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B-BOUNDED SEMIGROUPS AND C-EXISTENCE FAMILIES

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Abstract. The aim of this paper is to investigate links of the recently introduced B-bounded semigroups [5] with other generalizations of semigroups, like C-regularized semigroups and C-existence families.

1. Introduction

Let us consider the standard abstract Cauchy problem in a Banach space X:

$$\frac{du}{dt} = Au, \qquad \lim_{t\to 0^+} u(t) = \overset{\circ}{u} \,. \tag{1.1}$$

Very often the existence of a semigroup $(\exp(tA))_{t\geq 0}$ describing the evolution of this system is established in a non-constructive way. This is especially the case when the positivity methods are employed. Then, very little quantitative information on the evolution is available. On the other hand, there may exist an operator B such that $t \mapsto Be^{tA}$ can be calculated constructively, yielding some information about the evolution. An interesting example of this type, pertaining to the transport equation with multiplying boundary conditions, was analysed in [11] and has prompted one of the authors to define a class of evolution families which behave well if looked at through the "lenses" of another operator. Such families, called B-bounded semigroups, have been introduced in [5], and analysed and applied to various problems in a few papers [1, 3, 6, 7]. Recently it was observed that B-bounded semigroups can be applied to implicit evolution equations with irregular operators – the paper [4] is devoted to this topic.

The reasoning described above is similar to that leading to C-regularized semigroups (see, e.g., [8, 9]); also some results bear a formal resemblance. This created

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some misunderstandings (see, e.g., the reviews MR#98c:47049 and MR#98k:47075, where the reviewer claims that B-bounded semigroups are a very special case of C-existence families.)

This paper was prompted partly by such a misreading of the theory of B-bounded semigroups, and partly is intended to show the proper place of this theory amongst the variety of recent generalizations of C_0-semigroup theory.

To achieve this aim we first present some necessary developments of the theory and in the final sections we show two groups of comparison results - one answering the question when a B-bounded semigroup generated by A is the C-existence family for A with $B = C$, and the second addressing a more general question when a B-bounded semigroup is a C-existence family with possibly different defining operators.

The rough answer to both questions is that whenever B-bounded semigroup and C-existence family coincide, A (or some operator related to A) generates a C_0-semigroup in the original space X, which shows that these two families are quite distinct.

2. Basic Notations and Definitions

The definition of B-quasi bounded semigroups was introduced in [6] and (with some modifications due to the author of this paper) reads as follows.

Definition 2.1. Let $(A, D(A))$ be a linear operator in a Banach space X, $(B, D(B))$ be another linear operator from X to another Banach space Z, and for some $\omega \in \mathbb{R}$ the resolvent set of A satisfies

$$\rho(A) \supset]\omega, \infty[. \tag{2.1}$$

A one-parameter family of operators $(Y(t))_{t\geq 0}$ from X to Z, which satisfies:

1. $D(Y(t)) =: \Omega \supseteq D(B)$, and for any $t \geq 0$ and $f \in D(B)$,

$$\|Y(t)f\|_Z \leq M \exp(\omega t)\|Bf\|_Z, \tag{2.2}$$

2. the function $t \mapsto Y(t)f$ is in $C([0, \infty[, Z)$ for any $f \in \Omega$,
3. for any $f \in \Omega_0 := \{f \in D(A) \cap D(B);\ Af \in \Omega\} \subset D(A) \cap D(B)$,

$$Y(t)f = Bf + \int_0^t Y(s)Afds, \qquad t \geq 0, \tag{2.3}$$

is called a B-quasi bounded semigroup generated by A and B.

Since it will not cause any misunderstanding, in this paper the original name of B-quasi bounded semigroup will be replaced by B-bounded semigroup.

If A generates a B-bounded semigroup satisfying the above conditions, then we write $A \in B - \mathcal{G}(M, \omega, X, Z)$; we shall also say that $(Y(t))_{t\geq 0}$ defined in Definition 2.1 is simply a B-bounded semigroup generated by A. The notation $A \in \mathcal{G}(M, \omega, X)$ means that A is the generator of a C_0-semigroup in X with the Hille-Yosida constants M and ω.

Remark 2.1. A closer scrutiny of the considerations of [3] (see also [1]) shows that the assumption (2.1) can be replaced by the requirement that for $\lambda > \omega$ we have

$$(\lambda I - A) : D_B(A) \to D(B), \tag{2.4}$$

where $D_B(A) = \{x \in D(A) \cap D(B);\ Ax \in D(B)\}$, is bijective. Note that this requirement is purely algebraical. All the considerations below are therefore valid if this assumption holds.

The main role in the considerations is played by the space X_B which is the completion of the quotient space $D(B)/N(B)$ with respect to the norm $\|\cdot\|_B = \|B\cdot\|_Z$. It is known that then $D(B)/N(B)$ is isometrically isomorphic to a dense subspace of X_B, say $\mathcal{X}$. The canonical mapping of $D(B)$ into X_B (and onto $\mathcal{X}$) will be denoted by $\mathfrak{p}$. In a standard way, B can be shifted to $\mathcal{X}$ and extended by density to an isometry $\mathfrak{B} : X_B \to Z$.

It follows [3, Lemmas 3.1 and 3.2] that if A generates a B-bounded semigroup, then A preserves cosets of $D(B)/N(B)$ and therefore it can be defined to act from $\mathfrak{p}D_B(A) \subset \mathcal{X}$ into $\mathcal{X}$. Thus in what follows we shall always assume that the operator A has this property which can be expressed as

$$A(N(B) \cap D(A)) \subset N(B). \tag{2.5}$$

We denote by A_B the part of A in $D(B)$, i.e., $A_B = A|_{D_B(A)}$. It can be proved [3] that if $A \in B - \mathcal{G}(M, \omega, X, Z)$, then the shift $\hat{A}_B$ of A to X_B is closable in X_B; its closure $\overline{\hat{A}_B}$ in X_B is denoted by $\mathfrak{A}$.

Let us introduce the subspace $Z_B = \overline{R(B)}$ (the closure of the range of B in Z). The main result of [3] (Theorem 4.1) reads as follows.

Theorem 2.1. *If $A \in B - \mathcal{G}(M, \omega, X, Z)$ and $\overline{B[D_B(A)]}^Z = Z_B$, then $\mathfrak{A} \in \mathcal{G}(M, \omega, X_B)$. Conversely, if there is $\mathcal{A} \supset \hat{A}_B$ such that $\mathcal{A} \in \mathcal{G}(M, \omega, X_B)$, then $\mathcal{A} = \mathfrak{A}$ and $A \in B - \mathcal{G}(M, \omega, X, Z)$. The B-bounded semigroup $(Y(t))_{t\geq 0}$ is given by*

$$Y(t)x = \exp(t\mathfrak{B}\mathfrak{A}\mathfrak{B}^{-1})Bx = \mathfrak{B}\exp(t\mathfrak{A})\mathfrak{p}x, \tag{2.6}$$

for $x \in D(B)$.

The assumption that $B[D_B(A)]$ is dense in Z_B can be discarded if Z (and consequently Z_B) are reflexive spaces [3, Corollary 4.1]. Recently, Arlotti [1] (see also the proof of Theorem 3.1) proved that if the B-bounded semigroup satisfies an additional condition:

$$\forall_{x\in D(B)} \quad Y(0)x = Bx, \tag{2.7}$$

then $B[D_B(A)]$ is dense in Z_B (or equivalently, $D_B(A)$ is dense in X_B). Note that (2.3) gives (2.7) only for $x \in \Omega_0$ which in most cases reduces to $D_B(A)$. It is easy to see that the converse is also true. Thus if (2.7) holds, then the density assumption in Theorem 2.1 can be omitted.

In the next section we shall see that there exist objects satisfying the assumptions 1–3 of Definition 2.1 but associated with much more general operators than those specified in this definition. Following this observation, we shall formulate and prove a generalization of Theorem 2.1. It is also worthwhile to note that similar results, though through other methods, have been recently obtained by Arlotti in the forthcoming paper [2].

3. New Generation Theorem

Let us first observe that the existence of a B-bounded semigroup is no longer related to the existence of $(\exp(tA))_{t\geq 0}$, as was the case in the motivating example of [5]. Moreover, assumption (2.1) can be replaced by a weaker one (2.4). In the example below, we shall show that even this assumption is too restrictive and can be relaxed even further.

Example 3.1. Let us consider $X = L_2(\mathbb{R}, e^{x^2}dx)$, $Au = \partial_x u$ on the maximal domain, and $(Bu)(x) = \exp(-x^2/2))u(x)$. Clearly, $B : X \to X$ is a continuous operator. Moreover, $\|Bu\|_X = \|u\|_{L_2(\mathbb{R})}$ and since $C_0^\infty(\mathbb{R}) \subset X$, we can identify X_B with $L_2(\mathbb{R})$. Let us consider the closure $\mathfrak{A}$ of A, that is, we take a sequence $(u_n)_{n\in\mathbb{N}}$ of elements of $D(A)$ such that $u_n \to u$ and $\partial_x u_n \to g$ as $n \to \infty$ in $L_2(\mathbb{R})$. However, this is the same as the closure of $D(A)$ in $W_2^1(\mathbb{R})$, and as $C_0^\infty(\mathbb{R}) \subset D(A)$ is dense in $W_2^1(\mathbb{R})$, we obtain that $\mathfrak{A}u = \partial_x u$ for $u \in W_2^1(\mathbb{R})$. Thus, $\mathfrak{A}$ generates a semigroup in X_B and $(Y(t)u)(x) = (\exp(-x^2/2))u(t + x)$ satisfies conditions 1-3 of Definition 2.1.

On the other hand, $\lambda I - A : D_B(A) = D(A) \to D(B) = X$ is not bijective. To prove this, we note that for any $\phi \in C_0^\infty(\mathbb{R})$ we have

$$\int_{-\infty}^{\infty} \partial_x u(x)\phi(x)e^{x^2}dx = -\int_{-\infty}^{\infty} u(x)(\partial_x\phi(x) + 2x\phi(x))e^{x^2}dx,$$

and therefore $A^{\#}\phi = -\partial_x\phi - 2x\phi$ is the formal adjoint of A in X. Let us consider the equation $\lambda\Phi + 2x\Phi + \partial_x\Phi = 0$ in X; we see that $\Phi(x) = \exp(-x^2 - \lambda x)$ is its solution. We have $\Phi \in X$ and we must prove that $\Phi \in D(A^*)$. Let $\phi_n \in C_0^\infty(\mathbb{R})$, $n = 1, 2, \ldots$, be such that $\phi_n(x) = 1$ for $|x| \leq n$ and $\phi_n(x) = 0$ for $|x| \geq n+1$ with $|\partial_x\phi_n(x)| \leq M$ for $n \leq |x| \leq n+1$. Then, integrating by parts, we have for any $u \in D(A)$,

$$\int_{-\infty}^{\infty} \partial_x u(x)\Phi(x)\phi_n(x)e^{x^2}dx = -\int_{n\leq|x|\leq n+1} u(x)\partial_x\phi_n(x)e^{-\lambda x}dx + \lambda\int_{-\infty}^{\infty} u(x)\phi_n(x)e^{-\lambda x}dx.$$

Since $u, \Phi \in X$ so that $u\Phi \in L_1(\mathbb{R}, e^{x^2}dx)$, it follows that $u(x)e^{-\lambda x} = u(x)\Phi(x)e^{x^2} \in L_1(\mathbb{R})$. Hence passing to the limit with $n \to \infty$ we obtain

$$\int_{-\infty}^{\infty} \partial_x u(x)\Phi(x)e^{x^2}dx = \lambda\int_{-\infty}^{\infty} u(x)e^{-\lambda x}dx = \lambda\int_{-\infty}^{\infty} u(x)\Phi(x)e^{x^2}dx.$$

This shows that $\Phi \in D(A^*)$ and $\Phi \in N(\lambda - A^*) = R(\lambda - A)^\perp$, and therefore $(\lambda - A) : D(A) \to X$ is not a surjective operator; even more, it is not a surjection onto any dense subspace of X. ■

Remark 3.1. This example was used in [9] to motivate the introduction of C-existence and uniqueness families. Here we have seen an alternative way of regularizing this problem.

Example 3.1 shows that we should be able to replace the assumption 2.1 by one even weaker than (2.4), which would require only the bijectivity of a suitable extension of the shift of A (remember that (2.5) holds so that the shift $\hat{A}$ can be defined). In fact, in the proof of Theorem 2.1, (2.1) was used to show that $[\omega, \infty[\subset \rho(\mathfrak{A})$. Thus, what we really need is that the Hille-Yosida estimate holds on some dense subspace $\mathfrak{X}$ of X_B. Moreover, as we use the pseudo-resolvent identity, we require $D_\lambda = (\lambda I - A)^{-1}\mathfrak{X} \subset \mathfrak{X}$ for $\lambda > \omega$. Finally, as our starting point is the space X and the operators defined in it, the space $\mathfrak{X}$ must be accessible from X in the sense of the operator closure in X_B.

Before we formulate the suitable assumption, we note that the above requirements make our choice limited to certain extent. We have the following simple proposition.

Proposition 3.1. *$\forall_{\lambda>\omega}\, D_\lambda = D$ if and only if $\forall_{\lambda>\omega}\, D_\lambda \subset \mathfrak{X}$.*

Proof. To prove necessity, let us take $\lambda \neq \lambda' > \omega$, then for any $x' \in D_{\lambda'}$ there exists $x \in D_\lambda$ such that for some $y \in \mathfrak{X}$ we have $\lambda x - Ax = y = \lambda' x' - Ax'$. This can be written as

$$\lambda(x - x') - A(x - x') = (\lambda' - \lambda)x'.$$

Now, assume that $D_{\lambda'} \subset D_\lambda$, then $x - x' \in D_\lambda$, and therefore $\lambda(x-x') - A(x-x') \in \mathfrak{X}$. Thus, $x' \in \mathfrak{X}$. Since x' is arbitrary, $D_{\lambda'} \subset \mathfrak{X}$. Clearly, the converse is also true. Since the argument is symmetric with respect to primed and un-primed objects we conclude the proof. ■

These considerations lead to the following new assumption on A.

(2.1′) The shift $\hat{A}_B$ of the operator A_B is closable in X_B, i.e., if the sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $D_B(A)$ is such that $Bx_n \to 0$ and $BAx_n \to y$ in Z as $n \to \infty$, then $y = 0$. Denoting $\mathfrak{A} = \overline{\hat{A}_B}^{X_B}$, we assume further that there exist subspaces: $\mathfrak{X}$ satisfying $D(B)/N(B) \subseteq \mathfrak{X} \subseteq X_B$, and $D_B(A)/N(B) \subseteq D \subseteq \mathfrak{X} \cap D(\mathfrak{A})$ such that $(\lambda - \mathfrak{A}|_D) : D \to \mathfrak{X}$ is bijective for all $\lambda > \omega$.

We have then the following theorem.

Theorem 3.1. *Let the operators A and B satisfy the conditions of Definition* 2.1 *with assumption* (2.1) *replaced by assumption* 2.1′. *Then $A \in B - \mathcal{G}(M, \omega, X, Z)$ and* (2.7) *holds if and only if the following conditions are satisfied*:

1. *$B(D)$ is dense in Z_B*,
2. *there exist $M > 0$ and $\omega \in \mathbb{R}$ such that for any $\mathfrak{y} \in \mathfrak{X}$, $\lambda > \omega$ and $n \in \mathbb{N}$*:

$$\|\mathfrak{B}(\lambda I - \mathfrak{A}|_D)^{-n}\mathfrak{y}\|_Z \leq \frac{M}{(\lambda - \omega)^n}\|\mathfrak{B}\mathfrak{y}\|_Z. \tag{3.1}$$

If we don't assume (2.7), *then the assumption* 1. *is sufficient but not necessary.*

Proof. The proof essentially consists in checking that the new assumption (2.1′) is sufficient to mimic the crucial steps from proofs of Theorems 2.1 and 3.2. However, to make the paper self-contained we shall provide all the necessary details.

As in [5, 6] we introduce the operators $J_n(\lambda) : \Omega \longrightarrow Z$ as follows

$$J_n(\lambda)x = \frac{1}{(n-1)!}\int_0^\infty t^{n-1}\exp(-\lambda t)Y(t)x\, dt, \quad \lambda > \omega. \tag{3.2}$$

The integral exists since the function $t \mapsto Y(t)x$ is continuous and satisfies (2.2). In particular, for $x \in D(B)$, we have from the definition that

$$\|J_n(\lambda)x\|_Z \leq \frac{1}{(n-1)!}\int_0^\infty t^{n-1}\exp(-\lambda t)\|Y(t)x\|_Z\, dt \leq \frac{M\,\|x\|_B}{(\lambda-\omega)^n}. \tag{3.3}$$

By (3.3) and (2.2), we can extend by continuity the operators $J_n(\lambda)$, $n = 1, 2 \ldots$, $\lambda > \omega$, and $Y(t)$, $t \geq 0$, to bounded linear operators $\mathfrak{J}_n(\lambda) : X_B \to Z$ and $\mathfrak{Y}(t) : X_B \to Z$. Let $(x_n)_{n\in\mathbb{N}} \subset \mathcal{X}$ satisfy $x_n \to \mathfrak{x}$ in X_B. Then $\|Y(t)x_n - \mathfrak{Y}(t)\mathfrak{x}\|_Z \leq Me^{\omega t}\|x_n - \mathfrak{x}\|_B$, and thus $t \to \mathfrak{Y}(t)\mathfrak{x}$ is continuous for any $\mathfrak{x} \in X_B$. Moreover, by (3.3) we can pass to the limit in (3.2) to get

$$\mathfrak{J}_n(\lambda)\mathfrak{x} = \frac{1}{(n-1)!}\int_0^\infty t^{n-1}\exp(-\lambda t)\mathfrak{Y}(t)\mathfrak{x}\, dt, \qquad \lambda > \omega \tag{3.4}$$

for any $\mathfrak{x} \in X_B$. Thus, similarly to [5, Section 3], we obtain for all $\mathfrak{x} \in D(\mathfrak{A})$:

$$\mathfrak{J}_1(\lambda)(\lambda I - \mathfrak{A})\mathfrak{x} = \lambda\int_0^\infty \exp(-\lambda t)\mathfrak{Y}(t)\mathfrak{x}dt - \int_0^\infty \exp(-\lambda t)\mathfrak{Y}(t)\mathfrak{A}\mathfrak{x}.$$

To evaluate the first integral, we observe that (2.3) can be extended by density to

$$\mathfrak{Y}(t)\mathfrak{x} = \mathfrak{B}\mathfrak{x} + \int_0^t \mathfrak{Y}(s)\mathfrak{A}\mathfrak{x}ds, \qquad t \geq 0, \tag{3.5}$$

where $\mathfrak{x} \in D(\mathfrak{A})$. Inserting (3.5) into the first integral and carrying out the integration, we obtain

$$\mathfrak{J}_1(\lambda)(\lambda I - \mathfrak{A})\mathfrak{x} = \mathfrak{B}\mathfrak{x}. \tag{3.6}$$

Using assumption (2.1$'$) we obtain

$$\mathfrak{J}_1(\lambda)\mathfrak{y} = \mathfrak{B}(\lambda I - \mathfrak{A}|_D)^{-1}\mathfrak{y} \tag{3.7}$$

for all $\mathfrak{y} \in \mathfrak{X}$ and by the estimate (3.3) we obtain that for those $\mathfrak{y}$ we have

$$\|(\lambda I - \mathfrak{A}|_D)^{-1}\mathfrak{y}\|_{X_B} \leq \frac{M}{(\lambda - \omega)}\|\mathfrak{y}\|_{X_B}. \tag{3.8}$$

Next iterating the procedure used to derive (3.6), we obtain from (3.4) the formula

$$\mathfrak{J}_n(\lambda)\mathfrak{y} = \mathfrak{B}(\lambda I - \mathfrak{A}|_D)^{-n}\mathfrak{y}, \tag{3.9}$$

valid for $\mathfrak{y} \in \mathfrak{X}$, $\lambda > \omega$. Finally from Eqs. (3.9), (3.3) and (3.4) we obtain

$$\begin{aligned}\|\mathfrak{B}(\lambda I - \mathfrak{A}|_D)^{-n}\mathfrak{y}\|_Z &\leq \frac{1}{(n-1)!}\int_0^\infty t^{n-1}\exp(-\lambda t)\|\mathfrak{Y}(t)\mathfrak{y}\|_Z\, dt \\ &\leq \frac{M\ \|\mathfrak{y}\|_{X_B}}{(\lambda-\omega)^n}\end{aligned} \tag{3.10}$$

which gives (3.1).

To prove property 1, we extend the argument of [1, Theorem 2.1]. Let $\mathfrak{x} \in \mathfrak{X}$ and $\mathfrak{y}_\lambda = \lambda(\lambda - \mathfrak{A}|_D)^{-1}\mathfrak{x} \in D$. By (3.7),

$$\mathfrak{B}\mathfrak{x} - \mathfrak{B}\mathfrak{y}_\lambda = \mathfrak{B}\mathfrak{x} - \lambda\mathfrak{B}(\lambda I - \mathfrak{A}|_D)^{-1}\mathfrak{x} = \int_0^\infty \lambda e^{-\lambda t}(\mathfrak{B} - \mathfrak{Y}(t))\mathfrak{x} dt.$$

By (2.7), $Y(0)x = Bx$ for all $x \in D(B)$. Since the shifts of both operators to X_B can be extended by continuity to X_B, and $D(B)/N(B)$ is dense in X_B, we have $\mathfrak{Y}(0)\mathfrak{x} = \mathfrak{B}\mathfrak{x}$ for all $\mathfrak{x} \in \mathfrak{X}$. As we observed earlier, for such $\mathfrak{x}$ the function $t \mapsto \mathfrak{Y}(t)\mathfrak{x}$ is continuous, and hence for any $\epsilon > 0$ we can find $\delta > 0$ such that $\sup_{0\le t\le\delta} \|(\mathfrak{B} - \mathfrak{Y}(t))\mathfrak{x}\|_Z \le \epsilon$. Thus

$$\begin{aligned}\|\mathfrak{B}\mathfrak{x} - \mathfrak{B}\mathfrak{y}_\lambda\|_Z &\le \int_0^\infty \lambda e^{-\lambda t}\|(\mathfrak{B} - \mathfrak{Y}(t))\mathfrak{x}\|_Z dt \\ &\le \epsilon + (1+M)\int_\delta^\infty \lambda e^{-(\lambda-\omega)}\|\mathfrak{x}\|_{X_B} dt \le 2\epsilon,\end{aligned}$$

provided λ is sufficiently large. Hence, $\mathfrak{B}(D)$ is dense in $\mathfrak{B}(\mathfrak{X}) = R(B)$ and so in Z_B, in Z topology or, equivalently, D is dense in $\mathfrak{X}$ in X_B topology. However, as $D(B) \subset \mathfrak{X}$ is dense in X_B, $\mathfrak{X}$ is dense in X_B and therefore $\overline{D}^{X_B} = X_B$. This proves the necessity of the conditions 1 and 2.

To prove the sufficiency, we note first that the resolvent equation is of purely algebraic character and therefore for $\lambda, \mu \in [\omega, \infty[$ and $\mathfrak{x} \in \mathfrak{X}$ we have

$$(\lambda - \mathfrak{A}|_D)^{-1}\mathfrak{x} - (\mu - \mathfrak{A}|_D)^{-1}\mathfrak{x} = (\mu - \lambda)(\lambda - \mathfrak{A}|_D)^{-1}(\lambda - \mathfrak{A}|_D)^{-1}\mathfrak{x}, \tag{3.11}$$

where we used the assumption that $D \subset \mathfrak{X}$ coming from (2.1′).

Since $\mathfrak{B}(D)$ is dense in Z_B, we see that D is dense in X_B and so is $\mathfrak{X}$ by the assumption $D \subset \mathfrak{X} \subset X_B$ of (2.1′). From the assumption (3.1), it follows that for each $\lambda > \omega$ the operator $(\lambda - \mathfrak{A}|_D)^{-1}$ can be extended by continuity to a bounded operator $\mathfrak{R}(\lambda) : X_B \to X_B$, which satisfies for any $\mathfrak{y} \in X_B$,

$$\|\mathfrak{R}(\lambda)\mathfrak{y}\|_{X_B} \le \frac{M}{(\lambda - \omega)}\|\mathfrak{y}\|_{X_B}. \tag{3.12}$$

Thus, equation (3.11) can be extended onto the whole of X_B preserving its structure, and hence the family of operators $\mathfrak{R}(\lambda)$ is a pseudo resolvent. The range of each $\mathfrak{R}(\lambda)$ contains D, and therefore is dense in X_B. Thanks to (3.12) we can use Theorem 9.4 of [10] to conclude that $\mathfrak{R}(\lambda)$ is the resolvent of a unique densely defined closed operator in X_B. Denote this operator by $\mathcal{A}$. Since $((\lambda - \mathcal{A})^{-1})^{-1}\mathfrak{x} =$

$((\lambda - \mathfrak{A}|_D)^{-1})^{-1}\mathfrak{x}$ for $\mathfrak{x} \in D$ and $\mathcal{A} = \lambda I - \left((\lambda - \mathcal{A})^{-1}\right)^{-1}$, we obtain that $\mathcal{A}$ is an extension of $\mathfrak{A}_D := \mathfrak{A}|_D$. Hence $\mathfrak{A}_D$ is closable and $\overline{\mathfrak{A}_D} \subset \mathcal{A}$.

Let now $\mathfrak{x} \in D(\mathcal{A})$; then $\mathfrak{x} = (\lambda - \mathcal{A})^{-1}\mathfrak{y}$ for some $\mathfrak{y} \in X_B$. This means that $\mathfrak{x} = \lim\limits_{n\to\infty}(\lambda - \mathfrak{A}_D)^{-1}\mathfrak{y}_n$ for $\mathfrak{y}_n \in \mathfrak{X}$ and $\mathfrak{y}_n \to \mathfrak{y}$. In other words, $\mathfrak{x}_n = (\lambda - \mathfrak{A}_D)^{-1}\mathfrak{y}_n \in D$ converges to $\mathfrak{x}$. Solving this equation we get $\mathfrak{A}_D\mathfrak{x}_n = \lambda\mathfrak{x}_n - \mathfrak{y}_n$ and $(\mathfrak{A}_D\mathfrak{x}_n)_{n\in\mathbb{N}}$ converges to $\lambda\mathfrak{x} - \mathfrak{y}$. Hence $\mathfrak{x} \in D(\overline{\mathfrak{A}_D})$ and $\overline{\mathfrak{A}_D}\mathfrak{x} = \lambda\mathfrak{x} - \mathfrak{y} = \lambda x - \left((\lambda - \mathcal{A})^{-1}\right)^{-1}\mathfrak{x} = \mathcal{A}\mathfrak{x}$. This shows $\mathcal{A} \subset \overline{\mathfrak{A}_D}$. Consequently, we have $\mathcal{A} = \overline{\mathfrak{A}_D}$ and $\mathfrak{R}(\lambda) = (\lambda - \overline{\mathfrak{A}_D})^{-1}$.

Therefore, Eq. (3.12) can be written as $\|(\lambda - \overline{\mathfrak{A}_D})^{-1}\mathfrak{x}\|_{X_B} \le (\lambda-\omega)^{-1}M\|\mathfrak{x}\|_{X_B}$ valid for any $\mathfrak{x} \in X_B$ and $\lambda > \omega$. Since clearly $(\lambda - \overline{\mathfrak{A}_D})^{-n} = \overline{(\lambda - \mathfrak{A}_D)^{-n}}$, it follows from (3.1) and the density of $\mathcal{X}$ in X_B that writing $(\lambda - \mathfrak{A}_D)^{-n} = (\lambda - \mathfrak{A}_D)^{-1}(\lambda - \mathfrak{A}_D)^{-n+1}$ and using induction in $n \in \mathbb{N}$ we have

$$\|(\lambda - \overline{\mathfrak{A}_D})^{-n}\mathfrak{x}\|_{X_B} \le \frac{M}{(\lambda-\omega)^n}\|\mathfrak{x}\|_{X_B}$$

for all $n \in \mathbb{N}$ and $\mathfrak{x} \in X_B$. This shows that $\overline{\mathfrak{A}_D}$ generates a semigroup in X_B and thanks to the assumption $D_B(A)/N(B) \subset D \subset D(\overline{\mathfrak{A}_D})$, it is straightforward to prove that the family $(Y(t))_{t\ge0} = \left(\mathfrak{B}e^{t\overline{\mathfrak{A}_D}}\right)_{t\ge0}$ satisfies the conditions of Definition 2.1. ■

Remark 3.2. The assumption that $D_B(A)/N(B) \subset D$ may seem too restrictive as what we need and use is that $D_B(A)/N(B) \subset D(\overline{\mathfrak{A}_D})$ (otherwise property 3 of Definition 2.1 would be satisfied on a smaller set than required). However, the proposition below shows that this is precisely what we need.

Let us consider the relations between the operators appearing in this theorem. We have the original operator A, its B-closure $\mathfrak{A}$ (i.e., the closure in X_B of the shift $\hat{A}_B$ of the part A_B of A in $D(B)$), the restriction of $\mathfrak{A}$ to D, $\mathfrak{A}_D$, and the generator $\mathcal{A} = \overline{\mathfrak{A}_D}$. We can prove the following proposition.

Proposition 3.2. *The following are equivalent*:

(i) $\mathcal{A} = \mathfrak{A}$,

(ii) $\hat{A}_B \subset \overline{\mathfrak{A}_D}$,

(iii) *for some* $\lambda > \omega$ *the operator* $\lambda - \mathfrak{A}$ *is injective*,

(iv) $\hat{A}_B \subset \mathfrak{A}_D$.

Proof. (i) ⇔ (ii). Let $\mathcal{A} = \mathfrak{A}$. Then $\mathcal{A} = \overline{\mathfrak{A}_D} = \mathfrak{A}$ yields $\hat{A}_B \subset \overline{\mathfrak{A}_D}$. Conversely, from (ii) we have $\mathfrak{A} = \overline{A} \subset \overline{\overline{\mathfrak{A}_D}} = \mathcal{A}$. This and $\mathcal{A} \subset \overline{\mathfrak{A}_D} \subset \overline{\mathfrak{A}} = \mathfrak{A}$ yield $\mathcal{A} = \mathfrak{A}$.

For any operator K, let us introduce the notation $K_\lambda = \lambda I - K$.

To prove (i) $\Leftarrow$ (iii), we assume that $\mathfrak{A}_\lambda$ is one-to-one. Clearly, $\mathfrak{A} \supset \mathcal{A}$. We show the converse inclusion. Since $\mathcal{A}_\lambda$ acts onto X_B, for any $x' \in D(\mathfrak{A})$ there is $x \in D(\mathcal{A})$ such that $\mathfrak{A}_\lambda x' = \mathcal{A}_\lambda x$. Since $\mathcal{A} \subset \mathfrak{A}$, we have also $\mathfrak{A}_\lambda \supset \mathcal{A}_\lambda$ so that $\mathfrak{A}_\lambda x = \mathcal{A}_\lambda x$. Therefore $\mathfrak{A}_\lambda x = \mathfrak{A}_\lambda x'$, and by the injectivity of $\mathfrak{A}_\lambda$ we obtain $x = x' \in D(\mathcal{A})$. The converse implication is obvious.

For (i) $\Leftarrow$ (iv), we see that if $\hat{A}_B \subset \mathfrak{A}_D$, then $\mathfrak{A} = \overline{\hat{A}_B} \subset \overline{\mathfrak{A}_D} = \mathcal{A}$ and hence $\mathcal{A} = \mathfrak{A}$. Conversely, if $\mathfrak{A} = \mathcal{A}$, then by (iii) $\mathfrak{A}_\lambda$ is a one-to-one operator for some λ, and therefore A_λ and $(\mathfrak{A}_D)_\lambda$ are one-to-one. Let $x \in D_B(A)/N(B) \setminus D$. Then $\mathfrak{A}_\lambda x = A_\lambda x = y \in D(B)/N(B)$ and since $D(B)/N(B) \subset \mathfrak{X}$, by (2.1$'$) there is $x \neq x' \in D$ such that $\mathfrak{A}_\lambda x = (\mathfrak{A}_D)_\lambda x' = y$. However, since $x \neq x'$ and $\mathfrak{A}_\lambda$ is injective, this is impossible. Thus $D_B(A)/N(B) \subset D$ which is equivalent to (iv).■

Thus we see that the assumption $D_B(A)/N(B) \subset D$ is necessary and sufficient for the semigroup $(\exp(t\mathcal{A}))_{t\geq 0}$ to define a B-bounded semigroup (see Remark 3.2). Another consequence of this proposition is that the B-bounded semigroup $(Y(t))_{t\geq 0}$ is uniquely determined by A and B, being defined by the semigroup generated by the B-closure of A restricted to $D_B(A)$.

It can be checked that Theorem 3.1 is a generalization of the similar theorem, proved in [3] for the case of operators satisfying all the assumptions of Definition 2.1, which reads as follows.

Theorem 3.2. *Let operators A and B satisfy the conditions of Definition* 2.1 *together with* (2.5). *Then A is the generator of a B-quasi bounded semigroup satisfying* (2.7) *if and only if the following conditions hold*:

1. $B[D_B(A)]$ *is dense in* Z_B,
2. *there exist* $M > 0$ *and* $\omega \in \mathbb{R}$ *such that for any* $x \in D(B)$, $\lambda > \omega$ *and* $n \in \mathbb{N}$:

$$\|B\,(\lambda I - A)^{-n}\,x\|_Z \leq \frac{M}{(\lambda-\omega)^n}\|Bx\|_Z. \tag{3.13}$$

If we don't require $(Y(t))_{t\geq 0}$ to satisfy (2.7), *then condition* 1. *is sufficient but not necessary.*

Proof. We take $\mathfrak{X} = D(B)/N(B)$ and $D = D_B(A)/N(B)$. By (2.5), it is possible to define the shift $\hat{A}_B$ of A_B into X_B. Next, we see that if (2.4) holds, then the same is true for $\hat{A}_B$. In fact, that $\lambda - \hat{A}_B$ is surjective is obvious. To show that $\lambda - \hat{A}_B$ is injective, let $\hat{x} \in D_B(A)/N(B)$ be such that $(\lambda - \hat{A}_B)\hat{x} = 0$. This shows that for some $x \in D_B(A)$ we have $(\lambda - A)x = g \in N(B)$, and thus $x = (\lambda - A)^{-1}g$. Since $x \in D(B)$, we get $Bx = B(\lambda - A)^{-1}g$ and by (3.13) we obtain

$$\|Bx\|_Z = \|B(\lambda - A)^{-1}g\|_Z \leq M(\lambda-\omega)^{-1}\|Bg\|_Z = 0,$$

which shows that $x \in N(B)$ and consequently $\hat{x} = 0$. Therefore $\lambda - \hat{A}_B$ is bijective from $D = D_B(A)/N(B)$ to $\mathfrak{X} = D(B)/N(B)$. To complete the proof that the assumptions of Theorem 3.2 yield those of Theorem 3.1, we note that by [3, Lemma 4.1] from (3.13) it follows that $\hat{A}_B$ is closable in X_B. ■

Remark 3.3. In the recent papers [1, 2], the author showed a construction of B-bounded semigroups without passing through the space X_B and the related operators. In this way, the assumption (2.5) does not appear directly in the construction of the B-bounded semigroup but it is imbedded in the assumptions adopted by the author. In any case, it is a consequence of A being the generator of a B-bounded semigroup.

3.1 The case $X_B \hookrightarrow X$

It is of interest to determine conditions under which X_B is not an abstract space but can be identified with a subspace of X. This will play an important role in comparing B-bounded semigroups and C-existence families. The following theorem was proved in [4, Theorem 2.4].

Theorem 3.3. *Let $B : X \to Z$ be an injective operator. The following conditions are equivalent*:

(i) X_B *has the following properties*:

(i′) *each coset $\tilde{x} \in X_B$ contains a sequence $(x_n)_{n\in\mathbb{N}}$ converging in the norm of X to some $x \in X$, and x is the limit of no other X-Cauchy sequence in $\tilde{x}$*,

(i″) *if $(x_n)_{n\in\mathbb{N}} \in \tilde{x}$, $(y_n)_{n\in\mathbb{N}}$ satisfy $\|x_n - y_n\|_X \to 0$ as $n \to \infty$ and $(y_n)_{n\in\mathbb{N}} \in \tilde{y}$ for some $\tilde{y} \in X_B$, then $\tilde{x} = \tilde{y}$*,

(ii) *the operator B is closable and B^{-1} is bounded*,

(iii) *there is an isomorphism $T : X_B \to X'_B \hookrightarrow X$ which satisfies $T|_{D(B)} = Id$.*

If we have the case described in the theorem above, the operator $\mathfrak{A}$ also becomes much simpler. The proof of the following theorem can be found in [4, Theorem 2.5].

Theorem 3.4. If B is a closable operator such that B^{-1} is bounded, and A is closable in X with $\lambda I - \bar{A}$ injective for some $\lambda > \omega$, and moreover $A \in B - \mathcal{G}(M, \omega, X, Z)$, then

$$\mathfrak{A} = \bar{A}|_{D_{\bar{B}}(\bar{A})}, \tag{3.14}$$

where

$$D_{\bar{B}}(\bar{A}) = \{x \in D(\bar{A}) \cap D(\bar{B});\ \bar{A}x \in D(\bar{B})\}.$$

Example 3.2. Let us consider $X = L_2(\mathbb{R}, e^{-x^2}dx)$, $Au = \partial_x u$ on the maximal domain, and $(Bu)(x) = \exp(x^2/2))u(x)$. $B : X \to X$ is an unbounded operator and since $\|Bu\|_X = \|u\|_{L_2(\mathbb{R})}$, we see that $D(B) = L_2(\mathbb{R})$. Since $B(D(B)) = X$, we obtain that $X_B = D(B) = L_2(\mathbb{R})$ by Theorem 3.3. Then $D_B(A) = W_2^1(\mathbb{R})$ and A_B generates a contraction semigroup, say $(T(t))_{t\geq 0}$, in $L_2(\mathbb{R})$. Thus $Y(t)u = BT(t)u = \exp(x^2/2))u(t + x)$ is the B-bounded semigroup generated by (A, B).

Note that here neither $D(A) \subset D(B)$, nor $\rho(A) \supset [\omega, \infty[$ (see similar considerations in Example 3.1), but the assumption (2.4) is satisfied. ■

4. Relation to *C*-regularized Semigroups and Similar Objects

The philosophy and appearance of B-bounded semigroups are similar to C-semigroups and related objects, like C-existence and uniqueness families, and this has caused some misunderstanding. It follows, however, that these objects are quite different, as we shall see, to the extent that the only objects which can be simultaneously C-semigroups and B-bounded semigroups in the same space X are C_0-semigroups in X.

To understand a link between C-existence families and B-bounded semigroups, we begin with noting that by Theorem 2.1 any B-bounded semigroup solves an abstract Cauchy problem in X_B. Since the very concept of C-existence families is that they provide solutions for the initial values taken from a subspace of the original space X, we are placed in the situation described in Theorem 3.3 with $C = \overline{B}^{-1}$. The first result in this direction is the following.

Proposition 4.1. *Assume that $A : D(A) \to X$ is a closed operator with no eigenvalues in $]\omega, \infty[$ for some $\omega \in \mathbb{R}$, and $A \in B - \mathcal{G}(M, \omega, X)$. If B^{-1} is densely defined and $X_B \hookrightarrow X$, then*

$$(W(t))_{t\geq 0} = \left(\bar{B}^{-1}Y(t)\bar{B}^{-1}\right)_{t\geq 0}$$

is a mild $\bar{B}^{-1}$-existence family for A. It is a strong $\bar{B}^{-1}$-existence family for A if

$$\bar{B}^{-1}(D(A)) \subset D(A). \tag{4.1}$$

If $\bar{B}^{-1}A \subset A\bar{B}^{-1}$, then $\left(\bar{B}^{-1}Y(t)\bar{B}^{-1}\right)_{t\geq 0}$ is the $\bar{B}^{-1}$-regularized semigroup generated by A.

Proof. From Theorem 3.3, B is closable and B^{-1} is bounded and we can identify X_B with $D(\bar{B})$ and Im $\bar{B} = X$ (by the density of $D(B^{-1}) =$ Im B in X). Moreover, we have then by Theorem 3.4 that $\mathfrak{A} = A$ restricted to $D_{\bar{B}}(A) = \{x \in D(A) \cap D(\bar{B});\ Ax \in D(\bar{B})\}$, and the B-bounded semigroup is given by $Y(t) = \bar{B}e^{tA}$, where the semigroup acts in $D(\bar{B})$. Clearly, then $W(t) = e^{tA}\bar{B}^{-1} =$

$\bar{B}^{-1}Y(t)\bar{B}^{-1}$, $t \geq 0$, is a strong C-existence family for A. Indeed, since $X_B \hookrightarrow X$, $(W(t))_{t\geq 0}$ is a family of bounded operators in X and $t \mapsto W(t)x$ is continuous for any $x \in X$. Since $(\exp(tA))_{t\geq 0}$ is a semigroup in X_B, we have for any $y \in X_B$,

$$e^{tA}y = y + A\left(\int_0^t e^{sA}yds\right), \qquad t \geq 0,$$

and therefore for any $x \in X$ such that $x = \bar{B}y$ we have

$$W(t)x = e^{tA}\bar{B}^{-1}x = \bar{B}^{-1}x + A\left(\int_0^t e^{sA}\bar{B}^{-1}xds\right), \tag{4.2}$$

which is the mild C-existence family identity (note that again due to $X_B \hookrightarrow X$, the integral and A can be considered as X-space operations).

Next note that for a mild C-existence family to be a strong C-existence family it is necessary to leave $D(A)$ invariant [9, Definition 2.4], so if we have a semigroup acting in a subspace of X which is accessible by an operator C, then we must have $Cx \in D(A)$ whenever $x \in D(A)$, which in our case translates into Eq. (4.1). If this is the case, then using again the fact that $(\exp(tA))_{t\geq 0}$ is the semigroup generated by A, we have from $\bar{B}^{-1}x \in D(A)$ that $e^{sA}\bar{B}^{-1}x \in D(A)$ and $\left(e^{sA}\bar{B}^{-1}\right)_{t\geq 0}$ is a strongly continuous family of operators in $D(A)$ with graph norm. Thus A commutes with the integral in (4.2) and $(W(t))_{t\geq 0}$ is a strong $\bar{B}^{-1}$-existence family for A.

Finally, if the commutativity property is satisfied (and then (4.1) follows automatically), then for any $x \in D(A)$ and $t \geq 0$ we have $W(t)Ax = e^{tA}\bar{B}^{-1}Ax = Ae^{tA}\bar{B}^{-1}x = AW(t)x$ and by Theorem 3.7 of [9], $(W(t))_{t\geq 0}$ is a C-regularized semigroup generated by an extension of A and since $\rho(A) \neq \emptyset$, by Proposition 3.9 of [9] we obtain $(W(t))_{t\geq 0}$ is generated by A. ■

This proposition suggests that C-evolution families are related to C^{-1}-bounded semigroups rather that C-bounded semigroups. The following theorem shows that the choice is quite limited.

Theorem 4.1. *Assume that $A : D(A) \to X$ is a closed operator with no eigenvalues in $]\omega, \infty[$ for some $\omega \in \mathbb{R}$, B^{-1} is densely defined and $X_B \hookrightarrow X$. Let $(W(t))_{t\geq 0}$ be a mild $\bar{B}^{-1}$-existence family for A. The formula*

$$Y(t)x = \bar{B}W(t)\bar{B}x \tag{4.3}$$

defines a $\bar{B}$-bounded semigroup if and only if $\bar{B}A\bar{B}^{-1}$ generates a C_0-semigroup in X and $(W(t))_{t\geq 0}$ is exponentially bounded. Then $\overline{BAB^{-1}} = \bar{B}A\bar{B}^{-1}$, $\bar{B}W(t) = e^{\bar{B}A\bar{B}^{-1}t}$, $t \geq 0$, and $(Y(t))_{t\geq 0}$ is generated by A.

Proof. If Eq. (4.3) defines a $\bar{B}$-bounded semigroup, then, since $\bar{B}D(\bar{B}) = X$,

$$\forall_{t\geq 0}\, W(t)X \subset D(\bar{B}). \tag{4.4}$$

Also, since by Theorem 2.1, $Y(t) = e^{tK}\bar{B}$ for some K acting in X, $(\bar{B}W(t))_{t\geq 0}$ is a semigroup in X. To find K we use the definition of existence families to obtain $\forall_{t\geq 0, x\in X} \int\limits_0^t W(s)xds \in D(A)$ and

$$W(t)x = \bar{B}^{-1}x + A\left(\int\limits_0^t W(s)xds\right). \tag{4.5}$$

By (4.4), all the terms above are in $D(\bar{B})$ and we have

$$\bar{B}W(t)x = x + \bar{B}A\left(\int\limits_0^t \bar{B}^{-1}\bar{B}W(s)xds\right). \tag{4.6}$$

Since $\bar{B}^{-1}$ is bounded, we have $\int\limits_0^t \bar{B}W(s)xds \in D(\bar{B}A\bar{B}^{-1})$ and

$$\bar{B}W(t)x = x + \bar{B}A\bar{B}^{-1}\left(\int\limits_0^t \bar{B}W(s)xds\right). \tag{4.7}$$

By Eq. (4.3), $t \mapsto \bar{B}W(t)x = Y(t)\bar{B}^{-1}x$ and since $\bar{B}^{-1}x \in D(\bar{B})$, this is a continuous function by Definition 2.1, property 2. Therefore $t \mapsto \bar{B}W(t)x$ is a mild solution of the Cauchy problem

$$\partial_t u = \bar{B}A\bar{B}^{-1}u, \qquad u(0) = x. \tag{4.8}$$

Moreover, by property 1 of Definition 2.1, $\|\bar{B}W(t)x\|_X = \|Y(t)\bar{B}^{-1}x\|_X \leq Me^{\omega t}\|x\|_X$, and hence the solutions to (4.8) are exponentially bounded. Since $\bar{B}^{-1}$ is a bounded operator, we obtain also $\|W(t)x\|_X = \|\bar{B}^{-1}\bar{B}W(t)x\| \leq M'e^{\omega t}\|x\|_X$, and hence $(W(tt))_{t\geq 0}$ is exponentially bounded.

By Proposition 2.9 of [9], all exponentially bounded mild solutions are unique. Since we have exponentially bounded mild solution for any $x \in X$, by Theorems 5.5 and 5.16 of [9], the operator $\bar{B}A\bar{B}^{-1}$ generates a C_0-semigroup on X.

Next we obtain $\forall_{t\geq 0, x\in X}\, \bar{B}W(t)x = e^{t\bar{B}A\bar{B}^{-1}}x$ and consequently

$$\forall_{t\geq 0, x\in D(\bar{B})} \quad Y(t)x = e^{t\bar{B}A\bar{B}^{-1}}\bar{B}x. \tag{4.9}$$

By the semigroup property $\forall_{t\geq 0, x\in D(\bar{B})}$ $Y(t)x = e^{t\bar{B}A\bar{B}^{-1}}\bar{B}x = \bar{B}x + \int_0^t e^{s\bar{B}A\bar{B}^{-1}}$ $\bar{B}A\bar{B}^{-1}\bar{B}x = \bar{B}x + \int_0^t Y(s)Axds$ and by the uniqueness of B-bounded semigroups [5, Theorem 1], $(Y(t))_{t\geq 0}$ is generated by A, and from Eq. (4.9) of [3] it follows that $\overline{BAB^{-1}} = \bar{B}A\bar{B}^{-1}$.

Conversely, if $\bar{B}A\bar{B}^{-1}$ generates a C_0-semigroup in X, then repeating the considerations above we obtain that A generates a $\bar{B}$-bounded semigroup $(Y(t))_{t\geq 0}$, and by Proposition 4.1, $W'(t) = \bar{B}^{-1}Y(t)\bar{B}^{-1}$ defines an exponentially bounded $\bar{B}^{-1}$-existence family for A. Since $(W(t))_{t\geq 0}$ is also an exponentially bounded mild $\bar{B}^{-1}$-existence family for A, $(W''(t))_{t\geq 0} = (W(t) - W'(t))_{t\geq 0}$ is also exponentially bounded. However, we have for any $x \in X$, $W''(t)x = A\left(\int_0^t W''(s)xds\right)$, that is, $t \mapsto W''(t)x$ is an exponentially bounded mild solution to the homogeneous problem (1.1). By Proposition 2.9 of [9], $W''(t)x \equiv 0$, and hence $(W(t))_{t\geq 0} = (W'(t))_{t\geq 0}$ and the formula (4.3) holds. ■

Remark 4.1. From the proof of the above theorem, it follows that the "only if" part can be proved under a weaker assumption that mild solutions of (1.1) in X are unique. Note that the fact that A generates a semigroup in $X_B = D(\bar{B})$ is not sufficient for that purpose as it gives only uniqueness in a smaller space $D(\bar{B})$.

Corollary 4.1. *Let the assumptions of the previous theorem be satisfied and let $(W(t))_{t\geq 0}$ be a $\bar{B}^{-1}$ regularized semigroup generated by A. The formula*

$$Y(t)x = \bar{B}W(t)\bar{B}x \tag{4.10}$$

defines a $\bar{B}$-bounded semigroup if and only if $(\bar{B}W(t))_{t\geq 0}$ is a semigroup in X generated by A.

Proof. Since a $\bar{B}^{-1}$-regularized semigroup generated by A is a mild $\bar{B}^{-1}$-existence family for A [9, Theorem 3.5], we obtain from Theorem 4.1 that $(\bar{B}W(t))_{t\geq 0}$ is a semigroup generated by $\bar{B}A\bar{B}^{-1}$ which, since $\bar{B}^{-1}A \subset A\bar{B}^{-1}$, is an extension of A. However, using the definition of the generator we obtain that for $x \in D(\bar{B}A\bar{B}^{-1})$,

$$\bar{B}A\bar{B}^{-1}x = \lim_{t\to 0^+} \frac{\bar{B}W(t)x - x}{t} = \lim_{t\to 0^+} \bar{B}\frac{W(t)x - \bar{B}^{-1}x}{t}.$$

Since $\bar{B}^{-1}$ is bounded, the existence of the left-hand side limit yields the existence of the limit of $t^{-1}(W(t)x - \bar{B}^{-1}x)$, as $t \to 0^+$, which is $A\bar{B}^{-1}x \in D(\bar{B}) = R(\bar{B}^{-1})$. Thus by the definition of the generator we have $x \in D(A)$ and $Ax = \bar{B}A\bar{B}^{-1}x$. Consequently, $A = \bar{B}A\bar{B}^{-1}$ and the semigroup $(\bar{B}W(t))_{t\geq 0}$ is generated by A.

The converse follows as in Theorem 4.1 with the sole difference that we use the uniqueness of solutions of Cauchy problem (1.1) ensured by Theorem 3.5 of [9], as noted in Remark 4.1. ■

In [9, Chapter 6] the author develops the theory of C-regularized semigroups in extrapolation spaces (obtained by completion of X with respect to the norm $\|C\cdot\|_X$ – compare our approach to B-bounded semigroups). This allows to develop a link between C-regularized semigroups and B-bounded semigroups with a different set of assumptions on B.

Theorem 4.2. *Let $B : X \to X$ be a bounded, injective operator, $A : D(A) \to X$ be a closed operator which generates a B-bounded semigroup $(Y(t))_{t\geq 0}$ and satisfies*

$$BA \subset AB. \tag{4.11}$$

Then the extension of A, given by $B^{-1}AB$, generates a B-regularized semigroup $(W(t))_{t\geq 0}$ on X which is given by

$$\forall_{x\in X, t\geq 0} \quad W(t)x = Y(t)x. \tag{4.12}$$

If $\rho(A) \neq \emptyset$, then $A = B^{-1}AB$.

Proof. We check the following points. $\mathfrak{B} : X_B \to X$ is a bounded extension of B which satisfies $\mathfrak{B}(X_B) = \overline{\mathrm{Im}B}^X \hookrightarrow X \hookrightarrow X_B$ (the last embedding follows from the construction of the completion and boundedness of B). Moreover, $\mathfrak{A}$ generates a strongly continuous semigroup on X_B. Since $\mathfrak{A}$ is the closure of A in X_B, any $\mathfrak{x} \in D(\mathfrak{A})$ is defined by $\mathfrak{B}\mathfrak{x} = \lim_{n\to\infty} Bx_n$, $x_n \in D(A)$, and $\mathfrak{B}\mathfrak{A}\mathfrak{x} = \lim_{n\to\infty} BAx_n = \lim_{n\to\infty} ABx_n$, where in the last equality we used (4.11). Since $(Bx_n)_{n\in\mathbb{N}}$ converges (in X) and from above $(ABx_n)_{n\in\mathbb{N}}$ also converges, by closedness of A we obtain $\mathfrak{B}\mathfrak{A}\mathfrak{x} = A\mathfrak{B}\mathfrak{x} = \mathfrak{A}\mathfrak{B}\mathfrak{x}$ as, by the definition, $\mathfrak{A}$ is the closure of A in X_B. Therefore, $e^{t\mathfrak{A}}\mathfrak{B} = \mathfrak{B}e^{t\mathfrak{A}}$. Thus all the assumptions of Proposition 6.4 of [9] are satisfied and $B^{-1}AB$ generates a C-regularized semigroup on X given by

$$\forall_{x\in X, t\geq 0} \quad W(t)x = \mathfrak{B}e^{t\mathfrak{A}}x,$$

which, by Theorem 2.1, yields Eq. (4.12).

Note, that condition (4.11) ensures only that $A \subset B^{-1}AB$ as there can be $x \in X \setminus D(A)$ satisfying $Bx \in D(A)$ and $ABx \in \mathrm{Im}B$.

The last statement follows from Proposition 3.9 of [9]. ■

The second set of comparison results stems from the formal similarity of Eq. (2.3) and the formula (2) of Definition 2.4 of [9] which suggest that a B-bounded semigroup could be a C-existence family with $C = B$ subject to additional conditions. The following theorem shows that again this is possible only for a very restricted class of operators.

Theorem 4.3. *Let us assume that* $A : X \to X$ *is a closed operator such that* $[\omega, \infty[$ *does not contain its eigenvalues*, $B : X \to X$ *is a bounded operator with the range* ImB *dense in* X *and* $(W(t))_{t\geq 0}$ *is a mild* B*-existence family for* A*. Then* $(W(t))_{t\geq 0}$ *is a* B*-bounded semigroup* $(Y(t))_{t\geq 0}$ *satisfying* (2.7), *generated by some operator* $\mathcal{D}$ *if and only if* A *generates a semigroup in* X*. In such a case*,

$$\forall_{t\geq 0, x\in X} \quad W(t)x = e^{tA}Bx = e^{t\mathfrak{B}\mathfrak{D}\mathfrak{B}^{-1}}Bx = Y(t)x, \tag{4.13}$$

where $\mathfrak{B}, \mathfrak{D}$ *are the closures of* B *and* $\mathcal{D}$, *respectively*, *in* X_B.

Proof. Let $t \mapsto u(t, Bx) = W(t)x$ be a mild solution to (1.1) and $(W(t))_{t\geq 0}$ is a B-bounded semigroup. From the property 1 of Definition 2.1 we have for any $x \in X$, $\|u(t, Bx)\| = \|Y(t)x\| \leq Me^{\omega t}\|Bx\| \leq M'e^{\omega t}\|x\|$. Hence we can use [9, Lemma 2.10] to get

$$\forall_{x\in X, \lambda>\omega} \quad (\lambda - A)\int\limits_0^\infty e^{-\lambda t}u(t, Bx)dt = Bx. \tag{4.14}$$

On the other hand, from the original version of (3.6) (see (9) of [5]) we obtain

$$\forall_{x\in X, \lambda>\omega} \quad \int\limits_0^\infty e^{-\lambda t}Y(t)xdt = B(\lambda - \mathcal{D})^{-1}x. \tag{4.15}$$

Combining (4.14) and (4.15), we have

$$\forall_{x\in X, \lambda>\omega} \quad (\lambda - A)^{-1}Bx = B(\lambda - \mathcal{D})^{-1}x. \tag{4.16}$$

From Lemma 3.1 of [3], we know that $\lambda I - \mathcal{D}$ reduces cosets $X/N(B)$; therefore Eq. (4.16) can be written for $\mathfrak{B}$, restricted to X (where $\mathfrak{B}$ is the extension by density of B to the completion X_B of X with respect to the seminorm $\|B \cdot\|$; see Section 2):

$$\forall_{x\in X, \lambda>\omega} \quad (\lambda - A)^{-1}\mathfrak{B}x = \mathfrak{B}(\lambda - \mathcal{D})^{-1}x. \tag{4.17}$$

Let $x \in X_B$ be such that $\mathfrak{B}x = z \in \text{Im}\,B$; then from (4.17) we have for all $z \in \text{Im}\,B, \lambda > \omega$,

(4.18)
$$\|(\lambda - A)^{-1}z\|_X = \|\mathfrak{B}(\lambda - \mathcal{D})^{-1}\mathfrak{B}^{-1}z\|_X \leq \frac{M}{\lambda - \omega}\|\mathfrak{B}^{-1}z\|_{X_B} \leq \frac{M}{\lambda - \omega}\|z\|_X.$$

Since A is closed, $(\lambda - A)^{-1}$ is also closed and, being defined on a dense subspace Im $B \subset X$ and bounded, it is defined on the whole X. Therefore $(\lambda - A)^{-1}$ is the resolvent of A. Furthermore,

$$(\lambda - A)^{-2}z = (\lambda - A)^{-1}\left((\lambda - A)^{-1}z\right) = (\lambda - A)^{-1}\left(\mathfrak{B}(\lambda - \mathcal{D})^{-1}\mathfrak{B}^{-1}z\right)$$
$$= \mathfrak{B}(\lambda - \mathcal{D})^{-2}\mathfrak{B}^{-1}z$$

and, using (4.18) and (4.15),

$$\|(\lambda - A)^{-2}z\|_X \leq \frac{M}{(\lambda - \omega)^2}\|z\|_X. \tag{4.19}$$

By induction we see that A satisfies the Hille-Yosida estimates in X.

Since $(Y(t))_{t\geq 0}$ satisfies (2.7), $B(D(\mathcal{D}))$ is dense in X, and hence by Eq. (4.16) $D(A) = (\lambda I - A)^{-1}X \supset (\lambda I - A)^{-1}(\text{Im } B)$ is dense in X. Therefore A generates a semigroup in X and from (4.17) we obtain that

$$A = \mathfrak{B}\mathcal{D}\mathfrak{B}^{-1}. \tag{4.20}$$

Thus, by Theorem 2.1, the B-existence family is given by

$$W(t)x = e^{tA}Bx = Y(t)x = e^{t\mathfrak{B}\mathfrak{D}\mathfrak{B}^{-1}}Bx.$$

Conversely, assume that A generates a semigroup in X define by (4.20)

$$\forall_{\mathfrak{x}\in\mathfrak{B}^{-1}(D(A))} \quad \mathcal{D}\mathfrak{x} = \mathfrak{B}^{-1}A\mathfrak{B}\mathfrak{x}. \tag{4.21}$$

Since $D(A)$ is dense in X and $\mathfrak{B}$ is an isomorphism, $\mathfrak{B}^{-1}(D(A))$ is dense in X_B. Next, we obtain for any $x \in X$, $(\lambda I - A)^{-1}x = \mathfrak{B}(\lambda I - \mathcal{D})^{-1}\mathfrak{B}^{-1}x$. Therefore for any $\lambda > \omega$, $(\lambda - \omega)^{-1}M\|x\| \geq \|(\lambda I - A)^{-1}x\| = \|\mathfrak{B}(\lambda I - \mathcal{D})^{-1}\mathfrak{B}^{-1}x\|$, which is the same as $\|(\lambda I - \mathcal{D})^{-1}\mathfrak{x}\|_{X_B} \leq (\lambda - \omega)^{-1}M\|\mathfrak{x}\|_{X_B}$ for all $\mathfrak{x} \in X_B$. By induction we obtain all the Hille-Yosida estimates, and thus $\mathcal{D}$ generates a B-bounded semigroup in X. ■

Corollary 4.2. *If, in the statement of the previous theorem, $(Y(t))_{t\geq 0}$ is generated by an extension of A, then $(W(t))_{t\geq 0}$ is a B-regularized semigroup generated by A.*

Proof. By the closedness of A, the equation

$$(\lambda I - A)^{-1}z = \mathfrak{B}(\lambda I - \mathcal{D})^{-1}\mathfrak{B}^{-1}z \tag{4.22}$$

originally defined for $z \in \text{Im}B$ is valid (with the same operators) on the whole X. Therefore, $D(A) = \mathfrak{B}(D(\mathcal{D}))$. Moreover, $\mathcal{D}$ is an extension of A, that is,

$D(A) \subset D(\mathcal{D})$. Consequently, if $x \in D(A)$, then $\mathcal{D}x = Ax \in X$ and $\mathfrak{B}\mathcal{D}x = BAx$. Also, if $x \in D(A)$, then $Bx = \mathfrak{B}x \in D(A)$. Eq. (4.20) can be written as $\forall_{x\in\mathfrak{B}^{-1}(D(A))}\, A\mathfrak{B}x = \mathfrak{B}\mathcal{D}x$, which, by the considerations above, is equivalent to $\forall_{x\in D(A)}\, ABx = BAx$. Hence,

$$\forall_{x\in D(A)}\, W(t)Ax = e^{tA}BAx = Ae^{tA}Bx$$

and by Theorem 3.7 of [9], $(W(t))_{t\geq 0}$ is a B semigroup generated by an extension of A. However, since A itself is a generator, by Proposition 3.9 of *op. cit.*, $(W(t))_{t\geq 0}$must be generated by A. ■

These results allow to prove an interesting observation pertaining to B-bounded semigroups.

Corollary 4.3. *If $B : X \to X$ is a bounded, one-to-one operator satisfying $BA \subset AB$, and A generates a B-bounded semigroup $(Y(t))_{t\geq 0}$, then the extension of A, $B^{-1}AB$, generates a semigroup in X. If $\rho(A) \neq \emptyset$, then A generates a semigroup in X.*

Proof. By Theorem 4.2, there is a B-regularized semigroup $(W(t))_{t\geq 0}$ generated by an extension $B^{-1}AB$ of A, such that $\forall_{x\in X, t\geq 0}\, W(t)x = Y(t)x$. If $\rho(A) \neq \emptyset$, then $B^{-1}AB = A$ by Proposition 3.9 of [9]. By Theorem 3.5 of *op. cit.*, this B-regularized semigroup is a mild existence family for $B^{-1}AB$, or A, respectively. From Theorem 4.3 it follows then that $B^{-1}AB$ (or, resp. A) generates a semigroup in X. ■

Theorem 4.4. *Let us assume that $B : X \to X$ is a bounded one-to-one operator, $\overline{\mathrm{Im}B} = X$, and let $(W(t))_{t\geq}$ be a B-regularized semigroup in X generated by A. $(W(t))_{t\geq 0}$ is a B-bounded semigroup if and only if the semigroup $(B^{-1}W(t))_{t\geq 0}$ extends to a C_0-semigroup on X, generated by A.*

Proof. By Theorem 3.5 of [9], $(W(t))_{t\geq 0}$ is a strong B-existence family for A. If it is a B-bounded semigroup, then by Theorem 4.3, A generates a semigroup $\exp(tA))_{t\geq 0}$ such that $\forall_{x\in X, t\geq 0}\, W(t)x = e^{tA}Bx$. By Theorem 3.1 of *op. cit.* and the definition of B-regularized semigroup we have

$$BA \subset AB \tag{4.23}$$

and since B is bounded, from the exponential formula for e^{tA} we obtain $Be^{tA}x = e^{tA}Bx$ for any $x \in X$. Thus $W(t)x \subset \mathrm{Im}B$ for any $x \in X, t \geq 0$ and $B^{-1}W(t)x = e^{tA}x$ for any $x \in X, t \geq 0$. By (4.23) we have $B(\lambda I - A)^{-1} = (\lambda I - A)^{-1}B$. Indeed, if $y = B(\lambda I - A)^{-1}x$, $x \in X$, then $B^{-1}y \in D(A)$ and

$x = \lambda B^{-1}y - AB^{-1}y$. Eq. (4.23) is equivalent to saying that $B^{-1}Ay = AB^{-1}y$ whenever $B^{-1}y \in D(A)$ (and then clearly $y \in D(A)$). This is exactly the condition on y we have above, and thus $y = (\lambda I - A)^{-1}Bx$. Thanks to this, (4.16) can be written as

$$\forall_{x\in X,\lambda>\omega} \quad B(\lambda - A)^{-1}x = B(\lambda - \mathcal{D})^{-1}x$$

and from the invertibility of B we obtain $\mathcal{D} = A$. Applying Corollary 4.2 ends the proof of this part.

Conversely, if $(B^{-1}W(t))_{t\geq 0}$ is a C_0-semigroup generated by A, then we define

$$Y(t)x = e^{tA}Bx = Be^{tA}x = W(t)x.$$

Clearly, $\|Y(t)x\| \leq Me^{\omega}\|Bx\|_X$ and $t \mapsto Y(t)x$ is continuous for any $x \in X$. From the semigroup properties and (4.23), for any $x \in D(A)$, $Bx \in D(A)$ and

$$Y(t) = Bx + \int_0^t Ae^{sA}Bxds = Bx + \int_0^t e^{sA}BAxds,$$

which shows that $(Y(t))_{g\geq 0}$ is the B-bounded semigroup generated by A. ■

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