \to h_\nu(T) + \int_X ((1-\alpha^*)u(x) - q\phi(x))\, d\mu(x)$ is decreasing, then necessarily $\alpha^* \geq \sup\{\alpha^{\bar{\mu}}(q) : \bar{\mu} \in \mathcal{M}_e\}$.

On the other hand, if $\alpha < \alpha^*$ then $P_X((1-\alpha^*)u - q\phi|T) > 0$, implying that

$$\sup_{\nu\in\mathcal{M}}\left\{h_\nu(T) + \int_X ((1-\alpha)u(x) - q\phi(x))\ d\nu(x)\right\} > 0.$$

Thus, there necessarily exists a measure $\nu \in \mathcal{M}(T)$ such that $h_\nu(T) + \int_X((1-\alpha)u(x) - q\phi(x))d\nu(x) > 0$. Again, for $\nu \in \mathcal{M}(T)$ fixed and $u > 0$, the function $\alpha \to h_\nu(T) + \int_X((1-\alpha)u(x) - q\phi(x))d\nu(x)$ is decreasing, then $\alpha < \alpha^{\bar{\nu}}(q)$. In this way we obtain the reverse inequality, i. e., $\alpha^* \leq \sup\{\alpha^{\bar{\mu}}(q) : \bar{\mu} \in \mathcal{M}_e\}$, and the theorem follows. ■

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