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ON PERIODIC ORBITS OF RELAXATION OSCILLATIONS

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Abstract. Two nonlinear oscillators of relaxation type are studied for representing periodic orbits in terms of two inverse functions of x exp(x). The limit of the limit cycle of singularly perturbed van der Pol differential equation is approximated analytically; while the periodic orbit of singularly perturbed Lotka-Volterra system is represented in exact manner. These results are in an excellent agreement with numerical results computed via a stiff/nonstiff Maple ODE solver "NODES package" authored by Lawrence F. Shampine and Robert M. Corless. Some remarks are provided for the relaxation period.

1. Introduction

Rhythmic occurrence of events in biological, chemical, ecological, electrical, and mechanical systems has been very frequently observed to give rise to some nonlinear oscillations, which include both non-relaxation oscillations and relaxation oscillations (also known as fast-slow systems); see books such as Stoker [46], Minorsky [33], Chance, Pye, Ghosh, and Hess [5], Tyson [48], Nicolis and Prigogine [38], Mishchenko and Rosov [35], Field and Burger [14], Cronin [7], Edelstein-Keshet [11], Grasman [22], Goldbeter [21], Gray and Scott [24], Scott [41], Drazin [10], Murray [37], Mishchenko, Kolesov, Kolesov, and Rosov [34], Strogatz [47], Kolesov and Kolesov [28], Schneider and Miinster [40], Fowler [16], Epstein and Pojman [12], Keener and Sneyd [25], Koch and Segev [27], and Koch [26].

Two well-known examples of undamped oscillators are the linear harmonic oscillator

$$x^{''}(t) + x = 0$$

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or the system

$$x'(t) = y, \qquad y'(t) = -x, \tag{1.1}$$

and the simple pendulum

$$x^{''}(t) + \sin(x) = 0$$

or the system

$$x'(t) = y, \qquad y'(t) = -\sin(x).$$

We are interested in two singularly perturbed systems of first-order differential equations

$$\epsilon x'(t) = y + x - \frac{1}{3}x^3, \quad y'(t) = -x, \tag{1.2}$$

and

$$x'(t) = x\{1 - y\}, \quad \epsilon y'(t) = y\{x - 1\}, \tag{1.3}$$

where ϵ is a given positive constant satisfying $0 \ll \epsilon < 1$, and x, y are two unknown functions of the time t. The systems (1.2), (1.3) illustrate nonlinear oscillations of relaxation type. The main purpose of this work is to provide descriptions of periodic orbits governed by these equations in terms of two inverse functions of x exp(x).

The name "relaxation oscillations" was used by Balthasar van der Pol (1889-1959) [49, 1, 50, 51, 52], who investigated the differential equation

$$x^{''}(\tau) + \mu\{x^2 - 1\}x'(\tau) + x = 0 \tag{1.4}$$

with a positive constant $\mu \uparrow \infty$, as an application to the modeling of triode electric circuits and heartbeat. A relaxation oscillation is characterized by a jerky motion in a periodic orbit, which, as geysers in the Yellowstone National Park, exhibits two distinct and characteristic phases: one during which energy is stored up slowly and another in which the energy is discharged nearly instantaneously when a certain critical threshold potential is attained, as described in Stoker [46, p. 137]. The web site

`http://www.apmaths.uwo.ca/~bfraser/version1/vanderpol.html`

authored by Blair Fraser has an excellent JAVA animation of demonstrating nonlinear oscillations (1.4), both non-relaxation oscillations and relaxation oscillations, by changing the value of $\mu \in$ (0, 15) (ϵ in this web site).

From the viewpoint of singular perturbations for this nonuniform phenomenon, the regions of rapid change in time give internal layer problems; while slow phase with discontinuous segments observed by van der Pol correspond to the reduced problem. Since then, relaxation oscillations have been found to be important in many physical, chemical, biological, and engineering problems.

An equation related to (1.4) is the Rayleigh equation

$$y''(\tau) - \mu\{1 - y'(\tau)^2\}y'(\tau) + y = 0,$$

which becomes (1.4) on differentiating with respect to τ and on setting $x = \sqrt{3}y'(\tau)$. It was used in 1883 by Rayleigh [39] to model maintained vibrations of sound.

From the viewpoint of nonlinear chemical dynamics, Alfred J. Lotka (1880-1949) [30] devised in 1920 a hypothetical set of chemical reactions

$$\begin{aligned} R + X &\xrightarrow{k_1} 2X, \\ X + Y &\xrightarrow{k_2} 2Y, \\ Y &\xrightarrow{k_3} P, \end{aligned}$$

for the chemical reactions exhibiting periodic behavior in the chemical concentrations, where the reactant R is maintained constant. The law of mass action then yields the rate equations

$$x'(t) = k_1 rx - k_2 xy, \quad y'(t) = k_2 xy - k_3 y, \tag{1.5}$$

where $x(t)$, $y(t)$ are the concentrations of intermediates X and Y, respectively, and τ is the concentration of R. The system (1.5) is of the type

$$u'(t) = u\{a - bv\}, \quad v'(t) = v\{cu - d\}, \tag{1.6}$$

with positive constants a, b, c, d, which generates sustained temporal oscillations during the net overall reaction $R \to P$. On the other hand, from the viewpoint of nonlinear population dynamics, Lotka [31] in 1925 and Vito Volterra (1860-1940) [55] in 1926 studied independently the predator prey model (1.6), where $u(t)$ is the prey population, $v(t)$ that of the predator at time t. The number a relates to the birth rate of the prey, d to the death rate of the predator, and b, c to the interaction between the species.

The nonlinear system (1.6) becomes the singularly perturbed system (1.3) with $\epsilon = a/d$. The condition that ϵ is small was assumed in Grasman and Veling [23], Vasil'eva and Belyanin [53], Verhulst [54, p. 170], and Fowler [16, p. 34].

Both (1.2) and (1.3) possess the feature of two turning points. Existence of a turning point for a singularly perturbed equation has been known to exhibit typically internal layer behavior there. More precisely, (1.4) having $\mu \uparrow \infty$ corresponds to equation

$$\epsilon x''(t) + \{x^2 - 1\}x'(t) + x = 0, \tag{1.7}$$

with $\epsilon = \mu^{-2} \downarrow 0$ upon a change of variable $\tau = \mu t$. The singularly perturbed equation (1.7), which becomes the system (1.2) in the Liénard plane, is said to have a turning point at $x = -1$ and $x = 1$, respectively, by virtue of the fact that, for small ϵ, its leading term involving $x'(t)$ vanishes at $x = \pm 1$. The term "turning point" is clearly justified as illustrated in Figure 3 of §3 when the limit cycle makes a smooth turn at $x = \pm 1$ in the xy-plane. On the other hand, eliminating y in the system (1.3) gives the nonlinear singularly perturbed differential equation

$$\epsilon x x''(t) - \epsilon\{x'(t)\}^2 - x\{x-1\}x'(t) + x^2\{x-1\} = 0, \tag{1.8}$$

which has a turning point at $x = 0$ and $x = 1$, respectively. In particular, setting $\epsilon = 0$ in (1.8) gives

$$-x_0\{x_0 - 1\}x_0'(t) + x_0^2\{x_0 - 1\} = 0,$$

which has solutions $x_0 = 0$, $x_0 = 1$, and $x_0(t) = c\exp(t)$. Thus the nature of turning points of (1.8) seems to be different from (1.7), as shown in Figure 8 of §4. In fact, the slow phase of (1.3) satisfying initial data (4.15) is given by the solution $x_0 = x_\ell \exp(t)$, $y_0 = 0$, that is stable if $0 < x_\ell < 2/3$.

In this work, periodic orbits of two systems (1.2) and (1.3) are described by using two inverse functions $W(-k, x)$ of $x\exp(x)$ defined in §2. Giacomini and Neukirch [19, 20] used a sequence of polynomials of the form

$$h_n(x, y) = y^n + \sum_{k=0}^{n-1} q_{k,n}(x) y^k,$$

where n is an even integer, to approximate the limit cycle of (1.2); while our result in §3 gives the limit of the limit cycle of (1.2) as $\epsilon \downarrow 0$. In addition, our result in §4 provides some exact expressions for each periodic orbit of (1.3). Grasman and Veling [23] employed an implicit function theorem to construct four convergent series expansions for each periodic orbit of (1.3) as $\epsilon \downarrow 0$. Vasil'eva and Belyanin [53] used a method of matched a-symptotic expansions to construct outer functions as well as layer functions for each periodic orbit of (1.3) as $\epsilon \downarrow 0$.

2. Inverse Functions of x $\exp(x)$

2.1. Basic notations

To study van der Pol differential equation and Lotka-Volterra system, one is required to define two auxiliary functions. First of all, the function $x\exp(x)$ has the positive derivative $(x+1)\exp(x)$ if $x > -1$. Define the inverse function of

$x\exp(x)$ restricted on the interval $[-1,\infty)$ to be $W(0,x)$, which is a strictly increasing function mapping from $[-\exp(-1),\infty)$ to $[-1,\infty)$ so that the equivalence relation

$$x\exp(x) = y \Longleftrightarrow W(0,y) = x$$

holds for $x \in [-1,\infty)$, $y \in [-\exp(-1),\infty)$. Similarly, we define the inverse function of $x\exp(x)$ restricted on the interval $(-\infty,-1]$ to be $W(-1,x)$, which is a strictly decreasing function mapping from $[-\exp(-1),0)$ to $(-\infty,-1]$ so that the equivalence reiation

$$x\exp(x) = y \Longleftrightarrow W(-1,y) = x$$

holds for $x \in (-\infty,-1]$, $y \in [-\exp(-1),0)$. It then follows that

$$\begin{aligned} W(0,x\exp(x)) &= x, \qquad x \ge -1; \\ W(-1,x\exp(x)) &= x, \qquad x \le -1; \qquad (2.1) \\ W(0,x)\exp(W(0,x)) &= x, \qquad x \ge -\exp(-1); \end{aligned}$$

$$W(-1,x)\exp(W(-1,x)) = x, \qquad -\exp(-1) \le x < 0. \qquad (2.2)$$

Differentiating (2.1), (2.2) with respect to x gives

$$\begin{aligned} W'(0,x) &= \frac{\exp(-W(0,x))}{1+W(0,x)} = \frac{W(0,x)}{x\{1+W(0,x)\}}, \\ W'(-1,x) &= \frac{\exp(-W(-1,x))}{1+W(-1,x)} = \frac{W(-1,x)}{x\{1+W(-1,x)\}}, \end{aligned} \qquad (2.3)$$

for $x \neq -\exp(-1)$.

In what follows, we give some examples of using two inverse functions of $x\exp(x)$. In 1779, Euler obtained a series expansion for the solution of the trinomial equation $x^\alpha - x^\beta = (\alpha-\beta)vx^{\alpha+\beta}$ in the limiting case as $\alpha \to \beta$, which was proposed in 1758 by Lambert. In this case, the equation becomes $\log(x) = vx^\beta$, which has the solution $x = \exp(-W(0,-\beta v)/\beta)$. Here the notation log denotes the natural logarithmic function.

In the study of linear differential-difference equations with constant coefficients, one is led to solve some transcendental equations related to $W(-k,x)$ functions. As an illustration, Theorem 3.4 of Bellman and Cooke [2] is given below. The equation $a_0u'(t) + b_0u(t) + b_1u(t-\omega) = 0$ is satisfied by $\sum p_r(t)\exp(s_rt)$, where $\{s_r\}$ is any sequence of zeros of $a_0s + b_0 + b_1\exp(-\omega s)$, $p_r(t)$ is a polynomial of degree less than the multiplicity of s_r, and the sum is either finite or is infinite with suitable conditions to ensure convergence. For example, the equation $u'(t) = u(t-1)$ has a solution of the form $u(t) = \exp(st)$ with $s = W(0,1)$.

Fritsch, Shafer, and Crowly [18] provided an algorithm of computing $W(0, x)$ for $x > 0$. Shih [42] used $W(0, x)$ with $x > 0$ to describe a slowly moving shock of Burgers' equation in the quarter plane. A good reference of these functions is Corless, Gonnet, Hare, Jeffrey, and Knuth [6].

The functions $W(-k, x)$ are denoted by Lambert $W(-k, x)$ in the computer algebra system Maple V, release 5, and ProductLog$[-k, x]$ in Mathematica, version 3, respectively. Unfortunately, both of them produce erroneous asymptotic behavior for $W(-1, x)$ near $x = -\exp(-1)$.

2.2. Asymptotics

In this investigation, we are interested in the singular behavior of each function $W(-k, x)$ at the branch point $x = -\exp(-1)$. The classical Lagrange inversion formula of complex function theory is not applicable to the construction of an asymptotic expansion of $W(-k, x)$ at $x = -\exp(-1)$; see, for example, Fabijonas and Olver [13], because the first derivative of the function $x\exp(x)$ equals zero at $x = -1$.

For $x = -1$, we have

$$x\ \exp(x+1) = -1 + \frac{1}{2}(x+1)^2 + \frac{1}{3}(x+1)^3 + \frac{1}{8}(x+1)^4 + \frac{1}{30}(x+1)^5 + ...,$$

from which it follows that

$$\begin{aligned} x\exp(1) &= W(-k,x)\exp(W(-k,x)+1) \\ &= -1 + \frac{1}{2}\{W(-k,x)+1\}^2 + \frac{1}{3}\{W(-k,x)+1\}^3 \\ &\quad \frac{1}{8}\{W(-k,x)+1\}^4 + \frac{1}{30}\{W(-k,x)+1\}^5 + \dots . \end{aligned}$$

Thus we obtain the following asymptotic behavior of $W(-k, x)$ as $x \downarrow -\exp(-1)$:

$$(2.4) \quad W(-k,x) = \begin{cases} -1 + w - \frac{1}{3}w^2 + \frac{11}{72}w^3 - \frac{43}{540}w^4 + \frac{769}{17280}w^5 - \dots, & k = 0, \\ -1 - w - \frac{1}{3}w^2 - \frac{11}{72}w^3 - \frac{43}{540}w^4 - \frac{769}{17280}w^5 - \dots, & k = 1, \end{cases}$$

where $w = \sqrt{2\exp(1)\{x + \exp(-1)\}}$. In other words, we have

$$(2.5) \quad \frac{1}{1 + W(-k,x)} = \begin{cases} \frac{1}{w} + \frac{1}{3} - \frac{1}{24}w + \frac{2}{135}w^2 - \frac{23}{3456}w^3 - \dots, & k = 0, \\ \frac{-1}{w} + \frac{1}{3} - \frac{1}{24}w + \frac{2}{135}w^2 + \frac{23}{3456}w^3 - \dots, & k = 1, \end{cases}$$

as $x \downarrow -\exp(-1)$.

The behavior of $W(0,x)$ for x near $-\exp(-1)$ is equivalent to that of $W(0,w)$ for w near 0, with $w=\sqrt{2\exp(1)\{x+\exp(-1)\}}$. By virtue of (2.3), the function $W(0,w)$ satisfies the first-order differential equation

$$\{w^2-2\}\{1+f(w)\}f'(w)-2wf(w)=0.$$

Under the assumption

$$W(0,w)=\sum_{k=0}^{\infty}c_kw^k,$$

we then have an iterative relation

$$2(k+1)c_1c_k=(k-3)c_{k-1}+\sum_{j=1}^{k-1}jc_jc_{k-j-1}-2\sum_{j=2}^{k-1}jc_jc_{k-j+1},\quad k\geq 3, \tag{2.6}$$

with $c_0=-1$, $c_1=1$, and $c_2=-1/3$. Similarly, it follows that

$$W(-1,w)=\sum_{k=0}^{\infty}c_kw^k,$$

where c_k is defined by the iterative relation (2.6) with $c_0=-1$, $c_1=-1$, and $c_2=-1/3$.

Next, instead of deriving a general expression for (2.5) in the form

$$\frac{1}{1+W(0,x)}=\sum_{k=0}^{\infty}\alpha_kw^{k-1},\quad \text{as } x\downarrow -\exp(-1),$$

we proceed the following procedure in order to obtain an asymptotic behavior of $\Phi(s)$ in the period (4.11). Let

$$\psi(\sigma)=\frac{\sigma}{1+W(0,-\exp(-1-\frac{1}{2}\sigma^2))}. \tag{2.7}$$

Then, by using (2.3), $\psi(\sigma)$ satisfies the differential equation

$$\sigma\psi'(\sigma)=\psi(\sigma)+\sigma\psi(\sigma)^2-\psi(\sigma)^3. \tag{2.8}$$

Substituting the series

$$\psi(\sigma)=\sum_{k=0}^{\infty}\psi_k\sigma^k \tag{2.9}$$

into (2,8) and equating the like powers of σ give the recursive relation

(2.10) $$\psi_j = \frac{1}{j+2}\left\{\sum_{k=0}^{j-1}\psi_k\psi_{j-k-1} - \psi_0\sum_{k=1}^{j-1}\psi_k\psi_{j-k} - \sum_{i=1}^{j-1}\left[\psi_{j-i}\sum_{k=0}^{i}\psi_k\psi_{i-k}\right]\right\},$$

for $j \geq 2$, along with $\psi_0 = 1$, $\psi_1 = 1/3$ obtained from (2.5). Similarly, the function

$$\hat{\psi}(\sigma) = \frac{\sigma}{1 + W(-1, -\exp(-1 - \frac{1}{2}\sigma^2))}$$

has the series expansion $\hat{\psi}(\sigma) = \sum_{k=0}^{\infty}\hat{\psi}_k\sigma^k$, where the coefficients $\hat{\psi}_k$ satisfy (2.10) for $j \geq 2$, along with $\hat{\psi}_0 = -1, \hat{\psi}_1 = 1/3$ followed from (2.5). It can be shown by induction that $\hat{\psi}_j = (-1)^{j+1}\psi_j$ for $j \geq 0$. As an illustration, applying (2.10) recursively gives the following numerical values of ψ_j:

$$\psi_2 = \frac{1}{12}, \quad \psi_3 = \frac{2}{135}, \quad \psi_4 = \frac{1}{864}, \quad \psi_5 = \frac{-1}{2835}, \quad \psi_6 = \frac{-139}{777600}.$$

Thus combining (2.7) and (2.9) gives

(2.11) $$\frac{1}{1 + W(0, -\exp(-1 - s))} = \sum_{k=0}^{\infty} 2^{(k-1)/2}\psi_k s^{(k-1)/2}, \quad \text{as } s \downarrow 0.$$

Similarly, we have

(2.12) $$\frac{1}{1 + W(-1, -\exp(-1 - s))} = \sum_{k=0}^{\infty} 2^{(k-1)/2}\hat{\psi}_k s^{(k-1)/2}, \quad \text{as } s \downarrow 0.$$

It then follows from (2.11) and (2.12) that the function Φ defined by (4.12) has the following expansion

(2.13) $$\Phi(s) = \sum_{j=0}^{\infty} 2^{j+1/2}\psi_{2j}s^{j-1/2}, \quad \text{as } s \downarrow 0.$$

For the rest of this subsection, one is interested in asymptotic behavior of $W(k, x)$ for x near 0. We start with the definition $x\exp(x) = y$ if and only if $x = W(-1, y)$ for $x \in (-\infty, -1]$ and $y \in [-\exp(-1), 0)$. It follows that

$$\log(-y) = \log(-x\exp(x)) = \log(-x) + \log(\exp(x)) = \log(-x) + x,$$

which leads to the iterative scheme

$$x_{n+1} = \log(-y) - \log(-x_n),$$

with $x_0 = \log(-y)$. Thus we obtain

$$x_1 = \log(-y) - \log(-\log(-y)).$$

In other words, we have

$$W(-1,y) = \log(-y) - \log(-\log(-y)) + ..., \quad \text{as } y \uparrow 0. \tag{2.14}$$

The Taylor series expansion

$$W(0,x) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{k^{k-1}}{k!}x^k, \quad \text{as } x \to 0 \tag{2.15}$$

can be derived by using the Lagrange inversion formula of complex function theory; see, for example, de Bruin [9, p. 22].

3. van der Pol Differential Equation

The origin of relaxation oscillations can be described qualitatively by considering the linear differential equation with constant coefficients

$$x''(t) + 2bx'(t) + cx = 0, \tag{3.1}$$

where $c > b^2$. Then the general solution of (3.1) is

$$x(t) = A\exp(-bt)\sin(t\sqrt{c-b^2} + \alpha), \tag{3.2}$$

where A and α are some constants. If $b > 0$, the solution (3.2) represents a damped oscillation tending to zero. On the other hand, if $b < 0$, the solution (3.2) represents an oscillation with the increasing amplitude.

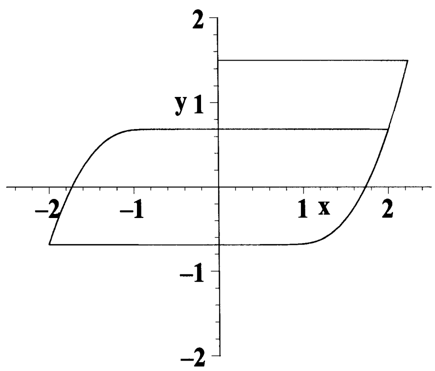


FIG. 1. Equation (3.4) in the Liénard plane

The equation (3.1) may be said to describe a system with positive (negative) damping if $b > 0$ ($b < 0$, respectively). Positive damping decreases the energy and thus the amplitude of an oscillation; while negative dampings increases it. Self-sustained oscillations arise in a system having both positive and negative dampings. Thus, qualitatively, one would expect the coefficient $2b$ of the resistance term to be a function of the amplitude itself, which is negative for small displacements, and is positive for large displacements.

The equation considered in detail by van der Pol since 1920 was

$$x^{''}(\tau) + \mu\{x^2 - 1\}x'(\tau) + x = 0, \tag{3.3}$$

with a given constant $\mu > 0$. It is clear that the nonlinear damping term is negative for $|x| < 1$, and positive for $|x| > 1$. The large values of μ correspond to the small values of the inductance L of the wires connected to the capacities in modeling triode electric circuits. The value of μ. estimated by van der Pol [50, p. 990] was about 3×10^5 when $L = 10$.

The existence and uniqueness of a limit cycle for equation (3.3) was proved in 1928 by Alfred Marie Liénard (1869-1958) [29], who studied a general differential equation of the form (known later as a Liénard differential equation)

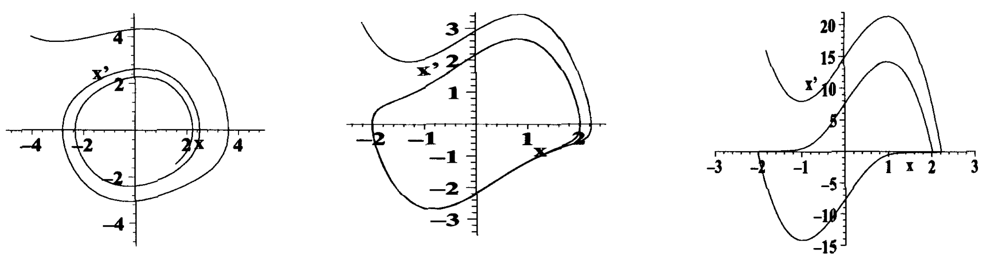
$$x^{''}(\tau) + f(x)x'(\tau) + x = 0,$$

in the setting of, instead of the usual phase plane, the so-called Liénard plane

$$x'(\tau) = y - F(x), \quad y'(\tau) = -x,$$

where $F(x) = \int_0^x f(s)ds$. The new dependent variable y is known as the Liénard variable in the literature.

Thus Liénard reformulated equation (3.3) as



(a) $\mu = 0.1$ (b) $\mu = 1$ (c) $\mu = 10$

FIG. 2. Equation (3.3) with $\mu = 0.01$, 1, 10 in phase plane

$$x'(\tau) = y + \mu\{x - \frac{1}{3}x^3\}, \quad y'(\tau) = -x. \tag{3.4}$$

Liénard [29, p. 952] might be the first person who plotted the limit cycle of van der Pol equation (3.3) in a plane in such a way that it remains *bounded* as $\mu \uparrow \infty$; see Figure 1. Using the Liénard plane to investigate the limit of the limit cycle of van der Pol equations as $\mu \uparrow \infty$ is a better choice than the phase plane illustrated by van der Pol in Figure 2 for three values $\mu = 0.1, 1, 10$. In fact, Cartwright [4] showed in 1950 that the solution $x(\tau)$ of (3.3) subject to the initial conditions $x(0) = x_0$, $x'(0) = x'_0$ satisfies

$$|x(\tau)| < M_0, \quad |x'(\tau)| < M_0\mu,$$

where M_0 is an absolute constant, for $t > t_0(x_0, x'_0, M_0)$. Such estimates give a justification of the Liénard plane employed by Liénard [29], instead of the phase plane used by van der Pol [50], on investigating analytically and numerically the limit cycle of (3.3).

In 1946, Flanders and Stoker [15] gave a rigorous proof that, as $\mu \uparrow \infty$, the limit of the cycles of (3.4) tends to the Jordan curve consisting of arcs of the curve $y = x^3/3 - x$ for $1 \le |x| \le 2$, and horizontal line segments $y = 2/3$ for $-1 \le x \le 2$, $y = -2/3$ for $-2 \le x \le 1$, as indicated by the heavy curve in Figure 3.

3.1. Approximation of the limit cycle

To construct an approximation of the limit cycle of van der Pol equation (3.3) as $\mu \uparrow \infty$, one considers the Liénard plane upon a change of variable. For the case $\mu \gg 1$, let $\epsilon = \mu^{-2} \ll 1$ and $t = \tau/\mu$. Then (3.3) becomes

$$\epsilon x''(t) + \{x^2 - 1\}x'(t) + x = 0,$$

which gives

$$\epsilon x'(t) = y + x - \frac{1}{3}x^3, \qquad y'(t) = -x, \tag{3.5}$$

in the Liénard plane. Note that the solution of the system (3.5) is symmetric with respect to the origin in the Liénard plane in the sense that if (x, y) is a solution, then so is $(-x, -y)$.

To construct the slow-motion arcs of the limit cycle, we obtain from (3.5) by letting $\epsilon = 0$

$$0 = y_0 + x_0 - \frac{1}{3}x_0^3, \qquad y'_0(t) = -x_0. \tag{3.6}$$

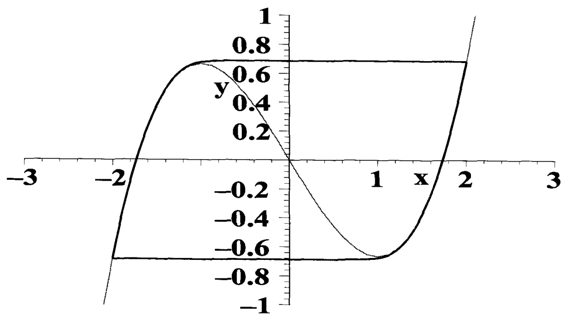


FIG. 3. Limit cycle of equation (3.4) and the function $\frac{1}{3}x^3 - x$

Combining two equations in the reduced system (3.6) gives a first-order separable differential equation

$$\{x_0^2 - 1\}x_0'(t) + x_0 = 0.$$

Note that this equation becomes singular if $x_0(t) = \pm 1$. Integrating gives rise to

$$t + c = \log(|x_0|) - \frac{1}{2}x_0^2, \tag{3.7}$$

where c is a constant of integration.

The right branch of the limit cycle satisfies $1 \leq x_0 \leq 2$. For the sake of investigating the singular behavior of the outer expansion at $x_0 = 1$, we set the initial condition to be $x_0(0) = 1$ in (3.7) to obtain $c = -1/2$ so that (3.7) becomes

$$t - \frac{1}{2} = \log(x_0) - \frac{1}{2}x_0^2,$$

which is solved to give

$$x_0(t) = \sqrt{-W(-1, -exp(2t - 1))},$$

with $t \in (-\infty, 0]$.

Thus the right branch of the limit cycle in the Liénard plane has a parametric representation

$$x_0(t) = \sqrt{-W(-1, -\exp(2t - 1))}, \quad y_0(t) = \frac{1}{3}x_0(t)^3 - x_0(t), \tag{3.8}$$

with $t \in [-3/2 + \log(2), 0]$. It starts at the point (2, 2/3) when $t = -3/2 + \log(2) \approx -0.8068528195$, and ends at the point $(1, -2/3)$ when $t = 0$ in the Liénard plane. Note that the point $x_0 = \sqrt{3}$, $y = 0$ corresponds to $t = -1 + \log(\sqrt{3}) \approx -0.4506938555$.

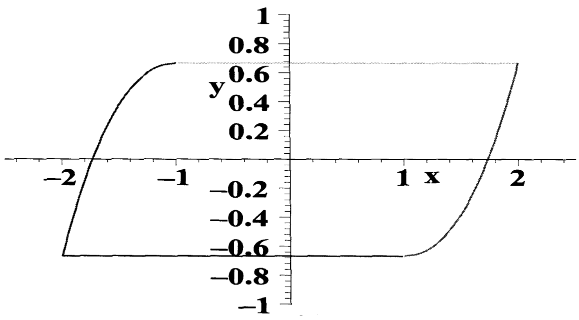


FIG. 4. Approximation of the limit cycle based on (3.8), (3.9), (3.12), (3.13)

From the asymptotic behavior (2.4) with $k = 1$, we have

$$x_0(t) = 1 + \sqrt{-t} + \frac{1}{6}(-t) + ..., \quad \text{as } t \uparrow 0.$$

This shows a singularity of derivatives of $x_0(t)$ at $t = 0$ employed in higher order approximations to the right branch of the limit cycle.

In a similar way, the left branch of the limit cycle in the Liénard plane having $-2 \le x \le -1$ is found to have a parametric representation

$$x_0(t) = -\sqrt{-W(-1, -\exp(2t-1))}, \quad y_0(t) = \frac{1}{3}x_0(t)^3 - x_0(t), \tag{3.9}$$

with $t \in [-3/2 + \log(2), 0]$. It starts at the point $(-2, -2/3)$ when $t = -3/2 + \log(2) \approx -0.8068528195$, and ends at the point $(-1, 2/3)$ when $t = 0$ in the Liénard plane.

The reduced system of (3.5) subject to two sets of initial conditions $(1, -2/3)$, $(-1, 2/3)$ gives slow-motion arcs of the limit cycle. Thus a rough approximation to the relaxation period is $3 - 2\log(2) \approx 1.613705639$, which was first obtained in 1928 by Liénard [29, p. 952].

Next, consider the fast-motion segments of the limit cycle. Using the stretched variable $\tau = t/\epsilon$ in the system (3.5) reads

$$x'(\tau) = y + x - \frac{1}{3}x^3, \quad y'(\tau) = -\epsilon x,$$

which yields by setting $\epsilon = 0$

$$X_0'(\tau) = Y_0 + X_0 - \frac{1}{3}X_0^3, \quad Y_0'(\tau) = 0. \tag{3.10}$$

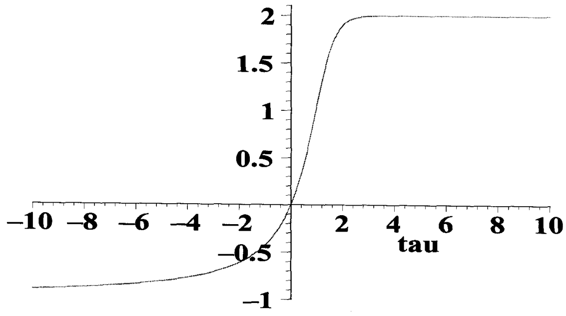


FIG. 5. Profile of function (3.11)

The system (3.10) implies that $Y_0(\tau)$ is a constant on fast-motion segments.

For the upper branch of the limit cycle connecting points $(-1, 2/3)$, $(2, 2/3)$, we have $Y_0(\tau) = 2/3$. Then $X_0(\tau)$ is defined by

$$X_0'(\tau) = \frac{3}{2} + X_0 - \frac{1}{3}X_0^3, \quad X_0(0) = 0.$$

Integration by using partial fractions gives the algebraic equation

$$\frac{1}{3}\log(\frac{X_0+1}{2-X_0}) - \frac{1}{X_0+1} = \tau - \frac{1}{3}\log(2) - 1,$$

which is solved to give

$$X_0(\tau) = \frac{2 - W(0, 2\exp(-3\tau+2))}{1 + W(0, 2\exp(-3\tau+2))}, \tag{3.11}$$

with the properties

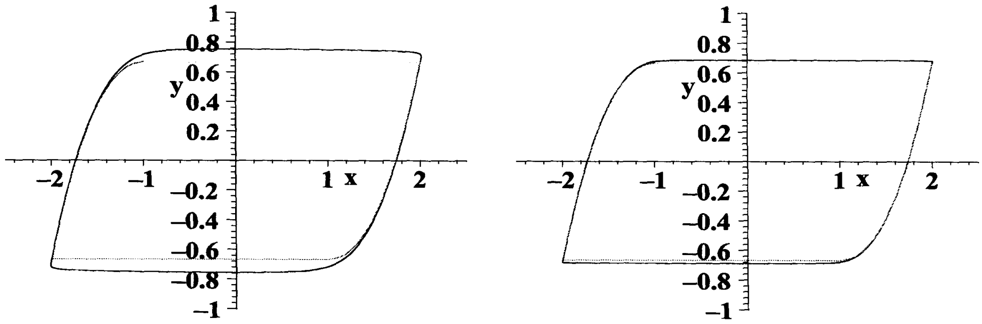
$$\lim_{\tau\downarrow-\infty} X_0(\tau) = -1, \quad \lim_{\tau\uparrow\infty} X_0(\tau) = 2,$$

as shown in Figure 5. Note that $X_0(\tau)$ tends to -1 algebraically as $\tau \downarrow -\infty$; while $X_0(\tau)$ goes to 2 exponentially as $\tau \uparrow \infty$.

The upper branch has a parametric representation

$$X_0(\tau) = \frac{2 - W(0, 2\exp(-3\tau+2))}{1 + W(0, 2\exp(-3\tau+2))}, \quad Y_0(\tau) = \frac{2}{3}, \tag{3.12}$$

with $\tau \in (-\infty, \infty)$. It travels from the point $(-1, 2/3)$ at $\tau = -\infty$ to the point (2, 2/3) at $\tau = \infty$.



(a) Computed result with $\epsilon = 0.01$ (b) Computed resuult with $\epsilon = 0.001$

FIG. 6. Constructed limit of limit cycle and computed limit cycle

In a similar way, for the lower branch of the limit cycle connecting points (1, −2/3), (−2, −2/3), we have $Y_0(\tau) = -2/3$. Then $X_0(\tau)$ is defined by

$$X_0'(\tau) = -\frac{2}{3} + X_0 - \frac{1}{3}X_0^3, \quad X_0(0) = 0.$$

Integration by using partial fractions yields an algebraic equation

$$\tau - \frac{1}{3}\log(2) - 1 = \frac{1}{3}\log(\frac{1 - X_0}{X_0 + 2}) + \frac{1}{X_0 - 1},$$

which is solved to give

$$X_0(\tau) = \frac{W(0, 2\exp(-3\tau + 2)) - 2}{1 + W(0, 2\exp(-3\tau + 2))}.$$

Note that

$$\lim_{\tau\downarrow-\infty} X_0(\tau) = 1, \quad \lim_{\tau\uparrow\infty} X_0(\tau) = -2.$$

Thus the lower branch is governed by the parametric representation

$$X_0(\tau) = \frac{W(0, 2\exp(-3\tau + 2)) - 2}{1 + W(0, 2\exp(-3\tau + 2))}, \quad Y_0(\tau) = -\frac{2}{3}, \tag{3.13}$$

with $\tau \in (-\infty, \infty)$. It travels from the point (1, –2/3) at $\tau = -\infty$ to the point (−2, 2/3) at $\tau = \infty$.

We have completed a description of an explicit representation for the limit, as $\epsilon \downarrow 0$, of the limit cycle of van der Pol equation; see Figure 4. The constructed approximation (3.8), (3.9), (3.12), (3.13) has shown in Figure 6 an excellent agreement with numerical computations based on a stiff/nonstiff Maple ODE solver "NODES package" '(containing five files: IVPx.mws, ivp.tex, solver.mpl, testnstf.mws, test-stf.mws), which is authored by Lawrence F. Shampine and Robert M. Corless, and is available at the web site

$$\texttt{ftp}://\texttt{cygnus.math.smu.edu/pub/shampine/maplecodes/}.$$

Higher order approximations to the limit cycle than what we present above in (3.8), (3.9), (3.12), (3.13) can be carried out with an additional expansion near each of two turning points (1, −2/3), (−1, 2/3).

4. Lotka-Volterra System

Before investigating a singularly perturbed Lotka-Volterra system, some results are provided for Lotka-Volterra system

$$u'(t) = u\{a - bv\}, \qquad v'(t) = v\{cu - d\}. \tag{4.1}$$

4.1. Algebraic equation and Morse lemma

Combining two equations of the system (4.1) yields the separable differential equation

$$\frac{dv}{du} = \frac{v\{cu - d\}}{u\{a - bv\}}.$$

An elementary integration gives a functional relation between u and v:

$$a\log(v) - bv - cu + d\log(u) + C = 0, \tag{4.2}$$

where C is the constant of integration. Defining $F_0(u, v) = cu + bv - d\log(u) - a\log(v)$, we then have (4.2) as

$$F_0(u, v) = C. \tag{4.3}$$

Thus the system (4.1) is conservative by virtue of the fact that $F_0(u, v)$ is a constant as a function of time with $C = F_0(u_0, v_0)$ on the trajectory passing through the initial data $u(0) = u_0 > 0$, $v(0) = v_0 > 0$. An elementary technique in calculus further shows that

$$C \geq a + d - a\log(\frac{a}{b}) - d\log(\frac{d}{c}),$$

and the minimum value takes place at $u = d/c$, $v = a/b$. In the notion of Hamiltonian systems, we write $C = a + d - a\log(a/b) - d\log(d/c) + E$ in (4.3) to obtain

$$F(u, v) = E, \tag{4.4}$$

where

$$F(u,v) = cu - d - d\log(\frac{c}{d}u) + bv - a - a\log(\frac{b}{a}v),$$

with the energy $E \geq 0$, and $E = 0$ at $u = d/c$, $v = a/b$.

Note that the given system (4.1) can be considered to be Hamiltonian in the sense of

$$p'(t) = \frac{\partial H}{\partial q}, \quad q'(t) = -\frac{\partial H}{\partial p};$$

by using a change of variables $p = \log(u)$, $q = \log(v)$, (4.1) becomes

$$p'(t) = a - b\exp(q), \quad q'(t) = c\exp(p) - d,$$

with $H = c\exp(p) - dp + b\exp(q) - aq$.

The system (4.1) has been known graphically to give a one-parameter family of periodic solutions (4.4) having $(d/c, a/b)$ as the center point as shown in Figure 7 since Lotka [30].

The system (4.1) has only one critical point (singular point, or equilibrium point) $(d/c, a/b)$ in the first quadrant of the uv-plane, which is the center of the linearized system of (4.1)

$$u'(t) = -\frac{bd}{c}\{v - \frac{a}{b}\}, \quad v'(t) = -\frac{ac}{b}\{u - \frac{d}{c}\}. \tag{4.5}$$

Moreover, the solution of the linearized problem (4.5) subject to the initial conditions $u(0) = u_0 > 0$, $v(0) = v_0 > 0$ is

$$u = \frac{d}{c} + r\cos(t\sqrt{ab} + t_*), \quad v = \frac{a}{b} + r\frac{c}{b}\sqrt{\frac{a}{d}}\sin(t\sqrt{ad} + t_*),$$

for some t_* satisfying

$$\sin(t_*) = \frac{1}{r}(v_0 - \frac{a}{b})\frac{b}{c}\sqrt{\frac{d}{a}}, \quad \cos(t_*) = \frac{1}{r}(u_0 - \frac{d}{c}),$$

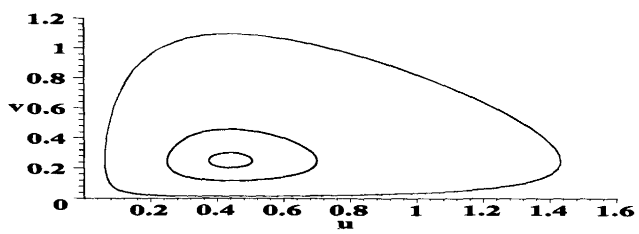


FIG. 7. Periodic orbrts of equation (4.1)

with

$$r = \sqrt{\left(u_0 - \frac{d}{c}\right)^2 + \left(v_0 - \frac{a}{b}\right)^2 \frac{b^2 d}{ac^2}}.$$

Hence the trajectory of the linearized problem (4.5) is an ellipse with the period $2\pi/\sqrt{ad}$ in the uv-plane. Note that the period of the linearized system is independent of the initial data.

The linearized theory, which appeared in both Lotka [30] and Volterra [55], may not predict what happens in the nonlinear system (4.1), but it is closely related to Frame [17] and Waldvogel [56, 57] in some sense. In fact, the period for the nonlinear system (4.1) will be shown to depend on the initial data.

In what follows, one invokes Morse lemma, which makes a connection between Lotka-Volterra oscillator (4.1) and a harmonic oscillator (1.1) via a nonlinear transformation.

Marston Morse (1892-1977) published in 1925 his first paper [36] on the distribution of critical points of a function. He returned to this subject in one form or another for the rest of his career with at least fifty papers presentating different settings of this theory, known as *Morse Theory*; see [32].

Let F be a smooth real-valued function on an open subset A of $\mathbb{R}^n$. A point $\alpha \in A$ is called a critical point of F if the gradient of F at α is zero. A critical point a is called non-degenerate if the Hessian matrix $\partial^2 F/\partial x_i \partial x_j(\alpha)$ is nonsingular. The critical point a has the index k if

$$F(x) = F(\alpha) - \sum_{k=1}^{\lambda} c_k (x_k - \alpha_k)^2 + \sum_{k=1}^{n-\lambda} c_k (x_{\lambda+k} - \alpha_{\lambda+k})^2 + \text{higher order terms},$$

with positive numbers $c_k, k = 1, 2, \dots, n$. Now we are ready to state Morse Lemma as follows: Let $A \subset \mathbb{R}^n$ be open and $F : A \to \mathbb{R}$ be a smooth function. Let α be a nondegenerate critical point for F with the index k. Then, in a neighborhood of α, there exists a neighborhood U of α and a neighborhood V of 0 in $\mathbb{R}^n$, and a smooth map $G : V \to U$ with a smooth inverse such that $F \circ G$ has the form

$$(F \circ G)(y) = F(\alpha) - \sum_{k=1}^{\lambda} y_k^2 + \sum_{k=1}^{n-\lambda} y_{\lambda+k}^2.$$

In the case of dimension $n = 2$ and index $k = 0$, we have closed orbits around the critical point.

Applying Taylor series of $\log(bv/a)$ at $v = a/b$

$$\log(\frac{b}{a}v) = \frac{b}{a}(v - \frac{a}{b}) - \frac{b^2}{2a^2}(v - \frac{a}{b})^2 + \frac{b^3}{3a^3}(v - \frac{a}{b})^3 + \cdots,$$

and that of $\log(cu/d)$ at $u = d/c$

$$\log(\frac{c}{d}u) = \frac{c}{d}(u - \frac{d}{c}) - \frac{c^2}{2d^2}(u - \frac{d}{c})^2 + \frac{c^3}{3d^3}(u - \frac{d}{c})^3 + \cdots,$$

to nonlinear algebraic equation (4.4) gives

$$\frac{c^2}{2d}(u - \frac{d}{c})^2 + \frac{b^2}{2a}(v - \frac{a}{b})^2 = E + \frac{c^3}{3d^2}(u - \frac{d}{c})^3 + \frac{b^3}{3a^2}(v - \frac{a}{b})^3 + \cdots,$$

which is an ellipse in the neighborhood of the critical point $u = d/c, v = a/b$.

In other words, the function F has a critical point at $u = d/c$, $v = a/b$, which is nonsingular, and has the index 0 as shown below:

$$F(u,v) = \frac{c^2}{2d}(u - \frac{d}{c})^2 + \frac{b^2}{2a}(v - \frac{a}{b})^2 - \frac{c^3}{3d^2}(u - \frac{d}{c})^3 - \frac{b^3}{3a^2}(v - \frac{a}{b})^3 + \cdots.$$

Thus, by virtue of Morse lemma, nonlinear algebraic equation (4.4) gives closed orbits around the critical point $u = d/c, v = a/b$.

The transformation a from new coordinate system (ξ, η) to old coordinate system (u, v) is given by

$$u = \begin{cases} -\frac{d}{c}W(-1, -\exp(1 - \frac{\xi^2}{d^2})), & \xi \geq 0, \\ -\frac{d}{c}W(0, -\exp(1 - \frac{\xi^2}{d^2})), & \xi \leq 0; \end{cases}$$

$$v = \begin{cases} -\frac{a}{b}W(-1, -\exp(1 - \frac{\eta^2}{a^2})), & \eta \geq 0, \\ -\frac{a}{b}W(0, -\exp(1 - \frac{\eta^2}{a^2})), & \eta \leq 0; \end{cases}$$

The inverse transformation G^{-1} from old coordinate system (u, v) to new coordinate system (ξ, η) is defined by

$$\xi = \begin{cases} \sqrt{cu - d - d\log(\frac{c}{d}u)}, & u \geq \frac{d}{c}, \\ -\sqrt{cu - d - d\log(\frac{c}{d}u)}, & u \leq \frac{d}{c}; \end{cases}$$

$$\eta = \begin{cases} \sqrt{cv - a - a\log(\frac{c}{d}u)}, & u \geq \frac{a}{b}, \\ -\sqrt{cv - a - a\log(\frac{b}{a}v)}, & v \leq \frac{a}{b}; \end{cases}$$

so that (4.4) becomes $\xi^2 + \eta^2 = E$.

4.2. Representations of periodic orbit

To express a periodic orbit of (4.1), Lotka [30] used Fourier series

$$u(t) = \sum_{k=0}^{\infty} a_k, \cos(knt) + b_k \sin(knt), \quad v(t) = \sum_{k=0}^{\infty} a'_k \cos(knt) + b'_k \sin(knt);$$

while Frame [17] used a change of variables for dependent variables u, v four times consecutively so that u, v are functions of $\cos(\theta)$, $\sin(\theta)$, respectively.

It seems in the literature that the algebraic equation (4.3) or (4.4) cannot be solved explicitly for either variable in terms of the other; see, for example, Simmons [45, p. 436] and Boyce and DiPrima [3, p. 475] among others. In Shih [43], we solved (4.4) explicitly for one variable in terms of the other, from which two integral representations of the period were obtained. The basic notations we employed are two inverse functions $W(0,x), W(-1,x)$ of $x\exp(x)$. Our method can be considered to be elementary but elegant; see Shih [44].

For the sake of completeness, we summarize some results of Shih [43]. Rewrite (4.4) as

$$-\frac{b}{a}v\exp(-\frac{b}{a}v) = -(\frac{c}{d}u)^{-d/a}\exp(\frac{c}{a}u - 1 - \frac{d}{a} - \frac{E}{a}),$$

which lies in the interval $[-\exp(-1), 0)$ for positive v. Solving this equation for v gives

$$v = g_k(u), \quad g_k(u) = -\frac{a}{b}W(-k, -(\frac{c}{d}u)^{-d/a}\exp(\frac{c}{a}u - 1 - \frac{d}{a} - \frac{E}{a})), \tag{4.6}$$

for $k = 0, 1$. Next, we determine the range of u. From

$$-(\frac{c}{d}u)^{-d/a}\exp(\frac{c}{a}u - 1 - \frac{d}{a} - \frac{E}{a}) \in [-\exp(-1), 0),$$

we get the inequality

$$-\frac{c}{d}u\exp(-\frac{c}{d}u) \le -\exp(-1 - \frac{E}{d}),$$

which is solved to give $u \in [u_{\min}, u_{\max}]$, where

$$u_{\min} = -\frac{d}{c}W(0, -\exp(-1 - \frac{E}{d})), \quad u_{\max} = -\frac{d}{c}W(-1, -\exp(-1 - \frac{E}{d})). \tag{4.7}$$

The orbit determined by (4.4) has four extreme points

$$P_w = (-\frac{d}{c}W(0, -\exp(-1 - \frac{E}{d})), \frac{a}{b}), \qquad P_e = (-\frac{d}{c}W(-1, -\exp(-1 - \frac{E}{d})), \frac{a}{b}),$$

$$P_s = (-\frac{d}{c}, -\frac{a}{b}W(0, -\exp(-1 - \frac{E}{a}))), \quad P_n = (-\frac{d}{c}, -\frac{a}{b}W(-1, -\exp(-1 - \frac{E}{d}))),$$

which divide the entire orbit into four segments: northeast with $u > d/c$, $v > a/b$, northwest with $0 < u < d/c$, $v > a/b$, southwest with $0 < u < d/c$, $0 < v < a/b$, and southeast with $u > d/c$, $0 < v < a/b$. Each segment has the following parametric representation $u(s)$, $v(s)$ for $s \in (0, E)$.

The northeastern segment defined by $v = g_1(u)$ on $(d/c, u_{\max})$ is represented by

$$u(s) = -\frac{d}{c}W(-1, -\exp(-1 - \frac{s}{d})), \quad v(s) = -\frac{a}{b}W(-1, -\exp(\frac{s}{a} - 1 - \frac{E}{a})),$$

which travels from the point P_n to the point P_e in the clockwise direction on $(0, E)$.

The northwestern segment defined by $v = g_1(u)$ on $(u_{\min}, d/c)$ is represented by

$$u(s) = -\frac{d}{c}W(0, -\exp(-1 - \frac{s}{d})), \quad v(s) = -\frac{a}{b}W(-1, -\exp(\frac{s}{a} - 1 - \frac{E}{a})),$$

which travels from the point P_n to the point P_w in the counterclockwise direction on $(0, E)$.

The southwestern segment defined by $v = g_0(u)$ on $(u_{\min}, d/c)$ is represented by

$$u(s) = -\frac{d}{c}W(0, -\exp(-1 - \frac{s}{d})), \quad v(s) = -\frac{a}{b}W(0, -\exp(\frac{s}{a} - 1 - \frac{E}{a})),$$

which travels from the point P_s to the point P_w in the clockwise direction on $(0, E)$.

The southeastern segment defined by $v = g_0(u)$ on $(d/c, u_{\max})$ is represented by

$$u(s) = -\frac{d}{c}W(-1, -\exp(-1 - \frac{s}{d})), \quad v(s) = -\frac{a}{b}W(0, -\exp(\frac{s}{a} - 1 - \frac{E}{a})),$$

which travels from the point P_s to the point P_e in the counterclockwise direction on $(0, E)$.

Substituting (4.6) into the first equation of (4.1) gives

$$dt = \frac{du}{u\{a - bg_k(u)\}}$$

for $k = 0, 1$. Then traveling along the lower branch described by $v = 9_0(u)$ from the point $(u_{\min}, a/b)$, with $t = t|P_w$, to the point $(u_{\max}, a/b)$, with $t = t|P_e$, in the counterclockwise direction yields

$$(4.8) \qquad t|_{P_e} - t|_{p_w} = \int_{u_{\min}}^{u_{\max}} \frac{du}{u\{a - bg_0(u)\}};$$

while traveling along the upper branch described by $v = g_1(u)$ from the point $(u_{\max}, a/b)$, with $t = t|_{P_e}$, to the point $(u_{\min}, a/b)$, with $t = t|_{P_w}$, in the counter-clockwise direction yields

$$t|_{P_w} - t|_{P_e} = \int_{u_{\max}}^{u_{\min}} \frac{du}{u\{a - bg_1(u)\}}. \tag{4.9}$$

Thus combining (4.8) and (4.9) yields an integral representation of the period for the closed trajectory determined by (4.4):

$$T = \int_{u_{\min}}^{u_{\max}} \left\{ \frac{1}{u\{a - bg_0(u)\}} + \frac{-1}{u\{a - bg_1(u)\}} \right\} du, \tag{4.10}$$

where the functions $g_0(u)$, $g_1(u)$ are defined by (4.6); and two endpoints $u_{\min}$, $u_{\max}$ of the integral are defined by (4.7). Note that the integral (4.10) has a weak singularity of the square root type at each endpoint of the integration.

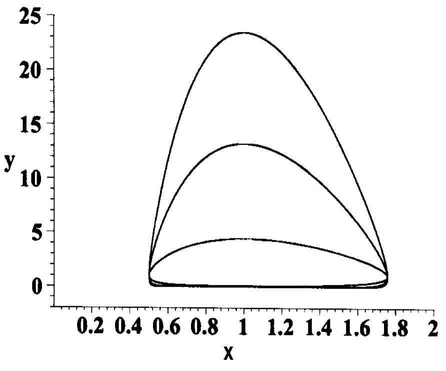
The period depends on the energy E, and thus on initial data u_0, v_0. This is different ftom the linearized problem.

With a splitting of the integration interval and a simple substitution, one can reduce the integral of the period (4.10) to be of the convolution type. Thus the period of the closed trajectory determined by (4.4) can be expressed as

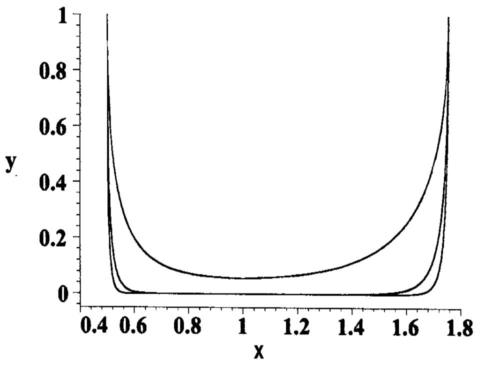
$$T = \frac{1}{ad} \int_0^E \Phi(\frac{s}{d}) \Phi(\frac{E - s}{a}) ds, \tag{4.11}$$

where

$$\Phi(s) = \{1 + W(0, -\exp(-1 - s))\}^{-1} - \{1 + W(-1, -\exp(-1 - s))\}^{-1}. \tag{4.12}$$



(a) y coordinate of P_n increases as $\epsilon \downarrow 0$



(b) y coordinate of P_s decreases as $\epsilon \downarrow 0$

FIG. 8. Periodic orbits of IVP (4.14), (4.15) with $\epsilon = 0.1, 0.02, 0.01$, $x_\ell = 0.5$

4.3. Singularly perturbed system

Grasman and Veling [23], Verhulst [54, p. 170], and Fowler [16, p. 34] used the substitutions

$$u = \frac{d}{c}x, \quad v = \frac{a}{b}y, \quad \tau = \frac{t}{a}, \quad \epsilon = \frac{a}{d},$$

to convert the system

$$u'(\tau) = u\{a - bv\}, \quad v'(\tau) = v\{-d + cu\}, \tag{4.13}$$

to a singularly perturbed system

$$x'(t) = x\{1 - y\}, \quad \epsilon y'(t) = y\{x - 1\}, \tag{4.14}$$

with $0 < \epsilon \ll 1$.

The system (4.14) seems simple but it is much harder (in terms of stiffness with chosen values of ϵ) to be computed numerically than van der Pol equation (1.2) by using the stiff/nonstiff Maple ODE solver "NODES package". Moreover, it is not trivial to approximate it analytically by using a method of matched asymptotic expansions. These observations, as shown in Figure 8, may give the reason why Vasil'eva and Belyanin [53] applied a change of variables twice to (4.13) under the same assumption that $\epsilon = a/d \ll 1$.

Using the initial conditions

$$x(0) = x_\ell, \quad y(0) = 1, \tag{4.15}$$

with $0 < x_\ell < 1$, Grasman and Veling [23] obtained the solution in an implicit form

$$x - \log(x) + \epsilon[y - \log(y)] + \log(x_\ell) - x_\ell - \epsilon = 0 \tag{4.16}$$

in the phase plane. Using an implicit function theorem, the closed curve determined by (4.16) is divided into four segments, each of which is represented by a convergent series expansion. Integrating four local expansions of either x or y over four segments of the curve, Grasman and Veling obtained an asymptotic formula for the period

$$\begin{aligned} T = {} & x_r - x_\ell + \left(\frac{-1}{1-x_\ell} + \frac{1}{1-x_r}\right)\epsilon\log(\epsilon) + \left\{\frac{1}{1-x_\ell} - \frac{1}{1-x_r}\right. \\ & + \frac{1}{1-x_\ell}\log[(x_\ell - 1)\log(x_\ell)] - \frac{1}{1-x_r}\log[(x_r - 1)\log(x_r)] \\ & \left. + I_\ell(x_\ell) + I_r(x_r)\right\}\epsilon + \mathcal{O}(\epsilon^2\log^2(\epsilon)), \end{aligned} \tag{4.17}$$

where the constant x_r is defined by

$$x_r - \log(x_r) = x_\ell - \log(x_\ell),$$

and I_ℓ, I_r are given by, respectively,

$$I_\ell(x_\ell) = \int_0^{-\log(x_\ell)} \left\{ \frac{1}{s + x_\ell(1 - \exp(s))} - \frac{1}{(1 - x_\ell)s} \right\} ds,$$

$$I_r(x_r) = \int_{-\log(x_r)}^{0} \left\{ \frac{1}{s + x_r(1 - \exp(s))} - \frac{1}{(1 - x_r)s} \right\} ds.$$

Table I provides a comparison of three values for the relaxation period. Both $T_{\rm as}$ and $T_{\rm num}$ obtained by Grasman and Veling are, respectively, for the asymptotic formula (4.17) and numerical results of integrating the system (4.14) by using a Zonneveld's Runge-Kutta scheme RK4na, which yields the same results in the required accuracy as a method using the implicit formula (4.16). For the sake of convenience, we also include numerical results T of an integral representation (4.18) for the period by using the Gauss-Tschebyscheff integration rule of the first kind (4.19).

The problem (4.14) studied by Grasman and Veling has the periodic orbit (4.16) with the energy

$$E = \frac{1}{\epsilon}\{x_\ell - \log(x_\ell) - 1\},$$

and four extreme points

$$P_w = (x_{\min}, 1), \quad P_e = (x_{\max}, 1); \quad P_s = (1, y_{\min}(\epsilon)), \quad P_n = (1, y_{\max}(\epsilon)),$$

where $x_{\min} = x_\ell$, $x_{\max} = -W(-1, -x_\ell \exp(-x_\ell))$; $y_{\min}(\epsilon) = -W(0, -\exp(-1 - \frac{1}{\epsilon}[x_\ell - \log(x_\ell) - 1]))$, $y_{\max}(\epsilon) = -W(-1, -\exp(-1 - \frac{1}{\epsilon}[x_\ell - \log(x_\ell) - 1]))$. Note that, as $\epsilon \downarrow 0$, $E \uparrow \infty$, $P_s \to (1, 0)$, and $P_n \to (1, \infty)$. For example, for $x_\ell = 0.5$, we have $x_{\max} = 1.756431208$, $y_{\min}(0.1) = 0.05641281135$, $y_{\min}(0.02) = 0.2352762013 \times 10^{-4}$, $y_{\min}(0.01) = 0.1504631140 \times 10^{-8}$, $y_{\max}(0.1) = 4.416912796$, $y_{\max}(0.02) = 13.24065073$, $y_{\max}(0.01) = 23.47046070$. Moreover, by virtue of asymptotic results (2.14) and (2.15), we obtain, as $\epsilon \downarrow 0$,

$$y_{\min}(\epsilon) \approx \exp(-1 - \frac{1}{\epsilon}[x_\ell - \log(x_\ell) - 1]),$$

$$y_{\max}(\epsilon) \approx 1 + \frac{1}{\epsilon}[x_\ell - \log(x_\ell) - 1] + \log(1 + \frac{1}{\epsilon}[x_\ell - \log(x_\ell) - 1]).$$

The upper branch of the orbit (4.16) is $y = g_1(x)$; while its lower branch is $y = g_0(x)$, where

$$g_k(x) = -W(-k, -\exp(-1 + \frac{1}{\epsilon}[x - \log(x) - x_\ell + \log(x_\ell)])),$$

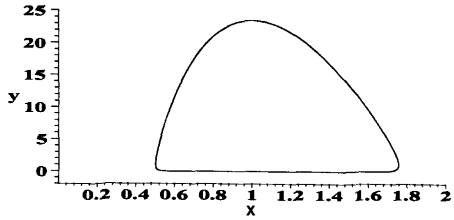


FIG. 9. Constructed orbit with $y = g_k(x)$ for $\epsilon = 0.01$

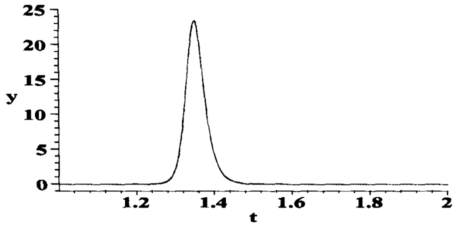
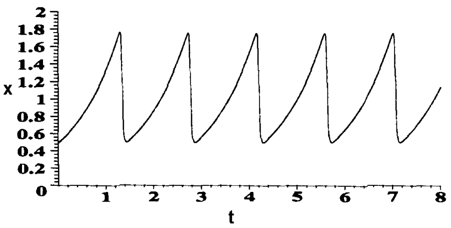
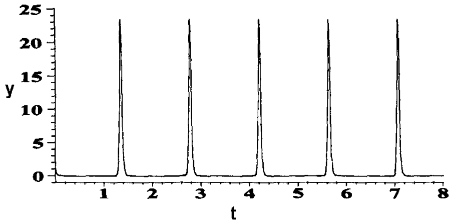


FIG. 10. Enlarged spike of y for $\epsilon = 0.01$



(a) Period of x in t



(b) Period of y in t

FIG. 11. Periodic behavior of (4.1) with $\epsilon = 0.01$

with $x \in [x_{\min}, x_{\max}]$. As shown in Figure 9 with $\epsilon = 0.01$, our expressions for the periodic orbit $y = g_k(x)$, with $k = 0$, 1, have an excellent agreement with numerical results by using NODES package, which gives wrong results when $\epsilon = 0.001$. The difficulty of approximating such orbit can be seen from Figures 4.11(a), 4.11(b), and 4.10. Specifically, the variable y is very close to zero most of the time (corresponding to slow phase in each cycle) except these spikes, each of which exists only for a short period of time (corresponding to fast phase in each cycle).

For the rest of this work, we illustrate computations of the period. In terms of the system (4.14) studied by Grasman and Veling, the period (4.11) becomes

$$\begin{aligned} T &= \epsilon \int_0^{\frac{1}{\epsilon}[x_\ell - \log(x_\ell) - 1]} \Phi(\epsilon s)\Phi(\frac{1}{\epsilon}[x_\ell - \log(x_\ell) - 1] - s)ds \\ &= \int_0^{x_\ell - \log(x_\ell) - 1} \Phi(\sigma)\Phi(\frac{1}{\epsilon}[x_\ell - \log(x_\ell) - 1 - \sigma])d\sigma, \end{aligned} \tag{4.18}$$

which has a weak singularity of square root type at endpoints $\sigma = 0$, $\sigma = x_\ell - \log(x_\ell) - 1$, respectively, as shown by the asymptotic result (2.13) of $\Phi(s)$ for s near 0.

To compute (4.18) numerically, it is converted to the form

$$T = \frac{1}{2}[x_\ell - \log(x_\ell) - 1] \int_{-1}^{1} \Phi(\frac{1}{2}[x_\ell - \log(x_\ell) - 1](1+s))\Phi$$
$$(\frac{1}{2\epsilon}[x_\ell - \log(x_\ell) - 1](1-s))ds.$$

Then the period is expressed in the form

$$T = \int_{-1}^{1} \frac{f(s)}{\sqrt{1-s^2}} ds,$$

with

$$f(s) = \frac{1}{2}[x_\ell - \log(x_\ell) - 1]\Phi(\frac{1}{2}[x_\ell - \log(x_\ell) - 1](1+s))\Phi$$
$$(\frac{1}{2\epsilon}[x_\ell - \log(x_\ell) - 1](1-s))\sqrt{1-s^2}.$$

Thus the period can be approximated numerically by using the Gauss-Tschebyscheff integration rule of the first kind

(4.19) $$T = \frac{\pi}{n}\sum_{i=1}^{n} f(s_i) + \frac{\pi}{2^{2n-1}(2n)!} f^{(2n)}(\xi),$$

where $s_i = \cos((2i-1)\pi/(2n)), i = 1, ..., n,$ and for some $\xi \in (-1, 1)$. A reference to Gauss-Tschebyscheff formula is Davis and Rabinowitz [8, p. 98].

Computational results of the period (4.19) given under columns T in Table I show an excellent agreement with numerical results T_{num} of Grasman and Veling.

TABLE I. Comparison of three values for the relaxation period

ϵ	$x_\ell = 0.50$			$x_\ell = 0.25$			$x_\ell = 0.10$		
	T_{as}	T_{num}	T	T_{as}	T_{num}	T	T_{as}	T_{num}	T
0.5	3.5359	4.6599	4.659884577	4.7247	5.1734	5.173375716	5.8567	6.0920	6.091989069
0.1	2.2470	2.3480	2.347993754	3.1303	3.1433	3.143266510	4.3014	4.3061	4.306122752
0.05	1.8668	1.8875	1.887492868	2.8015	2.8009	2.800912787	4.00945	4.00939	4.009387740
0.01	1.4320	1.4303	1.430285266	2.4612	2.4606	2.460594808	3.71766	3.71747	3.717469355
0.005	1.3557	1.3548	1.354840539	2.4058	2.4055	2.405499036	3.67143	3.67136	3.671357325
0.001	1.2816	1.2815	1.281540119	2.35364	2.35362	2.353620301	3.628628	3.628622	3.628621718

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