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A NOTE ON REDUCIBLE CYCLES IN MULTIPARTITE TOURNAMENTS*

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Abstract. [3] proves that if T is a strong c-partite tournament ($c \geq 3$), then there is a $(k-3)$-reducible k-cycle in T, for all $k = 3, 4, \cdots, c$. In this paper we investigate the smallest number of $(k-3)$-reducible k-cycles in strong c-partite tournaments for $3 \leq k \leq c$ and give some related problems.

1. INTRODUCTION

We assume that the reader is familar with the standard terminology on graphs and digraphs and refer the reader to [2].

A *digraph* $D = (V(D), A(D))$ is determined by its *set of vertices* $V(D)$, and its *set of arcs* $A(D)$. If xy is an arc of a digraph D, then we say that x *dominates* y and write $x \rightarrow y$. More generally, if A and B are two disjoint subdigraphs of D or subsets of $V(D)$ such that every vertex of A dominates every vertex of B, then we say that A *dominates* B and write $A \rightarrow B$. We use $A \Rightarrow B$ to denote the fact that there is no arc leading from B to A. By a *cycle* (*path*, resp.) we mean a directed cycle (directed path, resp.). A digraph D is *strong* if for any two vertices x and y there exists a path from x to y and a path from y to x in D. A cycle of length k is called a k-*cycle*. A cycle (path, resp.) of a digraph D is *Hamiltonian* if it includes all the vertices of D. A digraph D is *pancyclic* if it contains a k-cycle for all k between 3 and $|V(D)|$. A digraph D is *vertex pancyclic* if every vertex of D is contained in a k-cycle for all $k \in \{3, 4, \cdots, |V(D)|\}$. If S is a set of vertices in a digraph D, then $D[S]$ is the subgraph induced by S.

A c-*partite* or *multipartite tournament* is a digraph obtained from a complete c-partite graph by substituting each edge with an arc. Let T be a multipartite

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tournament and $v \in V(T)$. We use $V^c(v)$ to denote the partite set which v belongs to.

Let D be a digraph and let k be some integer. A cycle C_0 is *k-reducible* if there are cycles $C_1, C_2, \cdots, C_k$ such that for all $i = 0, 1, \cdots, k-1$ there is a vertex w_i in C_i such that $C_{i+1} = C_i[w_i^+, w_i^-]w_i^+$. Let $w \in V(D)$. Then a cycle C_0 is *(w, k)-reducible* if it is k-reducible and w belongs to all the cycles $C_1, C_2, \cdots, C_k$ (i.e., $w_i \neq w$ for all $i = 0, 1, \cdots, k-1$).

$[1]$ proves that if T is a strong c-partite tournament ($c \geq 3$), then there is a k-cycle in T for all $k = 3, 4, \cdots, c$. $[3]$ extends this result by showing that if T is a strong c-partite tournament ($c \geq 3$), then there is a $(k-3)$-reducible k-cycle in T for all $k = 3, 4, \cdots, c$. In this paper we investigate the smallest number of $(k-3)$-reducible k-cycles in strong c-partite tournaments for $3 \leq k \leq c$.

Theorem. *Let T be a strong c-partite tournament. Then the number of $(k-3)$-reducible k-cycles is at least $c-k+1$ for $3 \leq k \leq c$. Moreover, the lower bound is best possible.*

2. Proof of Theorem

Lemma 1 (Yeo [3]). *If T is a strong c-partite tournament $(c \geq 3)$, then there is a $(k-3)$-reducible k-cycle in T for all $k = 3, 4, \cdots, c$.*

Lemma 2 (Goddard & Oellermann [4]). *Every vertex of a strong c-partite tournament $(c \geq 3)$ belongs to a cycle which contains vertices from exactly q partite sets for each $q \in \{3, 4, \cdots, c\}$.*

Lemma 3 (Guo & Volkmann [5]). *Every partite set of a strong c-partite $(c \geq 3)$ tournament has at least one vertex which lies on a k-cycle for each $k \in \{3, 4, \cdots, c\}$.*

The following Lemma 4 is interesting in itself; it generalizes Moon's theorem on vertex pancyclicity in strong tournaments [6].

Lemma 4. *Let T be a strong c-partite tournament with partite sets $V_1, V_2, \cdots, V_c$. Then for any V_i there is a $(k-3)$-reducible k-cycle in T which contains at least one vertex of V_i for all $k = 3, 4, \cdots, c$.*

Proof. We prove the lemma by induction on k. When $k = 3$, Lemma 4 holds by Lemma 3. We assume $4 \leq k \leq c$ and V_i is given. By Lemma 1, there is a $(k-3)$-reducible k-cycle C_0 in T. If $V(C_0) \cap V_i \neq \emptyset$, we are done. So we assume that $V(C_0) \cap V_i = \emptyset$ and take a $(k-4)$-reducible $(k-1)$-cycle C_1 in T such that $V(C_1) \cap V_i = \emptyset$ (such a cycle exists by the reducibility of cycle C_0). If there is a vertex $v \in V_i$ with $v \not\Rightarrow V(C_1)$ and $V(C_1) \not\Rightarrow v$, then since v is adjacent to every vertex in $V(C_1)$ there exists a vertex $u \in V(C_1)$ such that $u^- \to v$ and $v \to u$. We obtain a k-cycle $C_1[u, u^-]vu$, which is $(k-3)$-reducible and contains a vertex v of V_i. Therefore we assume that for each $v \in V_i$ either $v \Rightarrow V(C_1)$ or $V(C_1) \Rightarrow v$.

Let $A_1 = \{v \in V_i | v \Rightarrow V(C_1)\}$ and $A_2 = \{v \in V_i | V(C_1) \Rightarrow v\}$. Clearly $A_1 \cup A_2 = V_i$. Since $V(C_1) \cap V_i = \emptyset$, it is easy to see that $A_1 \to V(C_1)$ and $V(C_1) \to A_2$. Let l be the length of a shortest path of all (C_1, A_1)-paths and (A_2, C_1)-paths. Without loss of generality, we assume that $P = y_0 y_1 \cdots y_l$ is a (C_1, A_1)-path of length l. If $y_1 \in V^c(y_l)$, then $l \geq 3$ and $y_1 \in A_2$. Let $x \in V(C_1) - V^c(y_2)$ be arbitrary. If $x \to y_2$, then the path $xP[y_2, y_l]$ is a shorter (C_1, A_1)-path than P, a contradiction. If $y_2 \to x$, then the path $y_1 y_2 x$ is a shorter (A_2, C_1)-path than P, a contradiction. So $y_1 \notin V^c(y_l)$. Similarly, from the minimality of l we obtain that $V^c(y_l) \cap \{y_1, y_2, \cdots, y_{l-1}\} = \emptyset$ and $y_l \to \{y_0, y_1, \cdots, y_{l-2}\}$. Let $C_2 = PC_1[y_0^{+l}, y_0]$. Since $y_l \to V(C_2) - \{y_{l-1}\}$, C_2 is a $(k-3)$-reducible k-cycle and contains a vertex y_1 of V_i.

This completes the proof of Lemma 4. ■

Corollary 5 (Moon [6]). *Every strong tournament is vertex pancyclic.*

Theorem 6. *Let T be a strong c-partite tournament. Then the number of $(k-3)$-reducible k-cycles is at least $c-k+1$ for $3 \leq k \leq c$.*

Proof. Let $V_1, V_2, \cdots, V_c$ be the partite sets of T and $3 \leq k \leq c$. We prove the theorem by induction on c. For $c = k$, the result follows from Lemma 1.

Suppose now that $c \geq k+1$ and that every strong $(c-1)$-partite tournament contains at least $(c-1)-k+1$ $(k-3)$-reducible k-cycles. According to Lemma 2, there exists a cycle C that contains vertices from exactly $c-1$ partite sets. $T[V(C)]$ is a strong $(c-1)$-partite tournament, which contains by the induction hypothesis at least $c-k$ $(k-3)$-reducible k-cycles. Without loss of generality, let V_1 be the partite set with $V_1 \cap V(C) = \emptyset$. By Lemma 4, T contains a $(k-3)$-reducible k-cycle C_2' with $V_1 \cap V(C_2') \neq \emptyset$. Clearly the $(k-3)$-reducible k-cycle C_2' is different from the $(k-3)$-reducible k-cycles in $T[V(C)]$. So T contains at least $c-k+1$ $(k-3)$-reducible k-cycles. ■

Corollary 7. *Let T be a strong c-partite tournament $(c \geq 3)$. Then there are at least $c-k+1$ pancyclic subgraphs of order k in T for all $k = 3, 4, \cdots, c$.*

Corollary 8 (Goddard & Oellermann [4]). *Let T be a strong c-partite tournament $(c \geq 3)$. Then T contains at least $c-2$ cycles of length 3.*

Corollary 9. *Let T be a strong c-partite tournament $(c \geq 3)$. Then T contains at least $\binom{c-1}{2}$ cycles.*

Corollary 10 (Moon [6]). *Let T be a strong tournament of order n. Then T contains at least $n-k+1$ cycles of length k for $3 \leq k \leq n$.*

Corollary 11 (Moon [6]). *Let T be a strong tournament of order n. Then T contains at least $\binom{n-1}{2}$ cycles.*

The tournament obtained by reversing the arcs of the unique Hamiltonian path in a transitive tournament T_n with n vertices is seen to have precisely $n-k+1$ $(k-3)$-reducible k-cycles for $3 \leq k \leq n$. This example shows that the estimation in Theorem 6 is best possible. We denote the above tournament by M_n.

We construct a 6-partite tournament T_6 with partite sets $V_i = \{V_i\}$, $i = 1, 2, 3, 4, 5$, and $V_6 = \{v_6, v_7\}$. Let $v_1 \to \{v_2, v_4, v_5, v_7\}$, $v_2 \to \{v_3, v_4, v_5, v_6, v_7\}$, $v_3 \to \{v_1, v_4, v_5, v_6, v_7\}$, $v_4 \to \{v_5, v_6\}$, $v_5 \to v_7$, $v_6 \to \{v_1, v_5\}$ and $v_7 \to v_4$. It is easy to see that T_6 is strong and T_6 contains no strong tournament with 6 vertices. This example shows that a strong n-partite tournament may not contain strong tournament with n vertices. So Theorem 6 is not a trivial generalization of Corollary 10.

Now we would like to give the following related problems.

Problem 12. *Are there examples of strong c-partite tournaments which are not tournaments with exactly $c-k+1$ $(k-3)$-reducible k-cycles for $4 \leq k \leq c$?*

The weak form of Problem 12 is also unsolved. We refer the reader who is interested in this problem to [7].

Problem 13 (Volkmann [7]). *Are there examples of strong c-partite tournaments which are not tournaments with exactly $c-k+1$ cycles of length k for $4 \leq k \leq c$?*

In [8], Yao proved that for T_n a strong tournament of order n, if there is an integer k $(3 < k < n)$ such that T_n contains exactly $n-k+1$ k-cycles, then $T_n \cong M_n$.

Problem 14. *How to characterize extremal strong c-partite tournaments containing minimum number of cycles?*

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