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A NEW PROOF OF $\ell_1 \overset{\vee}{\bigotimes} X \subseteq c_0 \overset{\wedge}{\bigotimes} X$

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Abstract. In this paper, we first show that for any Banach space X, $\ell_1[X] \subseteq \ell_\infty\langle X\rangle$ by Khinchin's inequality. Then by the relationship between Banach-valued sequence spaces and tensor products, we show that for any Banach space X, $\ell_1 \overset{\vee}{\otimes} X \subseteq c_0 \overset{\wedge}{\otimes} X$.

Let X be a Banach space over the complex field $\mathbb{C}$ or the real field $\mathbb{R}$ and X^* its dual. B_X denotes the closed unit ball of X. For $1 \le p \le \infty$, let p' be its conjugate, i.e., $1/p + 1/p' = 1$. For $1 \le p < \infty$, let $\ell_p[X]$ denote the space of weakly p-summable sequences on X, i.e.,

$$\ell_p[X] = \Big\{\overline{x} = (x_n)_n \in X^{\mathbb{N}} : \sum_{n=1}^{\infty} |x^*(x_n)|^p < \infty \text{ for all } x^* \in X^*\Big\}$$

and for every $\overline{x} \in \ell_p[X]$, let

$$\|\overline{x}\|_{[p]} = \sup\Big\{\Big(\sum_{n=1}^{\infty} |x^*(x_n)|^p\Big)^{1/p} : x^* \in B_{X^*}\Big\}.$$

For $p = \infty$, let

$$\ell_\infty[X] = \Big\{\overline{x} = (x_n)_n \in X^{\mathbb{N}} : \sup_{n\ge 1} |x^*(x_n)| < \infty \text{ for all } x^* \in X^*\Big\}$$

and for every $\overline{x} \in \ell_\infty[X]$, let

$$\|\overline{x}\|_{[\infty]} = \sup\{|x^*(x_n)| : x^* \in B_{X^*}, n \in \mathbb{N}\}.$$

Then $(\ell_p[X], \|\cdot\|_{[p]})(1 \le p \le \infty)$ is a Banach space (cf. [1, 8]).

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For $1 \le p \le \infty$, let $\ell_p\langle X\rangle$ denote the space of strongly p-summable sequences on X, i.e.,

$$\ell_p\langle X\rangle = \Big\{\overline{x} = (x_n)_n \in X^{\mathbb{N}} : \sum_{n=1}^{\infty} |x_n^*(x_n)| < \infty \text{ for all } (x_n^*)_n \in \ell_{p'}[X^*]\Big\}$$

and for every $\overline{x} \in \ell_p\langle X\rangle$, let

$$\|\overline{x}\|_{\langle p\rangle} = \sup \Big\{|\sum_{n=1}^{\infty} x_n^*(x_n)| : (x_n^*)_n \in B_{\ell_{p'}[X^*]}\Big\}.$$

Then $(\ell_p\langle X\rangle, \|\cdot\|_{\langle p\rangle})$ is a Banach space (cf. [1]).

Lemma 1. *Let X be a Banach space. Then for every $\overline{x} = (x_n)_n \in \ell_1[X]$ and every $\overline{x^*} = (x_n^*)_n \in \ell_1[X^*]$,*

$$\sum_{i=1}^{\infty} \Big(\sum_{k=1}^{\infty} |x_i^*(x_k)|^2\Big)^{1/2} \le \sqrt{2}\|\overline{x}\|_{[1]} \cdot \|\overline{x^*}\|_{[1]}.$$

Proof. Let $r_n(t)$ denote the Rademacher functions (see [5, p. 10]). For $n \in \mathbb{N}$, define functions f_n on [0, 1] by $f_n(t) = \sum_{k=1}^{n} r_k(t)x_k$. Then for every $n \in \mathbb{N}$ and every $t \in [0, 1]$,

$$\|f_n(t)\| = \sup_{x^* \in B_{X^*}} |x^*\Big(\sum_{k=1}^{n} r_k(t)x_k\Big)| \le \sup_{x^* \in B_{X^*}} \sum_{k=1}^{n} |x^*(x_k)| \le \|\overline{x}\|_{[1]}.$$

Now for $n, m \in \mathbb{N}$, by Khinchin's inequality (cf. [9, 10]).

$$\begin{aligned} \textstyle\sum_{i=1}^{m}(\sum_{k=1}^{n} |x_i^*(x_k)|^2)^{1/2} &\le \textstyle\sum_{i=1}^{m} \sqrt{2} \int_0^1 |\sum_{k=1}^{n} x_i^*(x_k) r_k(t)| dt \\ &= \textstyle\sqrt{2} \int_0^1 \sum_{i=1}^{m} |\langle x_i^*, f_n(t)\rangle| dt \\ &= \textstyle\sqrt{2}\|\overline{x}\|_{[1]} \int_0^1 \sum_{i=1}^{m} |\langle x_i^*, f_n(t)/\|\overline{x}\|_{[1]}\rangle| dt \\ &\le \textstyle\sqrt{2}\|\overline{x}\|_{[1]} \cdot \sup_{x \in B_X} \sum_{i=1}^{m} |\langle x_i^*, x\rangle| \\ &\le \sqrt{2}\|\overline{x}\|_{[1]} \cdot \|\overline{x^*}\|_{[1]}. \end{aligned}$$

Letting $n, m \longrightarrow \infty$, we have

$$\sum_{i=1}^{\infty} \Big(\sum_{k=1}^{\infty} |x_i^*(x_k)|^2\Big)^{1/2} \le \sqrt{2}\|\overline{x}\|_{[1]} \cdot \|\overline{x^*}\|_{[1]}.$$

The proof is completed. ■

Theorem 2. *Let X be a Banach space. Then $\ell_1[X] \subseteq \ell_\infty\langle X\rangle$ with $\|\cdot\|_{\langle\infty\rangle} \leq \sqrt{2}\|\cdot\|_{[1]}$.*

Proof. For $i \in \mathbb{N}$, let P_i denote the ith coordinate projection on ℓ_2. Then $\|P_i\| \leq 1$ for every $i \in \mathbb{N}$. Let $\overline{x} = (x_n)_n \in \ell_1[X]$ and $\overline{x^*} = (x_n^*)_n \in \ell_1[X^*]$. By Lemma 1,

$$\begin{aligned}\textstyle\sum_{i=1}^\infty |x_i^*(x_i)| &= \textstyle\sum_{i=1}^\infty |P_i((x_i^*(x_k))_k| \leq \sum_{i=1}^\infty \|P_i\|\ \|(x_i^*(x_k))_k\|_{\ell_2} \\ &\leq \textstyle\sum_{i=1}^\infty (\sum_{k=1}^\infty |x_i^*(x_k)|^2)^{1/2} \leq \sqrt{2}\|\overline{x}\|_{[1]} \cdot \|\overline{x}^*\|_{[1]}.\end{aligned}$$

Since $\overline{x^*}$ is arbitrary in $\ell_1[X^*]$, $\overline{x} \in \ell_\infty\langle X\rangle$ and

$$\|\overline{x}\|_{\langle\infty\rangle} = \sup\Big\{|\sum_{i=1}^\infty x_i^*(x_i)| : \overline{x^*} \in B_{\ell_1[X^*]}\Big\} \leq \sqrt{2}\|\overline{x}\|_{[1]}.$$

The proof is completed. ∎

Let

$$\ell_1[X]_G = \{\overline{x} \in \ell_1[X] : \lim_n \|\overline{x}(i > n)\|_{[1]} = 0\},$$

where $\overline{x}(i > n) = (0, \cdots, 0, x_{n+1}, x_{n+2}, \cdots)$, and let

$$c_0\langle X\rangle := (\ell_\infty\langle X\rangle)_G = \{\overline{x} \in \ell_\infty\langle X\rangle : \lim_n \|\overline{x}(i > n)\|_{\langle\infty\rangle} = 0\}.$$

Then we have

Corollary 3. *Let X be a Banach space. Then $\ell_1[X]_G \subseteq c_0\langle X\rangle$ with $\|\cdot\|_{\langle\infty\rangle} \leq \sqrt{2}\|\cdot\|_{[1]}$.*

Define

$$\begin{aligned}\psi : \ell_1 \otimes X(\text{or} c_0 \otimes X) &\longrightarrow X^{\mathbb{N}} \\ \textstyle\sum_{k=1}^n s^{(k)} \otimes x_k &\longmapsto (\textstyle\sum_{k=1}^n s_i^{(k)} x_k)_i.\end{aligned}$$

Then ψ is a well-defined linear map (cf. [2]).

Let $\ell_1 \overset{\wedge}{\otimes} X$ denote the completion of $\ell_1 \otimes X$ with respect to the injective tensor norm $\|\cdot\|_\vee$, and let $c_0 \overset{\wedge}{\otimes} X$ denote the completion of $c_0 \otimes X$ with respect to the projective tensor norm $\|\cdot\|_\wedge$ (cf. [6, p. 223-227]). Then by [3, Proposition 8.2] or [11], we have

Proposition 4. *Let X be a Banach space. Then $\psi(\ell_1 \overset{\wedge}{\otimes} X) = \ell_1[X]_G$ with the isometry ψ.*

Theorem 5. *Let X be a Banach space. Then* $\ell_1 \overset{\vee}{\otimes} X \subseteq c_0 \overset{\wedge}{\otimes} X$ *with* $\| \cdot \|_\wedge \leq \sqrt{2} \| \cdot \|_\vee$.

Proof. Let $u \in \ell_1 \overset{\vee}{\otimes} X$. Then there exist $u_n = \sum_{k=1}^n s^{(k)} \otimes x_k \in \ell_1 \otimes X$, $n = 1, 2, \cdots$, such that $\vee - \lim_n u_n = u$. By Proposition 4, there exists $\overline{x} \in \ell_1[X]_G$ such that $\overline{x} = \psi(u)$. By Corollary 3, $\overline{x} \in c_0\langle X \rangle$. So by Theorem 9 in [2], $u \in c_0 \overset{\wedge}{\otimes} X^{**}$. Now for every $n \in \mathbb{N}$,

$$\|u - u_n\|_\wedge = \|\psi(u) - \psi(u_n)\|_{\langle \infty \rangle} \leq \sqrt{2} \|\psi(u) - \psi(u_n)\|_{[1]} = \sqrt{2} \|u - u_n\|_\vee .$$

Notice that $u_n \in \ell_1 \otimes X \subseteq c_0 \otimes X$ for $n = 1, 2, \cdots$. So $u = \wedge - \lim_n u_n \in c_0 \overset{\wedge}{\otimes} X$ and $\|u\|_\wedge \leq \sqrt{2} \|u\|_\vee$. The proof is completed. ■

Remarks. (i) The result of Theorem 5 is also obtained by A. Grothendieck in [7] in a different way (also see [4]). (ii) By Proposition 10.8 in [3, p. 124], we have for each $\overline{x} \in \ell_1[X]$, $\|\overline{x}\|_{\langle \infty \rangle} \leq \|\overline{x}\|_{[1]}$. So $\sqrt{2}$ in the inequality in Theorem 5 can be replaced by 1, the best coefficient for the inequality.

References

1. H. Apiola, Duality between spaces of p-summable sequences, (p, q)-summing operators and characterization of nuclearity, *Math. Ann.* **219** (1976), 53-64.
2. Q. Y. Bu, On a sequential representation of the projective tensor product of ℓ_p and X, $1 < p < \infty$, *Quaestiones Math.*, to appear.
3. A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, North-Holland, Amsterdam, 1993.
4. J. Diestel, J. Fourie and J. Swart, *The Metric Theory of Tensor Products*, to appear.
5. J. Diestel, H. Jarchow, and A. Tonge, *Absolutely Summing Operators*, Cambridge Univ. Press, Cambridge, 1995.
6. J. Diestel and J. Uhl, *Vector Measures*, Math. Surveys Vol. 15, Amer. Math. Soc., Providence, 1977.
7. A. Grothendieck, Résumé de la théorie métrique des produits tensoriels topologiques, *Bol. Soc. Mat. São Paulo* **8** (1956), 1-79.
8. M. Gupta and Q. Y. Bu, On Banach-valued GAK-sequence spaces $\ell_p[X]$, J. Anal. **2** (1994), 103-113.
9. R. Latala and K. Oleszkiewicz, On the best constant in the Khinchin-Kahane inequality, *Studia Math.* **109** (1994), 101-104.
10. S. J. Szared, On the best constants in the Khinchin inequalty, *Studia Math.* **58** (1976), 197-208.

11. C. X. Wu and Q. Y. Bu, Köthe dual of Banach sequence spaces $\ell_p[X]$ and Grothendieck space, *Comment. Math. Univ. Carolin.* **34** (1993), 265-273.

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