

REMOTAL SETS REVISITED

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Abstract. Farthest point theory is not so rich and developed as nearest point theory, which has more applications. Farthest points are useful in studying the extremal structure of sets; see, e.g., the survey paper [14]. There are some interactions between the two theories; in particular, uniquely remotal sets in Hilbert spaces are related to the old open problem concerning the convexity of Chebyshev sets.

The aim of this paper is twofold: first, we indicate characterizations of inner product spaces and of infinite-dimensional Banach spaces, in terms of remotal points and uniquely remotal sets. Second, we try to update the survey paper [15], concerning uniquely remotal sets.

1. INTRODUCTION AND DEFINITIONS

Let $(X, \|\cdot\|)$ be a real Banach space; set, for $x \in X$ and $r \geq 0$: $B(x, r) = \{y \in X; \|y - x\| \leq r\}$; $B(X) = B(\Theta, 1)$; $S(X) = \{x \in X; \|x\| = 1\}$. Given a (nonempty) bounded set $A \subset X$, denote by $\delta(A)$ its diameter; throughout the paper, we shall always understand that A is nonempty and bounded. For $x \in X$, set $r(A, x) = \sup\{\|x - a\|; a \in A\}$; $F(A, x) = \{a \in A; \|x - a\| = r(A, x)\}$. The points in $F(A, x)$ (if they exist) are called *farthest points* to x from A . We say that A is *remotal* (*uniquely remotal*) if for every $x \in X$, $F(A, x) \neq \emptyset$ (respectively, $F(A, x)$ is a singleton). Given A , set $F_A : x \mapsto F(A, x)$; this multivalued map, whose codomain is the set of remotal points, is the *farthest point map*. We also set

$$r(A) = \inf\{r(A, x); x \in X\} \text{ (radius of } A\text{)}.$$

A *center* of A is a point c (if it exists) such that $r(A, c) = r(A)$.

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Notwithstanding many partial results, no general solution has been given to the following problem:

(Pb) Must a uniquely remotal set, in a Banach space X , be a singleton?

In particular, note that many results concerning a positive solution to (Pb) are known (see [15, §3]); some other results, concerning product spaces, have been given in [9] for ℓ_1 products, and in [12] for ℓ_∞ products.

In Section 2, we shall indicate results concerning farthest points in spaces with an inner product; we show that a characterization of centers, well-known in inner product spaces, in fact characterizes these spaces. In Section 3, we characterize infinite-dimensional Banach spaces in terms of farthest points. In Section 4, we indicate some facts concerning the continuity of the farthest point map.

As a starting point, we use the survey article [15] concerning uniquely remotal sets; we refer it for older results, while we try to update that paper by indicating here relevant further results in the area.

2. EUCLIDEAN AND HILBERT SPACES

In this section, we assume that the norm of X is derived from an inner product. The following result is well-known:

Theorem 2.1. *In a Hilbert space X , let c be the (unique) center of the set A . Then we have:*

$$(p) \quad r^2(A) + \|x - c\|^2 \leq r^2(A, x) \text{ for each } x \in X.$$

This result was first indicated only for uniquely remotal sets (see [4, Proposition 1]). In [16, Theorem 5], it was proved by only assuming that A is remotal, and it was noticed that (p) is also sufficient that c be a center; in fact, it is clear that, in any space X , (p) implies that c is the unique center of A . A general proof of Theorem 2.1 appears in [19, Lemma 3.1], then again in [5, Theorem 1]; in the last paper an example is given, in a two-dimensional smooth (non-strictly convex) space, showing that (p) is not a necessary condition for centers in non-Euclidean spaces. Finally, note that in [15] and in [5] the inequality (p) is considered only for points $x \in B(c, r(A))$ (which is not a true restriction).

In fact, the result says that the characterization of centers by (p) is only valid in Hilbert spaces, thus answering a question raised in [4, p. 1316], then also in [18, p. 55].

Theorem 2.2. *Let X be a normed space; then X is an inner product space if and only if for any nonempty, bounded set A with a center c , (p) is true.*

Proof. The “only if” part is known. We prove the “if” part. We assume that X is not an inner product space and we construct a compact set A with a center c , which does not satisfy (p).

According to [1, (6.9'')], under our assumption on X , there exists a pair x, y such that $\|x + y\| = \|x - y\|$ and $\|x + y\|^2 + \|x - y\|^2 < 2(\|x\|^2 + \|y\|^2)$; by scaling, and eventually exchanging x and y , we can assume $\|x\| \leq 1 = \|y\|$. Now let $A = \{-y, y, x, \Theta\}$. We have $r(A) = 1$, Θ is a center of A and y is farthest to x from A . But

$$r^2(A) + \|x - \Theta\|^2 = \|x\|^2 + 1 > \frac{1}{2}(\|x + y\|^2 + \|x - y\|^2) = \|x - y\|^2 = r^2(A, x),$$

so (p) is not satisfied. This proves the theorem. \blacksquare

We refer to [2, p. 365] for a simple example of a bounded, closed convex set A in a Hilbert space, such that for its center c we have $A \cap B(c, r(A)) = \emptyset$.

The following problem was studied in [10]: Given a compact, convex subset of a Euclidean space E^n , construct a minimal uniquely remotal set containing A .

Monotonicity properties of the map F_A in Hilbert spaces were studied in [21, §6]. In [16, Proposition 3], it was proved that in inner product spaces, F_A is pseudocontractive if A is uniquely remotal.

3. CHARACTERIZATIONS OF INFINITE-DIMENSIONAL BANACH SPACES

In finite-dimensional Banach spaces, a simple compactness argument shows that bounded closed sets are remotal. In infinite-dimensional spaces (even if they have some nice properties), this is not true.

We recall that some general properties of X can be characterized in terms of farthest points; for example, it is easy to see that X is strictly convex if and only if $B(X)$ is uniquely remotal with respect to $X - \{\Theta\}$ (see [3], where also other geometrical properties of the norm are characterized by using uniquely remotal sets).

In some classes of spaces, it is known that “most” points (e.g., in the sense of categories) admit farthest points (see, e.g., [11, §3]); but there are also spaces where bounded closed convex bodies A exist such that $F(A, x) = \emptyset$ for every $x \in X$ (in that case A is said to be *antiremotal*). For antiremotal sets, see also [6].

As proved in [20], every infinite-dimensional Banach space contains a convex body A such that $\Theta \in \text{int}(A)$ and $p_A(x - y) < \sup\{p_A(\xi - \eta); \xi, \eta \in A\}$ for every $x, y \in A$, p_A being the Minkowski functional of A .

Examples of sets lacking some good properties concerning remotality are scattered in the literature. The next result (which generalizes [7, Theorem 1] shows that some of them have indeed a general character,

Theorem 3.1. *The following properties are equivalent:*

- (a) $\dim(X) < \infty$;
 (b) if A is a bounded closed set with a center c such that $F(A, c)$ is a singleton, then A must be a singleton.

Proof. (b) \Rightarrow (a): We shall prove that not (a) \Rightarrow not (b).

By assuming $\dim(X) = \infty$, we construct a closed set A , not a singleton, as in (b). Since $S(X)$ is not compact, there exists a sequence $\{x_n\} \subset S(X)$ such that $\|x_i - x_j\| \geq 1/2$ for $i \neq j$. Set $A = \{x_1\} \cup \{\pm(1 - (1/n))x_n; n \geq 2\}$; this set is bounded and closed (it is a discrete set). We have $r(A) = 1$, Θ is a center of A , and $F(A, \Theta) = \{x_1\}$.

(a) \Rightarrow (b) (this generalizes Theorems 3.2 and 3.31 in [15]). Assume that $\dim(X) < \infty$. Let the set A have a center c such that $F(A, c)$ is a singleton; without loss of generality, we can assume that $c = \Theta$. Moreover, if A is not a singleton, we can also assume $r(A) = 1$, and then, if $F(A, \Theta) = \{q\}$, we have $\|\Theta - q\| = 1$. The set $A_\varepsilon = \{a \in A; 1 - \varepsilon \leq \|a\| \leq 1\}$ is closed and bounded, so it is compact; moreover (see, e.g., [8, Lemma 4.6]), $r(A_\varepsilon) = 1$. Now set $G_\varepsilon = A_\varepsilon \cap \{a \in A; \|a - q\| \geq 1/2\}$. Again, G_ε is bounded and nonempty for any $\varepsilon > 0$ (otherwise, $A_\varepsilon \subset \{a \in A; \|a - q\| < 1/2\}$ would imply $r(A_\varepsilon) \leq 1/2$). For any $n \in W$, we take $g_n \in G_{1/n} \subset A_{1/n}$ so that $\|g_n\| \in [1 - \varepsilon, 1]$ and $\|g_n - q\| \geq 1/2$. Since $\{g_n\}$ is bounded, we can find a convergent subsequence $\{g_{n_k}\}$, say, $g_{n_k} \rightarrow g_0 \in \bigcap_{n=1}^{\infty} G_{1/n} \subset A$. Then $\|g_0\| = 1$ and $\|g_0 - q\| \geq 1/2$. This implies that $g_0 \neq q$, and then $F(A, \Theta)$ would not be a singleton. ■

Remark 3.2. In other terms, our proof that (b) \Leftrightarrow (a) in Theorem 3.1 shows that $\dim(X) = \infty$ if and only if given a bounded, closed set A which is not a singleton, if c is a center of A , then $F(A, c)$ is either empty, or it contains at least two different points. Moreover (see [13, Theorem 2.2]), (a) is also equivalent to:

- (c) every bounded, closed set A is remotal.

Since A is remotal if and only if $co(A)$ is remotal, we can also indicate the following equivalent condition:

- (c') every bounded closed and convex set A is remotal.

This shows that if X is reflexive (and $\dim(X) = \infty$), then there exists in X a weakly compact, nonremotal set; but some reflexive spaces contain also antiremotal sets (see [6]; compare with [14, p. 62]).

4. THE ROLE OF CENTERS VERSUS UNIQUE REMOTENESS

Continuity and differentiability properties of the map F_A have been considered with respect to its single-valuedness; see, e.g., [11, Theorem 3.1].

Note that $y \in F(A, x)$ always implies $y \in F(A, x + t(y - x))$ for all $t < 0$. Moreover, when X is strictly convex, under the same assumption we can see that for

$t < 0$, $F(A, x + t(y - x)) = \{y\}$, and so $r(A, x + t(y - x)) = r(A, x) + |t| \cdot \|y - x\|$. For another result in strictly convex spaces, see [21, Proposition 3.1].

Several results concerning a (positive) solution to (Pb), in terms of continuity of F_A , have been proved; see, e.g., [15, Theorems 3.12 and 3.14]. In fact, sectional continuity of F_A suffices, and for remotal sets such continuity can be limited to single points (see [17, Corollary 3]). Moreover, concerning uniquely remotal sets with a center c , continuity at c plays a key role, and it is important to look at what happens along $[c, F_A(c)]$. In general (see [17, Theorem 1]), continuity of F_A at a point x can be studied considering only its restriction to the set

$$E_x = \{y \in X; r(A, y) \geq r(A, x)\}.$$

If we assume that $\{y_1, y_2\} \subset F(A, x)$ with $y_1 \neq y_2$, then F cannot be continuous at x (take sequences $x + (1/n)(x - y_1)$, $x + (1/n)(x - y_2)$ for n large).

The following conditions have been used in this context. If x is a remotal point of a set A , not a singleton, let:

$\mathcal{E}_x : u \in [x, F_A(x)]$ implies $u \in E_x$ (see [15, p.6]),

and say that $P(x, d)$ is true for some d , $0 < d < 1$, if $y \in F(A, x)$ implies

$$y \in F(A, x + t(y - x)) \text{ for } 0 < t \leq d \text{ (see [7]).}$$

Note that \mathcal{E}_x and $P(x, d)$ are mutually exclusive, and $P(x, d)$ implies sectional continuity of F_A at x along the ray containing $[x, F_A(x)]$. The condition \mathcal{E}_x implies that $r(A, x)$ has no directional derivative. According to the results in [21, p. 86], under some geometric assumptions on X (for example, X is a Hilbert space), F_A must be discontinuous at x . Moreover, according to [15, Theorem 3.25 (iii)], in Hilbert spaces uniquely remotal sets which are not singletons present a strong discontinuity at centers.

It is known (see [15, Theorem 3.14]) that a uniquely remotal set A must be a singleton if F_A is continuous at a center c of A . More generally, the same conclusion holds under the assumption that F_A is continuous at a point x satisfying \mathcal{E}_x .

Here are some remarks concerning $P(x, d)$:

- $P(c, d)$ cannot be true if c is a center, $F(A, c) = \emptyset$, A is not a singleton and $d > 0$ (we would obtain $r(A, c + d(y - c)) = r(A, c) - d\|y - c\|$ for $y \in F(A, c)$). In other words, if $P(x, d)$ holds for some x and some $d > 0$, then x is not a center of A .

- If X is strictly convex, then $P(x, d)$ cannot be true if $F(A, x)$ contains at least two elements y_1, y_2 (set $z = x + d(y_1 - x)$, $0 < d < 1$; $y_1 \in F(A, z)$ implies that y_1 is the unique farthest point to $x = z + d(x - y_1) = z + (d/(1 - d))(z - y_1)$ since $d/(1 - d) > 1$; thus $y_2 \notin F(A, z)$).

Note that in the above statement, instead of assuming $P(x, d)$ we could assume that $d > 0$ depends on the point $y \in F(A, x)$.

To conclude, we indicate the following result, which is a slightly stronger version of Theorem 9 in [7]:

Proposition 4.1. *Let A be a remotal set such that for some $\varepsilon > 0$, there exists $d > 0$ for which $P(y, d)$ is satisfied for all $y \in X$ satisfying $r(A, y) < r(A) + \varepsilon$. Then A must be a singleton.*

Proof. Under our assumptions, let $r(A) > 0$ and take c_d such that $r(A, c_d) < r(A) \cdot (1 + d)$. If $q_d \in F(A, c_d)$, take $c' = c_d + d(q_d - c_d)$. For all $x \in A$, we have $\|c' - x\| \leq \|c' - q_d\| = \|c_d - q_d\|d \cdot r(A, c_d) < r(A) + d \cdot r(A) - d \cdot r(A)$. This absurdity proves the result. ■

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