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A CONE-THEORETIC APPROACH TO THE SPECTRAL THEORY OF POSITIVE LINEAR OPERATORS: THE FINITE-DIMENSIONAL CASE

Bit-Shun Tam

Abstract. This is a review of a coherent body of knowledge, which perhaps deserves the name of the geometric spectral theory of positive linear operators (in finite dimensions), developed by this author and his co-author Hans Schneider (or S.F. Wu) over the past decade. The following topics are covered, besides others: combinatorial spectral theory of nonnegative matrices, Collatz-Wielandt sets (or numbers) associated with a cone-preserving map, distinguished eigenvalues, cone-solvability theorems, the peripheral spectrum and the core, the invariant faces, the spectral pairs, and an extension of the Rothblum Index Theorem. Some new insights, alternative proofs, extensions or applications of known results are given. Several new results are proved or announced, and some open problems are also mentioned.

CONTENTS

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1. Introduction

This paper is a considerably expanded version of an invited talk delievered by the author at the International Conference on Mathematical Analysis and Its Applications (ICMAA 2000) held at National Sun Yat-sen University in Taiwan in January 2000.

Here we are interested in linear operators or matrices that leave invariant a proper cone. By a *proper cone* we mean a nonempty subset K in a finite-dimensional real vector space V, which is a convex cone (i.e., $x,\ y \in K,\ \alpha,\ \beta \geq 0$ imply $\alpha x + \beta y \in K$), is pointed (i.e., $K \bigcap (-K) = \{0\}$), closed (with respect to the usual topology of V) and full (i.e., int $K \neq \emptyset$). A typical example is $\mathbb{R}^n_+$, the nonnegative orthant of $\mathbb{R}^n$. We denote by $\pi(K)$ the set of all linear operators (or matrices) A that satisfy $AK \subseteq K$. We refer to elements of $\pi(K)$ as *cone-preserving maps*. (Functional analysts usually call them positive or nonnegative linear operators.) Hereafter, unless stated otherwise, we always use K to denote a proper cone in $\mathbb{R}^n$.

1.1. Perron-Frobenius Theorems

Probably many of you have heard of the Perron-Frobenius theory of a nonnegative matrix discovered at the turn of the twentieth century. Let us recall the results.

In 1907, Perron [87, 88] gave proofs of the following famous theorem, which now bears his name, on positive matrices:

Theorem 1.1. (Perron Theorem). *Let A be a square positive matrix. Then $\rho(A)$ is a simple eigenvalue of A and there is a corresponding positive eigenvector. Furthermore, $\rho(A) > |\lambda|$ for all $\lambda \in \sigma(A),\ \lambda \neq \rho(A)$.*

Here we denote by $\sigma(A)$ the *spectrum* (the set of all eigenvalues) of a (square) matrix A, and by $\rho(A)$ the *spectral radius* of A, i.e., the quantity $\max\{|\lambda| : \lambda \in \sigma(A)\}$. By a *positive* (respectively, *nonnegative*) matrix we mean a real matrix each of whose entries is a positive (respectively, nonnegative) number. If A is a nonnegative (respectively, positive) matrix, we shall denote it by $A \geq 0$ (respectively, $A > 0$). We also call a nonnegative matrix A *primitive* if there exists a positive integer p such that $A^p > 0$.

In the years 1908–09, Frobenius [46, 47] also gave proofs of Perron's theorem. Then in 1912, Frobenius [48] extended the theorem to the class of irreducible nonnegative matrices.

An $n \times n$ matrix $A = (a_{ij})$ is said to be *reducible* if $n \geq 2$ and there exists a proper nonempty subset α of $\langle n \rangle := \{1, \dots, n\}$ such that $a_{ij} = 0$ for all $i \in \alpha$ and $j \in \langle n \rangle \backslash \alpha$; or, equivalently, if $n \geq 2$ and there exists a permutation matrix P such that

$$P^T AP = \begin{bmatrix} A_{11} & A_{12} \\ \mathbf{0} & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square matrices of order at least one. If A is not reducible, then it is said to be *irreducible*.

Theorem 1.2 (Frobenius Theorem). *Let $A \geq 0$ be irreducible. Then*

(i) *$\rho(A)$ is simple eigenvalue of A, and there is a corresponding positive eigenvector.*

(ii) *If A has h eigenvalues of modulus $\rho(A)$, then they are $\rho(A)e^{2\pi ti/h}$, $t = 0, \dots, h-1$. Moreover, the spectrum of A is invariant under a rotation about the origin of the complex plane by $2\pi/h$, i.e., $e^{2\pi i/h}\sigma(A) = \sigma(A)$. If $h > 1$, then there exists a permutation matrix P such that*

$$P^T AP = \begin{bmatrix} \mathbf{0} & A_{12} & & & \\ & \mathbf{0} & A_{23} & & \\ & & \mathbf{0} & \ddots & \\ & & & \ddots & A_{h-1,h} \\ A_{h1} & & & & \mathbf{0} \end{bmatrix},$$

where the zero blocks along the diagonal are square.

By a limiting argument, we readily obtain the Perron-Frobenius theorem of a nonnegative matrix; namely, that if A is a nonnegative matrix, then $\rho(A)$ is an eigenvalue of A and there is a corresponding nonnegative eigenvector.

The Perron-Frobenius theory of a nonnegative matrix has been one of the most active research areas in matrix analysis. Schneider [101, Section 2.5] has once referred to the theory as a branch point of (inward) matrix theory from abstract algebra. The theory has found numerous applications in mathematical, physical and social sciences. Nowadays, almost every textbook of matrix theory contains a chapter on nonnegative matrices (for instance, Gantmacher [51], Varga [138], Horn and Johnson [62], Lancaster and Tismenetsky [73], among others), and there are also monographs specially devoted to nonnegative matrices and their applications, such as Seneta [108], Berman and Plemmons [17], Minc [79], Berman, Neumann and Stern [16], and Bapat and Raghavan [4]. Recently, the Perron-Frobenius theory of nonnegative matrices has also found applications in the study of the Hausdorff dimension of some fractals (see Drobot and Turner [31], and Takeo [114]), and in the classification of surface homeomorphisms (see Bauer [14]).

1.2. Infinite-dimensional Results

A natural extension of the concept of a nonnegative matrix is that of an integral operator with a nonnegative kernel. The following extension of Perron's theorem is due to Jentzsch [71].

Theorem 1.3 (Jentzsch, 1912). *Let $k(\cdot,\cdot)$ be a continuous real function on the unit square with $k(s,t) > 0$ for all $0 \le s,t \le 1$. If $K : L^2[0,1] \to L^2[0,1]$ denotes the integral operator with kernel k defined by setting*

$$(Kf)(s) = \int_{[0,1]} k(s,t)f(t)dt, \;\; f \in L^2[0,1],$$

then

(i) *K has positive spectral radius*;

(ii) *the spectral radius $\rho(K)$ is a simple eigenvalue*, *with* (*strictly*) *positive eigenvector*;

(iii) *if $\lambda \ne \rho(K)$ is any other eigenvalue of K, then $|\lambda| < \rho(K)$.*

In 1948, in an abstract order-theoretic setting, in the important memoir [72] Krein and Rutman extended the theory to a compact linear operator leaving invariant a convex cone in a Banach space. They obtained the following:

Theorem 1.4 (Krein and Rutman). *Let A be a compact linear operator on a Banach space X. Suppose that $AC \subseteq C$, where C is a closed generating cone in X. If $\rho(A) > 0$, then there exists a nonzero vector $x \in C$ such that $Ax = \rho(A)x$.*

Since the advent of the above-mentioned paper by Krein and Rutman, the Perron-Frobenius theory has been extended to positive operators in various infinite-dimensional settings, including Riesz spaces (i.e., lattice ordered spaces), Banach spaces, Banach lattices, ordered topological vector spaces, ordered locally convex algebras, and C^*-algebras, etc. In case the underlying space is a Banach lattice, the theory is particularly rich. For references, we refer our reader to the expository papers by Dodds [30], Schaefer [95], Zerner [145], and also the books by Aliprantis and Burkinshaw [2], Meyer-Nieberg [78], Schaefer [96; 98, Appendix], Schwarz [107], and Zaanen [143], etc.

1.3. Cone-preserving Maps in Finite Dimensions

Early works on generalizations of the Perron-Frobenius theory were mainly done in the infinite-dimensional settings. In 1967, Birkhoff [19] gave an elementary proof of the Perron-Frobenius theorem for a cone-preserving map (in finite dimensions), using the Jordan canonical form. His paper attracted the attention of numerical analysts. Vandergraft [136] investigated the problem more closely. He considered also only finite-dimensional spaces, in order to obtain stronger results, with in mind applications to convergence theorems and comparison results (see also Marek [75], Rheinboldt and Vandergraft [91], and Vandergraft [137]). Vandergraft (and independently Elsner [33]) solved completely the problem of when an $n \times n$ real matrix leaves invariant some proper cone in the underlying space. (We shall return to this problem in a later part of this paper.) A nice account of Vandergraft's work has been summarized in the book by Berman and Plemmons [17, Chapter 1].

Based on the folllowing concept of a face, due to Hans Schneider, the concept of irreducibility was extended to cone-preserving maps:

A nonempty subset F of a proper cone K is a *face* of K if F is a subcone of K and in addition has the property that $y \geq^K x \geq^K 0$ and $y \in F$ imply $x \in F$. Here we use $\geq^K$ to denote the partial ordering on $\mathbb{R}^n$ induced by K, i.e. $y \geq^K x$ if and only if $y - x \in K$.

A matrix $A \in \pi(K)$ is said to be *K-irreducible* if A leaves invariant no nontrivial face of K. (The *trivial* faces of K are $\{0\}$ and K. Other faces of K are said to be *nontrivial*.) It is not difficult to show that an $n \times n$ nonnegative matrix is irreducible (in the sense we have already defined for a general square matrix) if and only if it is $\mathbb{R}^n_+$-irreducible.

The following equivalent conditions for K-irreducibility were obtained by Schneider and Vidyasagar [106]:

Theorem 1.5. *Let $A \in \pi(K)$. The following conditions are equivalent* :

(a) *A is K-irreducible.*

(b) *A has no eigenvector in the boundary of K.*

(c) $(I_n + A)^{n-1}(K\backslash\{0\}) \subseteq \text{int } K$.

Around that time, the concepts of positive and primitive matrices were also extended to cone-preserving maps: a matrix A is said to be *K-positive* if $A(K\backslash\{0\}) \subseteq$

$\mathrm{int}\ K$. In [5], Barker gave a definition for K-primitivity, which, he showed, is equivalent to the condition that A^p is K-positive for some positive integer p. He also obtained another equivalent condition for K-primitivity which is analogous to the definition of K-irreducibility; namely, A is K-primitive if and only if A leaves no subset of ∂K other than $\{0\}$ invariant.

In [106], Schneider and Vidyasagar proved that if K is a proper cone in $\mathbb{R}^n$, then $\pi(K)$ is a proper cone in the space of all $n \times n$ real matrices. Barker [5, Proposition 1] also proved the interesting fact that the interior of the cone $\pi(K)$ is composed of precisely all the K-positive matrices. Around that time, the geometry of the cone $\pi(K)$ (or, more generally, $\pi(K_1, K_2) := \{A : AK_1 \subseteq K_2\}$) has attracted the attention of many people, and in particular the determination of its extreme operators was a focus of interest. (See the review paper by Barker [7], and also the research-expository paper by Tam [125]. At the end of the latter paper, several open problems are posed.) A bit later, the algebraic structure of $\pi(K)$ as a semiring (under matrix addition and multiplication) was also investigated by a number of people (see Horne [63], Barker [6, Section 4], Tam [116], and the more recent Barker and Tam [10]).

In the above, we have mentioned a number of papers published in the '60–'70s, devoted to the study of the spectral theory of a cone-preserving map (in finite dimensions); namely, Birkhoff [19], Vandergraft [136, 137], Marek [75], Rheinboldt and Vandergraft [91], Elsner [33], and Barker [5]. Besides, there are also the papers by Barker and Turner [13] and by Barker and Schneider [8], which we shall come across later in this paper. Furthermore, we would also like to mention, in particular, the paper [90] by Pullman in 1971, in which he offered a geometric approach to the theory of nonnegative matrices via a study of its core. His paper has motivated our work in the '90s. Anyway, around the mid-'80s, people generally thought that the subject of positive linear operators (in finite or infinite dimensions) was more or less complete. Let us quote from the review papers by Barker and Dodds, respectively.

In 1981, in [7, p. 264] Barker put:

"... The spectral theory of cone preserving operators is now fairly complete and has been summarized in Barker and Schneider [8] and Berman and Plemmons [17, Chapter 1]. Since the subject of spectral theory is well summarized, we shall not consider it in detail in the main body of this paper ..." (The reference numbers given are changed to those used in our reference list.)

In 1995, Dodds [30, p.21] also wrote:

"Many of the central ideas and themes in the theory of positive compact operators may be traced back to the work, in the early part of this century, of Perron and Frobenius on the spectral theory of non-negative matrices (in the above sense) and to the related work of Jentzsch on integral equations with non-negative kernels. Only as recently as 1985 can it be said, in a certain sense, that the circle of ideas beginning with the work of Perron-Frobenius-Jentzsch has been finally completed ..."

1.4. More Recent Work

The past two decades have witnessed a rapid development of the combinatorial spectral theory of nonnegative matrices, where emphasis is put on the relation between the combinatorial structure and the spectral structure of the generalized eigenspace associated with the spectral radius or distinguished eigenvalues of a nonnegative matrix. Since our work was much motivated by the latter theory, we shall devote the whole Section 2 to a description of it.

The combinatorial spectral theory of nonnegative matrices also has had some impacts on the infinite-dimensional theory. From the early '80s to the mid-'90s, many of the graph-theoretic ideas (such as the concepts of classes, the notion of accessibility between states or classes) or of the combinatorial spectral results (such as the Nonnegative Basis Theorem and the Rothblum Index Theorem) have already been extended, first to the setting of an eventually compact linear integral operator on $L^p(\Omega, \mu)$, $1 \leq p < \infty$, with nonnegative kernel (see Victory [139, 140], Jang and Victory [67, 68]), and then to the setting of a positive, eventually compact linear operator on a Banach lattice having order continuous norm (see Jang and Victory [69, 70]). Their treatment, which employs mainly functional-analytic methods, is made possible by a decomposition of the positive operator under consideration in terms of certain closed ideals of the underlying space in a form which directly generalizes the Frobenius normal form of a nonnegative reducible matrix.

This author first got involved in the spectral theory of positive linear operators (in finite dimension) in the academic year 1986–87 when he supervised his master student Shiow-Fang Wu to write up her master thesis "Some results related to the class structure of a nonnegative matrix." In the subsequent year, he took his study leave at the University of Wisconsin-Madison, and with Hans Schneider, he embarked on a study of the Perron-Frobenius theory of a nonnegative matrix and its generalizations from the cone-theoretic viewpoint. After a prolonged study of more than ten years, in this subject, they produced, in succession, the following papers: Tam and Wu [133], Tam [121], Tam and Schneider [130, 131], and the intended future papers Tam and Schneider [132], and Tam [127, 128]. As a result, our knowldege of the spectral theory of cone-preserving maps (in finite dimensions) has grown considerably, to the extent that it probably deserves the name of the geometric spectral theory of positive linear operators.

1.5. Some Features of Our Work

In this paper, by "we" we often mean "this author and his co-author Hans Schneider (or S.F. Wu)".

The main feature of our work on positive linear operators is that, we approach the subject from a new viewpoint, the cone-theoretic geometric viewpoint, as opposed to the funcion-theoretic approaches used prevalently for infinite-dimensional spaces or the current graph-theoretic (combinatorial) approaches used for nonnegative matrices. We think that the geometric approach has interest in its own right and also that in applications, say, to linear dynamical systems, it is often important to understand the geometry.

In our study, the underlying positive cone is a proper cone (in a finite-dimensional space), which need not be lattice-ordered (or simplicial, in our terminology); that is, we are dealing with much more than the classical nonnegative matrix case. Apparently, with a general abstract proper cone, there is not much structure one can work with. (Indeed, until today we know very little about the elements of $\pi(K)$ which are not a nonnegative linear combination of its rank-one elements; see Tam [124, p.69].) But in order to achieve a better understanding, we chose to start our work in such a setting. Fortunately (and also quite surprisingly), we did obtain not a few useful, basic (nontrivial) results in this general setting. Then by confining ourselves to a narrower class of proper cones, such as the class of polyhedral (finitely generated) cones, we managed to obtain deeper results. We usually formulate and prove our results in a cone-theoretic language. The operator-invariant face, instead of the operator-invariant ideal, is one of our objects of interest; for a linear mapping preserving a general proper cone, we just don't have the concept of operator-invariant ideal. The Perron-Frobenius theorems, duality arguments, and a number of basic, simple (but nontrivial) results form our bag of tools. Many a time one can easily apply a simple useful fact – like, $A \in \pi(K)$ if and only if $A^T \in \pi(K^*)$, where K^* denotes the dual cone of K – without realizing that the used fact is actually nontrivial! The strength of our approach lies in the availability of a number of nontrivial simple useful results.

Another attracting feature of our work is that, we need only a few definitions. We have achieved a rather high # *of theorems produced* / # *of definitions needed* ratio.

Unlike the work of Jang and Victory, mentioned in Subsection 1.4, in our work we do not generalize the Frobenius normal form of a reducible nonnegative matrix. Indeed, we do not expect there is a natural generalization for a cone-preserving map on a general proper cone. Also, we do not use digraphs associated with a cone-preserving map, though we are aware that such digraphs have existed in the literature (see Barker and Tam [9] and Tam and Barker [129]); for one thing, such digraphs are not too natural and furthermore they are usually not finite. We believe that our cone-theoretic arguments are more susceptible to generalizations to infinite-dimensional spaces than graph-theoretic arguments.

Our geometric spectral theory can also offer an independent, self-contained, alternative approach to a large part of the recently developed combinatorial spectral theory of nonnegative matrices. This is an elegant, conceptual approach, where calculations are reduced to a minimum.

In our study, we borrowed many of our ideas from the nonnegative matrix theory. As a feedback, we obtained an unexpected, highly nontrivial necessary condition for the collection of elementary Jordan blocks corresponding to the eigenvalues lying in the peripheral spectrum of a nonnegative matrix (see Subsection 4.10). Furthermore, as an outcome of our investigation, we also discovered some new useful concepts, like spectral pairs, which have no counterparts in the original nonnegative matrix setting.

1.6. About This Paper

Now we describe the contents of this paper briefly. We devote Section 2 to a sketch of the combinatorial spectral theory of nonnegative matrices, which is a source of motivation for our work. Section 3 mainly describes our work in [133] and [121] in connection with the fundamental comcepts of Collatz-Wielandt sets, Collatz-Wielandt numbers, local spectral radius and distinguished eigenvalues. In Section 4, we describe our work in [130] on the core of a cone-preserving map. It is shown that there are close connections between the core, the peripheral spectrum, the Perron-Schaefer condition, and the distinguished invariant faces of a cone-preserving map. Section 5 is about the invariant faces of a cone-preserving map, which is the focus of interest in [131]. The important concepts of semi-distinguished invariant faces, and of spectral pairs of faces (or vectors) associated with a cone-preserving map are introduced. An extension of the Rothblum Index Theorem of nonnegative matrices to a linear mapping preserving a polyhedral cone in terms of semi-distinguished invariant faces is described. In Section 6, the last section, we describe our other related works in the papers [52, 126, 134], and also the future papers [127, 128, 132]. Of course, we mention also the related works of other people, and also open problems, where appropriate.

In this paper we focus on new ideas, concepts and results involving them. We hope the paper will serve as a quick introduction to the subject, from the basics up to research frontiers. Usually we do not give proofs to known results, unless we have a different proof, which is better or can illustrate something. At a few places, we do provide the original proofs, in order to show the interplay between various ideas and results.

This is a survey paper. However, we have included some original work (with proofs) in Subsections 4.8.1–4.8.3, and 6.1.1–6.1.4. In particular, we are able to describe fully the core of a K-irreducible matrix, and characterize all linear subspaces W for which there exist K-irreducible matrices A such that $\text{core}_K(A) = W \bigcap K$. We also discover the following interesting new result: *If K is a polyhedral cone* *with m maximal faces and if $A \in \pi(K)$, then there exists an $m \times m$ nonnegative matrix B and a B-invariant subspace W of $\mathbb{R}^m$, $W \bigcap \operatorname{int} \mathbb{R}^m_+ \neq \emptyset$, such that the cone-preserving maps $A \in \pi(K)$ and $B|_W \in \pi(W \bigcap \mathbb{R}^m_+)$ are equivalent* (in a natural sense to be defined later). Then with the help of the Preferred Basis Theorem (for singular M-matrices), we show that the above result can be used to rederive a crucial lemma needed in proving our extension of the Rothblum Index Theorem to the polyhedral cone case.

In Subsection 4.6, we also indicate how our results on the core of a nonnegative matrix can be used to rederive in a quick way the recent work of Sierksma [109] on limiting polytopes.

2. Combinatorial Spectral Theory

The (finite-dimensional) matrix case has two special features not shared by the infinite-dimensional case in general – a square matrix has a Jordan canonical form and also a directed graph naturally associated with it. So for a nonnegative matrix it is natural to investigate the connection between these two features. Since the mid-'70s, there has been a rapid growth in the combinatorial spectral theory of nonnegative matrices. Our work on positive operators by a cone-theoretic approach is partly influenced by this development. Excellent surveys on the combinatorial spectral theory of nonnegative matrices are already available in the literature; see Schneider [103] and Hershkowitz [56, 57]. In this section we are going to give a sketch of this topic. First, we begin with the necessary graph-theoretic definitions.

2.1. Definitions

Let A be an $n \times n$ matrix. By the *digraph of A*, denoted by $G(A)$, we mean as usual the directed graph with vertex set $\langle n \rangle := \{1, \ldots, n\}$, where (i, j) is an arc if and only if $a_{ij} \neq 0$. By a *class* of A we mean the vertex set of a strongly connected component of $G(A)$. The concept of a class was due to Rothblum [93]. We mainly follow his terminology, but sometimes we also borrow from Schneider [103]. If α, β are classes of A, we say α *has access to* β (or β *has access from* α) if either $\alpha = \beta$ or there is a path in $G(A)$ from some (and hence from all) vertex in α to some (and hence all) vertex in β; then we write $\alpha >= \beta$. We write $\alpha >- \beta$ if $\alpha >= \beta$ and $\alpha \neq \beta$. Sometimes we also write $i >= \alpha$, where $i \in \langle n \rangle$ and α is a class, with the obvious meaning. A class α is *initial* if it has no access from other class except from itself. Similarly, we can define the concept of a *final class*. If A is nonnegative, then we call a class α *basic* if $\rho(A_{\alpha\alpha}) = \rho(A)$, where $A_{\alpha\alpha}$ denotes the principal submatrix of A with rows and columns indexed by α. A class α of A is *distinguished* (respectively, *semi-distinguished*) if $\rho(A_{\alpha\alpha}) > \rho(A_{\beta\beta})$ (respectively, $\rho(A_{\alpha\alpha}) \geq \rho(A_{\beta\beta})$) for any class $\beta >- \alpha$. By applying a suitable permutation similarity to A, we can always put A into the familiar (lower-triangular block) *Frobenius normal form*

$$\begin{bmatrix} A_{11} & \mathbf{0} & \cdots & \mathbf{0} \\ A_{21} & A_{22} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1} & A_{p2} & \cdots & A_{pp} \end{bmatrix},$$

where the diagonal blocks $A_{11}, \ldots, A_{pp}$ are irreducible, each corresponding to a class of A. In case A is nonnegative, from its Frobenius normal form, we readily see that the number of basic classes A has is equal to the algebraic multiplicity of $\rho(A)$ as an eigenvalue of A, or, equivalently, the dimension of the *Perron generalized eigenspace* $\mathcal{N}((\rho(A)I_n - A)^n)$ of A. (In the literature, the Perron generalized eigenspace of a nonnegative matrix is also known as its *algebraic eigenspace*.)

A sequence of classes $\alpha_1, \ldots, \alpha_k$ is said to form a chain from α_1 to α_k if $\alpha_i >\!\!- \alpha_{i+1}$ for $i = 1, \ldots, k-1$. The *length* of a chain is the number of basic classes it contains. The *height* of a class β is the length of the longest chain of classes that terminates in β. It is clear that a basic class is of height one if and only if it is a distinguished class.

By the *reduced graph* of A, denoted by $\mathfrak{R}(A)$, we mean the directed graph with the classes of A as its vertices, such that (α, β) is an arc if and only if $\alpha \neq \beta$ and $A_{\alpha\beta} \neq 0$.

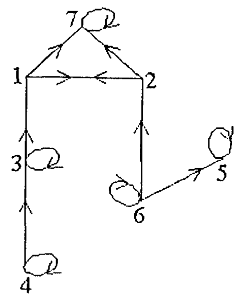
An example is in order.

Example 2.1. Consider the following 7×7 nonnegative matrix

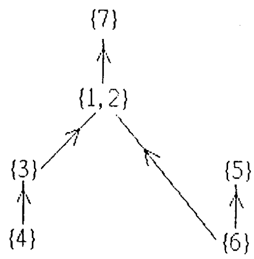
$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & * \\ 1 & 0 & 0 & 0 & 0 & 0 & * \\ * & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & * & 0 & 0 & * & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2/5 \end{bmatrix},$$

where each $*$ specifies a positive entry. The digraph $G(P)$ and the reduced graph $R(P)$ of P are given respectively by:

$G(P)$:



$R(P)$:



As can be readily seen, we have the following:
Classes of $P : \{1,2\},\ \{3\},\ \{4\},\ \{5\},\ \{6\}$ and $\{7\}$.
Initial classes of $P : \{4\}$ and $\{6\}$.
Final classes of $P : \{5\}$ and $\{7\}$.
$\rho(P) = 1$.
Basic classes of $P : \{1,2\},\ \{5\}$ and $\{6\}$.
Distinguished classes of $P : \{3\},\ \{4\}$ and $\{6\}$.
Semi-distinguished but not distinguished classes of $P : \{1,2\}$ and $\{5\}$.

2.2. Basic Results

For any square complex matrix A and any complex number λ, we denote by $\nu_\lambda(A)$ the least nonnegative integer k such that $\mathrm{rank}((\lambda I - A)^k) = \mathrm{rank}((\lambda I - A)^{k+1})$. If λ is an eigenvalue of A, then $\nu_\lambda(A)$ is the *index of* λ as an eigenvalue of A; otherwise, $\nu_\lambda(A) = 0$.

In 1956, Schneider [100] gave the following earliest results that connect the graph structure and the Jordan structure of a nonnegative matrix. The results as stated there are given in terms of a singular M-matrix (i.e., an $n \times n$ real matrix of the form $\rho(P)I_n - P$, where P is nonnegative), and graphs do not appear. Here we reformulate them in terms of a nonnegative matrix.

Theorem 2.2. *If $P \geq 0$, then $\nu_{\rho(P)}(P) = 1$ if and only if there are no chains of two or more basic classes of P.*

Theorem 2.3. *If $P \geq 0$, then the Jordan canonical form of P contains only one block corresponding to $\rho(P)$ if and only if any two basic classes of P are comparable with respect to the accessibility relation.*

Motivated by the results and technique developed by Schneider [100], in 1963 Carlson [25] studied the solvability of the equation $Ax = b$, $x \geq 0$, where A is a singular M-matrix and b is a given nonnegative vector. In 1973, Cooper [28] resumed the study of the spectral properties of nonnegative matrices from the combinatorial viewpoint by considering the distinguished (i.e., nonnegative) eigenvectors of a nonnegative matrix. In 1975, Rothblum [93], and later Richman and Schneider [92] independently, looked at the distinguished generalized eigenvectors of a nonnegative matrix corresponding to its spectral radius. The following results were obtained.

We call a matrix (also, a vector) *semipositive* if it is nonzero, nonnegative. A basis (for a linear subspace) which consists of semipositive vectors is called a *semipositive basis*.

Theorem 2.4. *Let P be an $n \times n$ nonnegative matrix with m basic classes $\alpha_1, \dots, \alpha_m$.*

(i) *The Perron generalized eigenspace $\mathcal{N}((\rho(P)I_n - P)^n)$ contains a semipositive basis $\{x^{(\alpha_1)}, \dots, x^{(\alpha_m)}\}$ such that $x_j^{(\alpha_i)} > 0$ if and only if $j >= \alpha_i$, where $x_j^{(\alpha_i)}$ denotes the jth component of the vector $x^{(\alpha_i)}$.*

(ii) *There exists a basis for $\mathcal{N}((\rho(P)I_n - P)^n)$ which, in addition to the property described in* (i), *satisfies the following : For $i = 1, \dots, m$, we have*

$$(P - \rho(P)I_n)x^{(\alpha_i)} = \sum_{k=1}^{m} c_{ik} x^{(\alpha_k)},$$

where c_{ik} is positive if $\alpha_k >\!\!- \alpha_i$ and equals 0 otherwise.

Part (i) of Theorem 2.4 is now usually referred to as the Nonnegative Basis Theorem, and is due to Rothblum [93, Theorem 3.1, Part I]. Part (ii) of the same theorem is called the Preferred Basis Theorem and first appears in Richman and Schneider [92]. In [58], Hershkowtiz and Schneider also extended the result to cover the case of a distinguished eigenvalue, i.e., an eigenvalue for which there is a corresponding semipositive eigenvector.

We would like to mention that the Perron eigenspace $\mathcal{N}(\rho(P)I_n - P)$ of a nonnegative matrix P, however, need not have a semipositive basis (as was, mistakingly, stated in [67, p.196, lines 23–24 and p.197, lines 15–16]). Cooper [28] proved that the dimension of the subspace spanned by the semipositive eigenvectors of a nonnegative matrix P is equal to the number of distinguished basic classes of P (cf. Theorem 2.7(iii), to be given later). He also showed that if the singular graph of P is a rooted forest (or, in other words, if for any basic class β of P, the collection $\{\alpha : \beta >= \alpha, \alpha \text{ a basic class}\}$ forms a chain, then the Perron eigenspace of P contains a semipositive basis. Later, Richman and Schneider [92, Corollary 5.8] (or, see Schneider [103, Corollary 8.6]) also generalized the latter result.

Theorem 2.5. (Rothblum Index Theorem). *If $P \geq 0$, then there exists a chain $\alpha_1 >\!\!- \alpha_2 >\!\!- \cdots >\!\!- \alpha_\nu$, where each α_k is a basic class of P and $\nu = \nu_{\rho(P)}(P)$, but there is no such chain with longer length.*

Let us illustrate the preceding two theorems by the nonnegative matrix P considered in Example 2.1. In this case, associated with the basic class $\{1, 2\}$, we have a generalized eigenvector of P corresponding to $\rho(P)$ with sign pattern of the form $(+, +, +, +, 0, +, 0)^T$. Also, $\{6\} >\!-\{1, 2\}$ is a chain of basic classes of P with maximum length. By the Rothblum Index Theorem, we have $\nu_{\rho(P)}(P) = 2$.

Part (ii) of Theorem 2.4 readily yields the following:

Corollary 2.6. *If $P \geq 0$, then there exists a semipositive vector x such that $(A - \rho(A)I_n)^\nu x = 0$ and $(A - \rho(A)I_n)^i x$ is semipositive for $i = 1, \ldots, \nu - 1$, where $\nu = \nu_{\rho(P)}(P)$.*

The chain of semipositive vectors

$$x, (P - \rho(P)I_n)x, \ldots, (P - \rho(P)I_n)^{\nu-1}x$$

considered in Corollary 2.6 is now usually referred to as a *semipositive Jordan chain* for P of ν vectors (corresponding to $\rho(P)$).

In 1985, Victory [141], in his study of the solvability of the equation $(\lambda I - P)x = b,\ x \geq 0$, where $P \geq 0,\ \lambda > 0$ and $b \geq 0$ are given, introduced the concept of a distinguished class, and gave a characterization of the distinguished eigenvalue and the support structure of a corresponding distinguished eigenvector for a nonnegative matrix. Here is his result, which is augmented by a third part.

Theorem 2.7 (Frobenius-Victory Theorem). *Let P be an $n \times n$ nonnegative matrix. Then :*

(i) *For any real number λ, λ is a distinguished eigenvalue of P (for $\mathbb{R}^n_+$) if and only if there exists a distinguished class α of P such that $\rho(P_{\alpha\alpha}) = \lambda$.*

(ii) *If α is a distinguished class of P, then there is a (up to multiples) unique semipositive eigenvector $x^\alpha = (\xi_1, \ldots, \xi_n)^T$ corresponding to $\rho(P_{\alpha\alpha})$ with the property that $\xi_i > 0$ if and only if $i >= \alpha$.*

(iii) *For each distinguished eigenvalue λ of P, the cone $\mathcal{N}(\lambda I_n - P) \bigcap \mathbb{R}^n_+$ is simplicial and its extreme vectors are precisely all the distinguished eigenvectors of P of the form x^α as given in* (ii), *where α is a distinguished class such that $\rho(P_{\alpha\alpha}) = \lambda$.*

If $S \subseteq K$, we denote by $\Phi(S)$ the *face of K generated by S*, i.e., the intersection of all faces of K including S. If $x \in K$, we write $\Phi(\{x\})$ simply as $\Phi(x)$; it is known that $\Phi(x) = \{y \in K : x \geq^K \alpha y$ for some $\alpha > 0\}$. A vector $x \in K$ is called an *extreme vector* if either x is the zero vector or x is nonzero and $\Phi(x) = \{\lambda x : \lambda \geq 0\}$; in the latter case, the face $\Phi(x)$ is called an *extreme ray*. A proper cone K is said to be *polyhedral* if it has finitely many extreme rays. By a *simplicial cone* we mean a polyhedral cone whose number of extreme rays is equal to the dimension of its linear span. A typical example of a simplicial cone is the nonnegative orthant $\mathbb{R}^n_+$.

Part (ii) of Theorem 2.7 first appeared in Schneider [99] and [100, Theorem 2] for the special case when α is a distinguished basic class. In a slightly weaker form and for the special case when $\lambda = \rho(P)$, part (iii) of Theorem 2.7 can also be found in Carlson [25, Theorem 2]. As noted by Schneider in [103, p.168], most of Section II of Frobenius [48] is devoted to what may, with hindsight, be regarded as a proof of Theorem 2.7. Thus, Theorem 2.7 is now usually referred to as the Frobenius-Victory Theorem.

By part (i) of Theorem 2.7, the distinguished eigenvalues of the nonnegative matrix P, considered in Example 2.1, are precisely 1/3, 1/2 and 1.

2.3. A Deep Result

The survey paper of Schneider [103] in 1986 stimulated a lot of research work, and subsequently many papers on the subject were produced. Let us quote from the Introductory section of the survey paper [57] by Hershkowtiz:

"The well-known Perron-Frobenius spectral theory of nonnegative matrices motivated an intensive study of the relationship between graph theoretic properties and spectral properties of matrices. While for about 70 years research focused on nonnegative matrices, in the past fifteen years the study has been extended to general matrices over an arbitrary field ..."

"... In his Ph.D. thesis Schneider discussed the combinatorial structure of the generalized eigenspace associated with the eigenvalue $\rho(A)$ of a *reduible* nonnegative matrix A or, equivalently, the generalized nullspace of a singular M-matrix. He observed that in two extreme cases, the height characteristic $\eta(A)$ of an M-matrix A, which describes the analytic structure of the generalized nullspace, is equal to the level characteristic $\lambda(A)$ of A, which is determined by the zero pattern of the block triangular Frobenius normal form of A. The equality of the two sequences does not hold for all M-matrices, and therefore Schneider asked what the relations between the two sequences for general M-matrices are. He also asked what the cases of equality are. These questions were answered about thirty six years later, when a majorization relation between the two sequences was established, and thirty five equivalent conditions were given to describe the equality case..."

Before we close this section, we would like to describe the "majorization relation" between the height and the level characteristic of an M-matrix, as mentioned near the end of the above paragraph. This is a deep result in this area, obtained by Hershkowtiz and Schneider [61] in 1991. Before we describe their result, we need more definitions.

Let A be an $n \times n$ nonnegative matrix. By the *level characteristic* of A associated with $\rho(A)$, denoted by $\lambda(A)$, we mean the finite sequence $(\lambda_1, \lambda_2, \dots)$ of positive integers, where for each j, λ_j denotes the number of basic classes of A of height j. By the *height characteristic* (also known as *Weyr characteristic*) of A associated with $\rho(A)$, denoted by $\eta(A)$, we mean the finite sequence $(\eta_1, \dots, \eta_\nu)$ given by

$$\eta_k = \dim\, \mathcal{N}((\rho(A)I_n - A)^k) - \dim\, \mathcal{N}((\rho(A)I_n - A)^{k-1})$$

for $k = 1, \dots, \nu$, where $\nu = \nu_{\rho(A)}(A)$. Or, put it in another way, η_k is the number of dots in the ith row of the Jordan diagram $J(A)$ of A associated with $\rho(A)$; the Jordan diagram $J(A)$ of A is the sequence of sizes of elementary Jordan blocks of A that correspond to $\rho(A)$ arranged as dots in columns of nonincreasing height from left to right. From the second alternative definition, it is clear that the sequence $\eta(A)$ is nonincreasing. For the nonnegative matrix P considered in Example 2.1, we have $\lambda(A) = (1, 2)$ and $\eta(A) = (2, 1)$.

We also need the usual concept of majorization. For any n-tuple of real numbers $x = (x_1, \dots, x_n)$, let $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ denote the components of x arranged in nonincreasing order. If $x,\ y \in \mathbb{R}^n$, we say that x is *majorized* by y (or y *majorizes* x), denoted by $x \prec y$, if

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \text{ for } k = 1, \dots, n-1,$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}.$$

Theorem 2.8 (Hershkowitz and Schneider). *Given two sequences of positive integers $\lambda = (\lambda_1, \dots, \lambda_p)$ and $\eta = (\eta_1, \dots, \eta_q)$, where $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_q$, in order that there exists a nonnegative matrix A such that $\lambda(A) = \lambda$ and $\eta(A) = \eta$, it is necessary and sufficient that $p = q$ and $\lambda \prec \eta$.*

The above theorem of Hershkowitz and Schneider says, in particular, that for a nonnegative matrix A, the sequences $\lambda(A)$ and $\eta(A)$ have the same length, which is the Rothblum Index Theorem. If $\lambda = (1, \dots, 1)$, then the only sequence η of positive integers (with the same length) that can majorize λ is clearly λ itself. On the other hand, if $\eta = (1, \dots, 1)$ the only sequence λ of positive integers that η majorizes is also η itself. Thus, Theorems 2.2 and 2.3 also follow from Theorem 2.8.

There are, indeed, many more deep results in this area. We refer the interested reader to the excellent survey papers [56] and [57] by Hershkowitz.

3. Collatz-wielandt Sets and Distinguished Eigenvalues

3.1. Collatz-Wielandt Sets (or Numbers)

In 1950, Wielandt [142] offered a new proof of the basic Perron-Frobenius theorem of an irreducible nonnegative matrix (i.e., part (i) of Theorem 1.2). For an $n \times n$ irreducible nonnegative matrix P, he considered the function $f_P : \triangle \to \mathbb{R}_+$ given by $f_P(x) = \min_{x_i \neq 0}(Px)_i/x_i$, where $\triangle$ is the set of all vectors in $\mathbb{R}^n_+$ the sum of whose components equals 1, and $(Px)_i$ denotes the ith component of Px. He showed that the function f_P always attains its maximum at a positive vector u, and furthermore we have $Pu = f_P(u)u$ and $f_P(u) \geq |\lambda|$ for any eigenvalue λ of P. Wielandt's proof is considerably shorter than the previously known proofs. It is so appealing that most textbooks written since 1950, which have a chapter on nonnegative matrices, follow Wielandt's approach. For a very interesting commentary on Wielandt's paper [142] (and its influences), see Schneider [104].

In order to extend Wielandt's method to the case of a cone-preserving map A, Barker and Schneider [8] introduced the following four sets, now known as the *Collatz-Wielandt sets associated with* A, when the underlying space need not be finite-dimensional. Then under certain existence assumptions, they obtained theorems of Perron-Frobenius type when A is strongly irreducible (which is the same as A being irreducible, in the finite-dimensional case).

$$\begin{aligned}
\Omega(A) &= \{\omega \geq 0 : \exists x \in K\backslash\{0\},\ Ax \geq^K \omega x\}.\\
\Omega_1(A) &= \{\omega \geq 0 : \exists x \in \text{int } K,\ Ax \geq^K \omega x\}.\\
\Sigma(A) &= \{\sigma \geq 0 :\ \exists x \in K\backslash\{0\},\ Ax^K \leq \sigma x\}.\\
\Sigma_1(A) &= \{\sigma \geq 0 :\ \exists x \in \text{int } K,\ Ax^K \leq \sigma x\}.
\end{aligned}$$

Closely related to the Collatz-Wielandt sets are the *lower and upper Collatz-Wielandt numbers* of a vector x (in K) with respect to A defined by

$$\begin{aligned}
r_A(x) &= \sup \{\omega \geq 0 :\ Ax \geq^K \omega x\},\\
R_A(x) &= \inf \{\sigma \geq 0 :\ Ax^K \leq \sigma x\},
\end{aligned}$$

where we write $R_A(x) = \infty$ if no σ exists such that $Ax^K \leq \sigma x$. It is ready to see that, in the nonnegative matrix case, $r_A(\cdot)$ is the same as the function $f_A(\cdot)$ mentioned above (with $A = P$).

3.1.1. Bounds for Collatz-Wielandt Sets

In Tam and Wu [133], we consider the values of the suprema or infima of the Collatz-Wielandt sets associated with a cone-preserving map. Some further works are followed in Tam [121] and the intended future paper Tam and Schneider [132]. Below we give a summary of the results obtained. (We are still assuming that K is a proper cone in $\mathbb{R}^n$.)

Theorem 3.1. *Let* $A \in \pi(K)$. *Then* :

(i) $\sup \Omega(A) = \inf \Sigma_1(A) = \rho(A)$.

(ii) $\inf \Sigma(A)$ *is equal to the least distinguished eigenvalue of* A *for* K.

(iii) $\sup \Omega_1(A) = \inf \Sigma(A^T)$, *and hence is equal to the least distinguished eigenvalue of* A^T *for* K^*.

(iv) *We always have* $\sup \Omega(A) \in \Omega(A)$ *and* $\inf \Sigma(A) \in \Sigma(A)$.

(v) *When* K *is polyhedral, we have* $\sup \Omega_1(A) \in \Omega_1(A)$. *In general, we may have* $\sup \Omega_1(A) \notin \Omega_1(A)$.

(vi) *For a nonnegative matrix* P, *we have* $\rho(P) \in \Sigma_1(P)$ *if and only if every basic class of* P *is final.*

(vii) $\rho(A) \in \Sigma_1(A)$ *if and only if* $K = \Phi((\mathcal{N}(\rho(A)I - A) \bigcap K) \bigcup C)$, *where* $C = \{x \in K : \ \rho_x(A) < \rho(A)\}$ *and* $\rho_x(A)$ *denotes the local spectral radius of* A *at* x.

Barker and Schneider [8] showed that for any $A \in \pi(K)$, we always have $\sup \Omega(A) \leq \inf \Sigma_1(A)$, and if in addition A is K-irreducible, then $\sup \Omega(A) = \inf \Sigma_1(A) = \rho(A)$ (and $\Omega(A) = \Omega_1(A), \ \Sigma(A) = \Sigma_1(A)$). Part (i) of Theorem 3.1 gives the complete result, as it appears in Tam and Wu [133, Theorem 3.1]. Parts (ii), (iii), (iv) and (vi) of Theorem 3.1 can also be found in Tam and Wu [133, Theorems 3.2, 3.3 and 4.7], whereas part (v) appears in Tam [121, Corollary 4.2 and Example 5.5]. Part (vii) is a new, deeper result, and will appear in Tam and Schneider [132]. At present, we *do not know* the answer to the question of when $\sup \Omega_1(A)$ belongs to $\Omega_1(A)$ for a general proper cone K.

In the irreducible nonnegative matrix case, part (i) of Theorem 3.1 reduces to the well-known max-min and min-max characterizations of $\rho(A)$ due to Wielandt. Schaefer [97] generalized the result to irreducible compact operators in L^p-spaces and, more recently, Friedland [43, 44] also extended the characterizations in the settings of a Banach space or a C^*-algebra. (Recall that $\mathbb{C}^n$ is a commutative C^*-algebra under coordinatewise multiplication, with the nonnegative orthant $\mathbb{R}^n_+$ as its cone of positive self-adjoint elements.)

3.1.2. A Comparison of Spectral Radii

The following slightly improves an early result, due to Vandergraft, which compares the spectral radii of two cone-preserving maps (see Vandergraft [136, Theorem 4.6], Rheinboldt and Vandergraft [91, Theorem 9] or Berman and Plemmons [17, Corollary 1.3.29]). Since we cannot find the result in the literature, we take the opportunity to include it here.

Let $\mathbf{0} \overset{\pi(K)}{\leq} A \overset{\pi(K)}{\leq} B$, *where* B *is* K*-irreducible and* $A \neq B$. *Then* $\rho(A) < \rho(B)$.

(In the literature, A is assumed to be K-irreducible instead.) For the nonnegative matrix case, the above assertion follows immediately from Wielandt's lemma (see Berman and Plemmons [17, Theorem 2.2.14]). The proof for the general case goes as follows:

Choose an eigenvector $x \in K$ of A corresponding to $\rho(A)$. Then $\mathbf{0} \overset{K}{\le} \rho(A)x = Ax \overset{K}{\le} Bx$, and so $\rho(A) \in \Omega(B)$. Hence we have $\rho(A) \le \sup \Omega(B) = \rho(B)$. If the desired strict inequality does not hold, we must have $\rho(A) = \rho(B)$. Then from the above, we have $\rho(B)x \overset{K}{\le} Bx$. Since B is K-irreducible, by a standard Wielandt-type argument (making use of condition (c) of Theorem 1.5), it follows that $\rho(B)x = Bx$, and hence necessarily $x \in \text{int } K$. Thus, $(B - A)x = \rho(B)x - \rho(A)x = \mathbf{0}$. Together with $B - A \in \pi(K)$ and $x \in \text{int } K$, this implies that $B - A = \mathbf{0}$, which is a contradiction.

3.1.3. Local Spectral Radius and Collatz-Wielandt Numbers

In part (vii) of Theorem 3.1, we need the concept of the *local spectral radius of* A *at* x for an $n \times n$ complex matrix A and a vector $x \in \mathbb{C}^n$. It can be defined as follows. If x is the zero vector, take $\rho_x(A)$ to be 0. Otherwise, define $\rho_x(A)$ in one of the following equivalent ways (see Tam and Wu [133, Theorem 2.3]):

(i) $\rho_x(A) = \lim\ \sup_{m \to \infty} \|A^m x\|^{1/m}$, where $\|\cdot\|$ is any norm of $\mathbb{C}^n$.

(ii) $\rho_x(A) = \rho(A|_{W_x})$, where W_x is the cyclic space relative to A generated by x, i.e., the linear subspace span $\{A^i x :\ i \ge 0\}$.

(iii) Write x uniquely as a sum of generalized eigenvectors of A, say, $x = x_1 + \cdots + x_k$, where $m \ge 1$ and $x_1, \ldots, x_k$ are generalized eigenvectors corresponding respectively to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Then define $\rho_x(A)$ to be $\max_{1 \le i \le k} |\lambda_i|$.

The lower and upper Collatz-Wielandt numbers are also related to the problem of determining nested bounds for the spectral radius. In the nonnegative matrix case, we have the well-known inequality

$$r_P(x) \le \rho(P) \le R_P(x),$$

due to Collatz [27] under the assumption that x is a positive vector and due to Wielandt [142] under the assumption that P is irreducible and x is semipositive. (This explains why $r_A(x)$ and $R_A(x)$ are referred to as Collatz-Wielandt numbers.) Extending the above result, in Tam and Wu [133, Theorem 2.4(i)] we obtain the following

$$r_A(x) \le \rho_x(A) \le R_A(x), \tag{3.1}$$

where $A \in \pi(K)$ and $x \in K$. In [38], Förster and Nagy also establish relations between the local spectral radius $\rho_x(T)$ and the upper and lower Collatz-Wielandt numbers $r_T(x)$ and $R_T(x)$, for a nonnegative linear continuous operator T in a partially ordered Banach space E and a nonzero nonnegative vector x in E. For the local spectral radius they adopt the "lim sup" definition. (We would like to add that in the finite-dimensional case we may replace "lim sup" by "lim" in the definition of local spectral radius. A proof of this can be found in an appendix to [132].) They use the local resolvent function $x_T(\cdot)$, given by $x_T(\mu) = \sum_{j=0}^{\infty} \mu^{-j-1} T^j x$ for $|\mu| > \rho_x(T)$, as a basic tool. They note that the inequality $r_T(x) \leq \rho_x(T)$ always holds, but in general the inequality $\rho_x(T) \leq R_T(x)$ is invalid. However, when the positive cone of E is normal, the latter inequality always holds.

3.1.4. Open Problems

Given $A \in \pi(K)$ and $0 \neq x \geq^K 0$, a natural question to ask is, when do we have $\lim_{i\to\infty} r_A(A^i x) = \rho(A) = \lim R_A(A^i x)$? Friedland and Schneider [45, Theorem 6.8] have completely answered this question for the nonnegative matrix case. For a general proper cone K, in the special case when A is K-irreducible, the question is solved by Tam and Wu [133, Theorem 5.2]. But when A is K-reducible, the question remains unsettled. Of course, an even more general question to ask is the following:

Given $A \in \pi(K)$ *and* $0 \neq x \in K$, *when do we have* (i) $\lim_{i\to\infty} r_A(A^i x) = \rho_x(A)$, (ii) $\lim_{i\to\infty} R_A(A^i x) = \rho_x(A)$, *or* (iii) $\lim_{i\to\infty} r_A(A^i x) = \rho_x(A) = \lim_{i\to\infty} R_A(A^i x)$?

Some works on these problems have also been done in the setting of a nonnegative linear continuous operator in a partially ordered Banach space. See Förster and Nagy [37] and Marek [76].

3.2. Cone-solvability Theorems

The study of the Collatz-Wielandt sets also has connections with cone solvability theorems (or theorems of the alternative over cones). For example, the question of whether $\omega \in \Omega_1(A)$ is equivalent to $(A - \omega I_n)(\text{int } K) \bigcap K \neq \emptyset$. This led us to the discovery of the following new cone solvability theorem.

Theorem 3.2 (Tam [121, Theorem 3.1]). *Let A be an $m \times n$ real matrix. Let K_1 be a closed full cone in $\mathbb{R}^n$, and let K_2 be a closed cone in $\mathbb{R}^m$. The following conditions are equivalent* :

(a) $A(\text{int } K_1) \bigcap K_2 \neq \emptyset$.

(b) cl $A^T(K_2^*) \bigcap (-K_1^*) = \{0\}$.

(c) $\mathbb{R}^n = A^{-1}K_2 - K_1$, where $A^{-1}K_2 = \{x \in \mathbb{R}^n : \ Ax \in K_2\}$.

In condition (b) of the above theorem, we use K^* to denote the *dual cone* of a cone K given by: $K^* = \{z : \ \langle z, y \rangle \geq 0 \ \forall y \in K\}$, where $\langle \, , \, \rangle$ is the usual inner product of the underlying euclidean space. It is well-known that if K is a proper cone, then so is K^* and furthermore we have $K^{**} = K$.

In the '70s, Berman and Ben-Israel have obtained theorems of the alternative for the following two types of linear systems over cones (see, for instance, Berman [15, Chapter 1, Section 4]):

(I) $Ax \in K_2,\ 0 \neq x \in K_1$;

(II) $Ax \in \text{int } K_2,\ x \in \text{int } K_1$.

They showed, in particular, that we always have either (I) is consistent or the system (II)$'$, obtained from (II) by replacing A, K_2 and K_1 respectively by A^T, K_1^* and $-K_2^*$, is consistent, but not both. Note that (II) is also equivalent to the system: $Ax \in \text{int } K_2,\ 0 \neq x \in K_1$. With the addition of Theorem 3.2, the investigation along this direction is now completed. For a recent theorem of the alternative for cones similar to Theorem 3.2, see Cain, Hershkowitz and Schneider [24, Theorem 2.7].

3.3. Distinguished Eigenvalues

If $A \in \pi(K)$ and $x \in K$ is an eigenvector (respectively, generalized eigenvector) of A, then x is called a *distinguished eigenvector* (respectively, *distinguished generalized eigenvector) of A for K*, and the corresponding eigenvalue is known as a *distinguished eigenvalue of A for K*. When there is no danger of confusion, we simply use the terms distinguished eigenvector, distinguished generalized eigenvector and distinguished eigenvalue (of A).

Our results on the Collatz-Wielandt sets also help partly in proving the following theorem (Tam [121, Theorem 5.1]), which provides equivalent conditions for the existence of a distinguished generalized eigenvector of A that lies in int K. The theorem extends a result in the earlier paper by Tam and Wu [133, Theorem 4.4] on nonnegative matrices, which in turn combines several known results.

Theorem 3.3. *Let $A \in \pi(K)$. Consider the following conditions* :

(a) *$\rho(A)$ is the only distinguished eigenvalue of A for K.*

(b) *$x \geq^K 0$ and $Ax \ {}^K\!\!\leq \rho(A)x$ imply that $Ax = \rho(A)x$.*

(c) *A^T has a generalized eigenvector (necessarily corresponding to $\rho(A)$) in* int *K^*.*

(d) *$\rho(A) \in \Omega_1(A^T)$.*

Conditions (a), (b) *and* (c) *are always equivalent and are implied by condition* (d). *When K is polyhedral, condition* (d) *is also an equivalent condition.*

In condition (d) of Theorem 3.3, when we talk about $\Omega_1(A^T)$, we are treating A^T as an element of $\pi(K^*)$ and using the basic simple fact that $A \in \pi(K)$ if and only if $A^T \in \pi(K^*)$. The proof of this fact relies on the observation that $A \in \pi(K)$ if and only if $\langle z,\ Ay\rangle \geq 0$ for all $y \in K$ and $z \in K^*$, which in turn depends on the result that $K^{**} = K$. Since a proof of the latter result usually invokes the use of a standard separation theorem for convex sets, the basic simple fact mentioned above is actually deeper than it looks.

We take this opportunity to offer the following more transparent argument for the equivalence of conditions (a) and (c) of Theorem 3.3:

Let K_1 be the cone $\mathfrak{R}((\rho(A)I_n - A)^n) \bigcap K$, where we denote by $\mathfrak{R}(B)$ the range space of a matrix B. Observe that $\mathfrak{R}((\rho(A)I_n - A)^n)$ is equal to the intersection of $\mathbb{R}^n$ with the direct sum of the generalized eigenspaces (in $\mathbb{C}^n$) of A corresponding to eigenvalues other than $\rho(A)$. Clearly we have $AK_1 \subseteq K_1$. If $K_1 \neq \{0\}$, then by the Perron-Frobenius theory, A must have an eigenvector in K_1, and hence in K, corresponding to an eigenvalue other than $\rho(A)$, i.e., condition (a) is not satisfied. Conversely, if A has a distinguished eigenvalue other than $\rho(A)$, then the corresponding distinguished eigenvector must belong to K_1, and hence $K_1 \neq \{0\}$. This shows that condition (a) is equivalent to the condition that $K_1 = \{0\}$. Now the latter condition says that the subspace $\mathfrak{R}((\rho(A)I_n - A)^n)$ meets K only at the zero vector. So by a generalization of the Gordan-Stiemke theorem (see, for instance, [11, Corollary 2.6]) the condition becomes:

$$\mathcal{N}((\rho(A)I_n - A^T)^n) \bigcap \mathrm{int}\ K^* (= [\mathfrak{R}((\rho(A)I_n - A)^n)]^{\perp} \bigcap \mathrm{int}\ K^*) \neq \emptyset,$$

which is condition (c). This proves the equivalence of conditions (a) and (c).

For distinguished eigenvectors, we have the following corresponding result.

Theorem 3.4. *Let $A \in \pi(K)$. The following conditions are equivalent* :

(a) $\rho(A)$ *is the only distinguished eigenvalue of A for K, and* $\nu_{\rho(A)}(A) = 1$.

(b) *For any vector $x \in \mathbb{R}^n$, $Ax \stackrel{K}{\leq} \rho(A)x$ implies that $Ax = \rho(A)x$.*

(c) A^T *has an eigenvector in* int K^* (*corresponding to* $\rho(A)$).

[Here we would like to point out that the implication (a)$\Longrightarrow$(c) of Theorem 3.4 follows immediately from Theorem 3.3, (a)$\Longrightarrow$(c), whereas the proof of (c)$\Longrightarrow$(b) should be corrected as follows: Apply Corollary 3.2 (of Tam [121]) with $K_1 = K^*$, $K_2 = \{0\}$, and A replaced by $(A - \rho(A)I_n)^T$.]

3.4. A Cone-theoretic Proof for the Nonnegative Basis Theorem

In [121], we also give an interesting existential proof for the Nonnegative Basis Theorem (Theorem 2.4(i)), using Theorem 3.3 and the Frobenius-Victory Theorem (Theorem 2.7). We rewrite the proof below:

The difficult part of the proof is to show that for any basic class α of the nonnegative matrix P, we can find a nonnegative generalized eigenvector x^α with the property that the ith component of x^α is positive if and only if the vertex i has access to the basic class α. Let Q denote the principal submatrix of P with rows and columns indexed by vertices of $G(P)$ that have access to α. Then the problem is reduced to constructing a positive generalized eigenvector of Q (corresponding to $\rho(Q)$ which equals $\rho(P)$). By Theorem 3.3, it suffices to show that $\rho(Q)$ is the only distinguished eigenvalue of Q^T. But that is clear; since Q^T has only one initial class which is also a basic class, namely α, by the Frobenius-Victory theorem, our contention follows.

3.5. Extensions to the Polyhedral Cone Case

The idea of using algebraic, matrix-theoretic arguments to study positive operators on polyhedral cones first appeared in a paper by Burns, Fiedler and Haynsworth [23]. In the paper they introduced the concept of a minimal generating matrix for a polyhedral cone. Subsequently, many other authors also use matrix-theoretic methods to study the positive operators between polyhedral cones (see, for instance, Fiedler and Pták [36], and Adin [1]).

Let K be a polyhedral cone in $\mathbb{R}^n$. We call an $n \times m$ real matrix X a *minimal generating matrix for* K if the columns of X form a set of distinct (up to multiples) extreme vectors of K. Note that for any $A \in \pi(K)$, there exists an $m \times m$ (not necessarily unique) nonnegative matrix B such that $AX = XB$.

To extend known results on the Perron generalized eigenspace of a nonnegative matrix to a matrix preserving a polyhedral cone, in [121] we also make use of a minimal generating matrix as a tool. However, since we prefer to a geometric approach, we are not employing matrix methods. We adopt the operator-theoretic viewpoint, noting that the above equation $AX = XB$ can be interpreted as an equation between cone-preserving maps, with $A \in \pi(K)$, $B \in \pi(\mathbb{R}^m_+)$, and $X \in \pi(\mathbb{R}^m_+,\ K)$ (such that $X(\mathbb{R}^m_+) = K$). The following interesting result (which is stated in a form slightly less general than that in [121, Theorem 7.3]) is obtained:

Theorem 3.5. *Let $K_1,\ K_2$ be proper cones in possibly different euclidean spaces. Let $A \in \pi(K_1)$, $B \in \pi(K_2)$, and $P \in \pi(K_2,\ K_1)$ be such that $AP = PB$, $PK_2 = K_1$, and $\mathcal{N}(P) \bigcap K_2 = \{0\}$. Then :*

(i) *Any representative matrices for A and $B^T|_{\mathfrak{R}(P^T)}$ are similar.*

(ii) *$\rho(A) = \rho(B) = \rho$, say.*

(iii) $\nu_\rho(A) = \nu_\rho(B)$.

(iv) *The set of distinguished eigenvalues of A for K_1 is equal to the set of distinguished eigenvalues of B for K_2.*

(v) *For any distinguished eigenvalue λ of B for K_2, P takes the generalized* *eigenspace of B corresponding to λ onto the generalized eigenspace of A corresponding to λ.*

We call a nonzero vector $x \in K$ a *K-semipositive vector*. A basis of a subspace which consists of K-semipositive vectors is called a *K-semipositive basis*.

Using Theorem 3.5 (with $K_1 = \mathbb{R}^m_+$, $K_2 = K$ and P equal to a minimal generating matrix for K), the Nonnegative Basis Theorem and Corollary 2.6, one readily deduces the following result (Tam [121, Theorem 7.5]):

Theorem 3.6. *Let $A \in \pi(K)$, where K is a polyhedral cone. Then* :

(i) *The Perron generalized eigenspace $\mathcal{N}((\rho(A)I_n - A)^n)$ of A contains a K-semipositive basis.*

(ii) *There exists a K-semipositive vector x such that $(A - \rho(A)I_n)^\nu x = 0$ and $(A - \rho(A)I_n)^i x$ is K-semipositive for $i = 1, \dots, \nu - 1$, where $\nu = \nu_{\rho(A)}(A)$.*

3.6. Semipositive Solutions of a Matrix Equation

In [54], Hartwig posed the following question:

Given nonnegative square matrices A, B (not necessarily of the same size), when does the matrix equation $AX = XB$ admit a semipositive solution X? If solutions exist, what do they look like?

In terms of the concept of a distinguished eigenvalue, this author [121, Theorem 8.1] offered the following answer to the first part of the above question:

Theorem 3.7. *Let K_1, K_2 be proper cones in possibly different euclidean spaces. Let $A \in \pi(K_1)$ and $B \in \pi(K_2)$. Then there exists a nonzero $X \in \pi(K_2, K_1)$ such that $AX = XB$ if and only if the set of distinguished eigenvalues of A for K_1 and the set of distinguished eigenvalues of B^T for K_2^* have a common element.*

4. The Core and the Peripheral Spectrum

4.1. Motivation

Another object of interest in our study is the core of a cone-preserving map. If $A \in \pi(K)$, then by the *core of A relative to K*, denoted by $\text{core}_K(A)$, we mean the convex cone given by $\text{core}_K(A) = \bigcap_{i=1}^\infty A^i K$. For the motivation of our study, let us quote from Tam and Schneider [130, p.480]:

"There are plausible reasons which explain why a study of the core of a cone-preserving map is worthwhile. First, in an initial study of the core of a nonnegative matrix (relative to the nonnegative orthant) Pullman [Pul] succeeded in rederiving the famous Frobenius theorem for an irreducible nonnegative matrix. This theorem, as we now know, is important to the treatment of nonnegative matrices by matrix-theoretic methods. Second, Birkhoff [Bir] gave an elementary proof of the Perron-Frobenius theorem for a cone-preserving map by considering the Jordan canonical form of a matrix. His method was later modified by Vandergraft [Van] to obtain an equivalent condition, now known as the Perron-Schaefer condition (which will be given in Section 2), for a matrix to have an invariant proper cone. Their proofs start by considering the limit of a convergent subsequence of $(A^i x/\|A^i x\|)_{i\in\mathbb{N}}$, where A is the cone-preserving map under consideration and x is an appropriate nonzero vector in the cone. But any such limit belongs to the core of A (relative to the cone). So it seems likely that the core of A contains much information about its spectral properties."

4.2. The Perron-Shaefer Condition

A square (complex or real) matrix A is said to satisfy the *Perron-Schaefer condition* if for any eigenvalue λ in the peripheral spectrum of A (i.e., $\lambda \in \sigma(A)$ such that $|\lambda| = \rho(A)$), we have $\nu_\lambda(A) \leq \nu_{\rho(A)}(A)$. Then clearly $\rho(A)$ is an eigenvalue of A.

In the '60s Schaefer proved that if T is a positive linear operator on an ordered complex Banach space with a normal reproducing cone, then the spectral radius $\rho(T)$ is an element of the spectrum of T, and if in addition $\rho(T)$ is a pole of the resolvent, then it is of maximal order on the spectral circle of T (see Schaefer [98, p. 311]). Restricted to the finite-dimensional case, Schaefer's result means that every cone-preserving map satisfies the Perron-Schaefer condition. In 1967, Birkhoff [19] gave an elementary proof for the Perron-Frobenius theorem of a cone-preserving map in the finite-dimensional case. His proof makes use of the Jordan canonical form of a matrix. Later, by modifying Birkhoff's argument, Vandergraft [136] established the Perron-Schaefer condition for a cone-preserving map in the finite-dimensional case. In the same paper, Vandergraft (and independently Elsner [33]) also proved the striking converse result: *If A is an $n \times n$ real matrix which satisfies the Perron-Schaefer condition, then there exists a proper cone K in $\mathbb{R}^n$ such that $A \in \pi(K)$.* (For more theorems of this type, which relate the spectral properties and the geometric properties of a matrix, see Djoković [29], Elsner [33, 35], Schneider [102], Stern and Wolkowicz [113], and Vandergraft [136].)

4.3. Basic Properties of the Core

A penetrating study of the core of a cone-preserving map is made in Tam and Schneider [130]. It is shown that there are close connections between the core, the peripheral spectrum, the Perron Schaefer condition, and the distinguished invariant faces of a cone-preserving map. The following is a basic result on the core of a cone-preserving map:

Theorem 4.1. *If* $A \in \pi(K)$, *then*

(i) *the set* $\text{core}_K(A)$ *is a closed, pointed cone*;

(ii) *the restriction of* A *to* $\text{span}(\text{core}_K(A))$ *is an automorphism of* $\text{core}_K(A)$;

(iii) *if* K *is polyhedral* (*respectively, simplicial*), *then so is* $\text{core}_K(A)$.

A matrix A is said to be *an automorphism of* K, denoted by $A \in \text{Aut}(K)$, if A is invertible, and $A,\ A^{-1}$ both belong to $\pi(K)$; or, equivalently, $AK = K$. (For works on $\text{Aut}(K)$ or subgroups of $\pi(K)$, see Horne [64] and Tam [117, Theorem 3.3; 118, Section 3; 123, Lemma and Theorem].)

The "polyhedral" part of Theorem 4.1(iii) is due to Pullman [90]. His proof depends on a compactness argument and invokes the use of a separation theorem for convex sets. The "simplicial" part follows from a result in [130] together with the Frobenius-Victory theorem, as we shall explain a bit later.

Given $A \in \pi(K)$, it is obvious that every distinguished eigenvector of A (or, of its positive powers) that corresponds to a nonzero distinguished eigenvalue belongs to $\text{core}_K(A)$. So, if we denote by $D_k(A)$ the cone generated by the distinguished eigenvectors of A^k corresponding to its nonzero distinguished eigenvalues, then $\bigcup_{i=1}^{\infty} D_i(A)$ is included in $\text{core}_K(A)$, and in fact it is equal to $D_k(A)$ for some positive integer k (see [130, Lemma 3.1]).

4.3.1. When the Core is Polyhedral

To obtain deeper results, we restrict our attention to the case when $\text{core}_K(A)$ is a polyhedral cone. By Theorem 4.1, this covers the case when K is a polyhedral cone, and hence also the important nonnegative matrix case. It is easy to see that $\text{core}_K(A)$ is the zero cone if and only if A is nilpotent. Hereafter, we assume that $\text{core}_K(A)$ is a nonzero polyhedral cone. In this case, A permutes the extreme rays of $\text{core}_K(A)$. We denote by τ_A the induced permutation. By the order of τ_A we mean, as usual, the smallest positive integer m such that τ_A^m is the identity permutation. It is not difficult to show the following [130, Theorem 3.2]:

Remark 4.2. Let $A \in \pi(K)$. Suppose that $\text{core}_K(A)$ is a nonzero, polyhedral cone. For each positive integer i, let $D_i(A)$ have the same meaning as before. Then:

(i) $\text{core}_K(A) = D_j(A)$ for some positive integer j.

(ii) For each positive integer i, $D_i(A) = \text{core}_K(A)$ if and only if i is a multiple of m, where m is the order of the induced permutation τ_A.

We can now obtain the "simplicial" part of Theorem 4.1(iii) as follows: By Remark 4.2(i), $\text{core}_K(A) = D_i(A)$ for some positive integer i. By definition of $D_i(A)$, we have

$$\text{core}_K(A) = \bigoplus [\mathcal{N}(\lambda I - A^i) \bigcap K],$$

where the direct sum is taken over all nonzero distinguished eigenvalues λ of A^i. (We say a cone K is a *direct sum* of the cones $K_1, \dots, K_p$ and write $K = K_1 \oplus \cdots \oplus K_p$ if each vector x in K can be expressed uniquely as $x_1 + \cdots + x_p$, where $x_j \in K_j$ for each j.) Since K is simplicial, it is linearly isomorphic with a nonnegative orthant. By applying the Frobenius-Victory theorem to A^i $(\in \pi(K))$, we infer that for each nonzero distinguished eigenvalue λ of A^i, the cone $\mathcal{N}(\lambda I - A^i) \bigcap K$ is simplicial, hence so is the cone $\text{core}_K(A)$.

According to [130, Corollary 3.4], when $\text{core}_K(A)$ is polyhedral, it cannot contain any generalized eigenvectors of A other than eigenvectors. In below we restate the result in a slightly stronger form and indicate a modified proof.

Remark 4.3. Let $A \in \pi(K)$. If $\text{core}_K(A) = D_i(A)$ for some positive integer i (which is the case if $\text{core}_K(A)$ is a nonzero polyhedral cone), then in span($\text{core}_K(A)$) there is no generalized eigenvectors of A (or of any positive powers of A) other than eigenvectors.

The reason is, in this case span($\text{core}_K(A)$) has a basis consisted of eigenvectors of A^i (corresponding to nonzero distinguished eigenvalues); that is, $(A|_{\text{span}(\text{core}_K(A))})^i$ is diagonalizable, and hence so is $A|_{\text{span}(\text{core}_K(A))}$ or any of its positive powers.

When $\text{core}_K(A)$ is a nonzero polyhedral cone, we can write the induced permutation τ_A as a product of disjoint cycles. As observed by Pullman [90], each cycle of τ_A gives rise to a distinguished eigenvector of A. To see this, let σ be one such cycle. By an abuse of language, we shall refer to the extreme rays of $\text{core}_K(A)$ which are not fixed by σ as the extreme rays in the cycle σ. Choose a nonzero vector, say x, from one of the extreme rays in the cycle σ. Let d be the length (and hence also the order) of σ. Since A^d maps the ray generated by x onto itself, we have $A^d x = \lambda^d x$ for some positive number λ. Let $v = \sum_{i=0}^{d-1} \lambda^{-i} A^i v$. Then v is a nonzero vector of K and a straightforward computation shows that $Av = \lambda v$. It is easy to check that (up to multiples) the vector v is independent of the choice of the vector x from an extreme ray in σ. We shall call v the *distinguished eigenvector of A associated with the cycle σ* for the eigenvalue λ.

4.4. Distinguished A-invariant Faces

Let $A \in \pi(K)$. We call a nonzero face F of K a *distinguished A-invariant face* (for the eigenvalue ρ_F) if F is A-invariant (i.e., $AF \subseteq F$), and for any nonzero A-invariant face G properly included in F, we have $\rho_G < \rho_F$, where we denote by ρ_F the spectral radius of the restriction map $A|_{\text{span } F}$. The concept of a distinguished A-invariant face is a natural analog of the concept of a distinguished class for a nonnegative matrix. Note that if F is a distinguished A-invariant face, then the eigenvector of A in F corresponding to ρ_F is (up to multiples) unique and must lie in the relative interior of F. The following is a main result in the earlier part of [130].

Theorem 4.4. *Let $A \in \pi(K)$. Suppose that* $\mathrm{core}_K(A)$ *is nonzero and simplicial. Let τ_A denote the permutation induced by A on the set of extreme rays of* $\mathrm{core}_K(A)$. *For any cycle σ of τ_A, let F_σ denote the face of K generated by the distinguished eigenvector of A associated with σ. Then*

(i) *the association $\sigma \mapsto F_\sigma$ gives a one-to-one correspondence between the set of cycles of τ_A and the set of distinguished A-invariant faces of K for nonzero distinguished eigenvalues*;

(ii) *eigenvalues in the peripheral spectrum of $A|_{\mathrm{span}\ F_\sigma}$ are simple, and are precisely ρ_{F_σ} times all the d_σth roots of unity, where d_σ is the length of σ.*

It is clear that if $A \in \pi(K)$ is K-irreducible, then there is only one distinguished A-invariant face of K, namely, K itself. So, if A is an irreducible nonnegative matrix, then by Theorem 4.4, the induced permutation τ_A is itself a cycle, and the eigenvalues in the peripheral spectrum of A are simple, and are precisely $\rho(A)$ times all the hth roots of unity, where h is the length of τ_A as a cycle. Thus, we recover part of Theorem 1.2(ii) and part of the following geometric equivalent condition for irreducibility of a nonnegative matrix, as obtained by Pullman [90]:

An $n \times n$ nonnegative matrix P is irreducible if and only if P has no zero columns, the induced permutation τ_P is itself a cycle, say, with length d, and $\mathbb{R}^n$ can be written as a direct sum of d coordinate subspaces each containing exactly one extreme ray of core$_{\mathbb{R}^n_+}(P)$ in its positive orthant.

The proof of Theorem 4.4 (i) depends on the existence of a one-to-one correspondence between the set of distinguished A-invariant faces of $\mathrm{core}_K(A)$ and the set of distinguished A-invariant faces of K for nonzero distinguished eigenvalues, assuming only that $\mathrm{core}_K(A)$ is a nonzero cone (see [130, Theorem 3.13]); whereas the proof of Theorem 4.4 (ii) relies on part (iv) of the following result [130, Theorem 3.9]:

Theorem 4.5. *Let $A \in \pi(K)$ with $\rho(A) > 0$ and $\nu_{\rho(A)}(A) = 1$. Let M (respectively, N) denote the intersection of $\mathbb{R}^n$ with the direct sum of all eigenspaces (respectively, generalized eigenspaces) of A corresponding to eigenvalues with modulus $\rho(A)$ (respectively, with modulus less than $\rho(A)$). Denote by P the projection of $\mathbb{R}^n$ onto M along N. Then we have* :

(i) *There exists a subsequence of $((A/\rho(A))^k)_{k\in\mathbb{N}}$ which converges to P; hence $P \in \pi(K)$.*

(ii) $M = \mathrm{span}(M \bigcap K)$.

(iii) $M \bigcap K \subseteq \mathrm{core}_K(A)$.

(iv) *The peripheral spectrum of A and that of the restriction of A to* $\mathrm{span}(\mathrm{core}_K(A))$ *are the same, counting algebraic multiplicities.*

4.5. The Core of a Nonnegative Matrix

By the *support of a vector* $x = (\xi_1, \ldots, \xi_n)^T$, denoted by $\mathrm{supp}(x)$, we mean the set $\{i \in \langle n \rangle : \ \xi_i \neq 0\}$. If α is a class of an $n \times n$ nonnegative matrix P, we denote by F_α the set of all vectors in $\mathbb{R}^n_+$ whose supports are included in the union of all classes having access to α. It is readily checked that F_α is a P-invariant face of $\mathbb{R}^n_+$. Making use of the Frobenius-Victory theorem, it is not difficult to show the following [130, Lemma 4.1]:

Remark 4.6. Let P be an $n \times n$ nonnilpotent nonnegative matrix. Then the association $\alpha \mapsto F_\alpha$ gives a one-to-one correspondence between the set of distinguished classes of P and the set of distinguished P-invariant faces of $\mathbb{R}^n_+$, both for nonzero distinguished eigenvalues.

Below is a complete description of the core of a nonnegative matrix, as given in [130, Theorems 4.2 and 4.7].

By the *index of imprimitivity* of an irreducible nonnegative matrix we mean the cardinality of its peripheral spectrum.

Theorem 4.7. *Let P be an $n \times n$ nonnegative matrix with positive spectral radius. For each distinguished class α of P, denote by h_α the index of imprimitivity of the irreducible submatrix $P_{\alpha\alpha}$, and also by x^α the extremal distinguished eigenvector of P associated with the class α as given in part* (ii) *of the Frobenius-Victory theorem. Then we have the following* :

(i) *For each cycle σ of the induced permutation τ_P we associate with it the distingushed class α of P with the property that x^α is the distinguished eigenvector of P associated with the cycle σ. Then this association is a one-to-one correspondence between the set of cycles of τ_P and the set of distinguished classes of P for nonzero distinguished eigenvalues. Furthermore, if σ is a cycle of τ_P and α is the corresponding distinguished class, then the length of the cycle σ is equal to h_α.*

(ii) $\mathrm{core}_{\mathbb{R}^n_+}(P)$ *is a simplicial cone with $\sum h_\alpha$ extreme rays, where the summation is taken over all distinguished classes α of P for nonzero distinguished eigenvalues.*

(iii) *Each distinguished class α of P for a nonzero distinguished eigenvalue gives rise to h_α distinct (up to multiples) extreme vectors of* $\mathrm{core}_{\mathbb{R}^n_+}(P)$, *which are precisely the extremal distinguished eigenvectors of P^{h_α} associated with the h_α noncomparable distinguished classes of P^{h_α} into which the class α of P splits.*

In part (iii) of Theorem 4.7, we are using certain facts that relate the classes of a nonnegative matrix to those of its positive powers. More specifically, if α is a class of a nonnegative matrix P such that the index of imprimitivity of the corresponding submatrix $P_{\alpha\alpha}$ is h_α, then for any positive integer q, the class α of P splits into d_α noncomparable classes of P^q, where d_α is the greatest common divisor of q and h_α. Furthermore, if α is a distinguished class of P, then the classes of P^q into which α splits are also all distinguished. The proof depends on the well-known fact that if P is an irreducible nonnegative matrix, then each of its positive powers is permutationally similar to a direct sum (possibly with only one summand) of irreducible matrices each having the same spectral radius. The usual proof for the latter fact is matrix computational. However, in [130, Lemma 4.5 and Corollary 4.6] we give conceptual proofs, that involve an interesting interplay between the geometric idea of the core and the combinatorial idea of classes.

4.6. An Application

In [130], we studied the core of a cone-preserving map, out of our interest in its spectral properties. Actually, the results we obtained on the core of a nonnegative matrix can also provide a quick way to rederive many known results on the limiting behaviour of Markov chains (see, for instance, Pullman [89], Chi [26], and Sierksma [109]). We illustrate this by considering the question treated in [109]. Let us quote the first paragraph of the Introduction of the paper:

"The following question was our motivation of looking at the limiting process of discrete-time Markov chains. Let V_0 be n points in the Euclidean space $\mathbb{R}^d$ ($n, d \geq 1$). Consider n convex combinations of these n vertices. Let V_1 be the set of points when these n convex combinations are successively applied on V_0. Clearly, $V_1 \subseteq \operatorname{conv}(V_0)$. Applying the same convex combinations again on V_1 leads to the set of points V_2 with $V_2 \subseteq \operatorname{conv}(V_1) \subseteq \operatorname{conv}(V_0)$, etc. Define $V_\infty = \bigcap_{i=1}^{\infty} \operatorname{conv}(V_i)$, called a limiting polytope. It is well-known, see, e.g., Pullman [1965], that V_∞ is in fact a polytope. Questions that may arise are: What is the 'limit' V_∞? When is V_∞ precisely one point? What are the extreme points of V_∞?"

To tackle the above problem, we first reformulate it in terms of matrices. Let P be a $d \times n$ real matrix whose column vectors constitute the set V_0, and let A be the $n \times n$ column stochastic matrix whose n column vectors correspond to the n covex combinations under consideration. (Since we represent points in $\mathbb{R}^d$ by column vectors, we work with a column stochastic matrix, instead of a stochastic matrix as in [109].) Then the set V_1 consists of the column vectors of the matrix PA, and for each $i = 1, 2, \ldots, \operatorname{conv}(V_i)$ is simply the convex hull of the column vectors of the matrix PA^i. Let $\triangle$ denote the standard simplex of $\mathbb{R}^n$. It is obvious that $\bigcap_{i=0}^{\infty} A^i \triangle = \operatorname{core}_{\mathbb{R}^n_+}(A) \bigcap \triangle$. But $\operatorname{core}_{\mathbb{R}^n_+}(A)$ is a simplicial cone (as A is a nonnegative matrix) and the affine hull of $\triangle$ does not contain the origin, it follows that $\bigcap_{i=0}^{\infty} A^i \triangle$ is a simplex. Let S denote a matrix with column vectors formed by the extreme points of this simplex. We contend that the set V_∞ is equal to

$PS(\triangle)$, the convex hull of the column vectors of the matrix PS. To see this, first choose any vector y from the latter set. Then y can be written as PSx for some probability vector x. Now $Sx \in \bigcap_{i=0}^{\infty} A^i\triangle$, so for each nonnegative integer i, we have $Sx = A^i u$ for some probability vector u, and hence $PSx = PA^i u \in \mathrm{conv}(V_i)$. This shows that $y \in V_\infty$. Conversely, if $y \in V_\infty$, then for each nonnegative integer i, there exists a probability vector u_i such that $y = PA^i u_i$. Since $(A^i u_i)_{i\in\mathbb{N}}$ is a bounded sequence of vectors in $\mathbb{R}^n$, it has a convergent subsequence with a limit, say v. Then $y = Pv$, and $v \in \bigcap_{i=0}^{\infty} A^i\triangle$ (see [130, Remark 3.10]). By definition of S, we have $v = Su$ for some probability vector u. Hence, $y = PSu \in PS(\triangle)$. This proves our contention. Since the core of a nonnegative matrix is already completely determined (see Theorem 4.7), we can readily determine the simplex $\bigcap_{i=1}^{\infty} A^i\triangle$, and then the matrix S, and hence the set V_∞. We leave to the reader to supply the details and to compare our approach with that given by Sierksma [109].

4.7. Characterizations in Terms of the Core

4.7.1. A Characterization of K-irreducibility

In [130, Theorem 5.7], we prove that if $A \in \pi(K)$ satisfies the (obvious necessary) conditions $\mathcal{N}(A)\bigcap K = \{0\}$ and $\mathrm{core}_K(A)\bigcap \mathrm{int}\ K \neq \emptyset$, then in order that A be K-irreducible it is necessary and sufficient that the restriction of A to span(core$_K(A)$) is irreducible with respect to core$_K(A)$. We also show that, in case core$_K(A)$ is simplicial, we may replace the latter condition simply by "the induced permutation τ_A is itself a cycle". (See [130, Corollary 5.8].) Then in the paper it is explained how we can recover the geometric condition for irreducibility of a nonnegative matrix, as obtained by Pullman [90, Theorem 6.1].

4.7.2. An Interesting Useful Result

In [131, Theorem 5.10], we give a list of equivalent conditions for a non-nilpotent $A \in \pi(K)$ to have spectral radius with index one. In addition, when K is polyhedral, a further equivalent condition is that, A is an automorphism of the cone $M\bigcap K$ when restricted to its linear span, where M denotes the intersection of $\mathbb{R}^n$ with the direct sum of all generalized eigenspaces of A corresponding to eigenvalues in its peripheral spectrum. One direction of the proof depends on the following result [130, Theorem 5.9], which has interest of its own:

Theorem 4.8. *Let A be an $n \times n$ real matrix. The following conditions are equivalent* :

(a) *A is nonzero, diagonalizable, all eigenvalues of A are of the same modulus and $\rho(A)$ is an eigenvalue of A.*

(b) *There exists a proper cone K such that $A \in \pi(K)$ and A has an eigenvector in* int K.

(c) *There exists a proper cone* K *such that* $A \in \pi(K)$, *and for any such cone* K, *we have*, $A \in \mathrm{Aut}(K)$.

(d) *There exists a proper cone* K *such that* $A \in \pi(K)$, *and for any such cone* K, *we have* $A \in \mathrm{Aut}(K)$ *and* A *has an eigenvector in* int K.

(e) *There exists a proper cone* K *such that* A *takes a complete compact cross-section of* K *onto itself.*

The proof of Theorem 4.8 as given in [130] depends on a number of previous results. To give the reader a feeling of the type of arguments involved, let us indicate a new direct proof for the implication (a)$\Longrightarrow$(d):

Clearly when condition (a) is fulfilled, A satisfies the Perron-Schaefer condition. So there exists a proper cone K such that $A \in \pi(K)$. For any such cone K, we now apply Theorem 4.5. Note that in this case the peripheral spectrum and the spectrum of A coincide; so the subspace M in Theorem 4.5 is simply $\mathbb{R}^n$ (= span K). By part (iii) of Theorem 4.5, it follows that $K = \mathrm{core}_K(A)$; in other words, $A \in \mathrm{Aut}(K)$. By condition (a), it is also clear that $\rho(A)$ is the only distinguished eigenvalue of A for K, and $\nu_{\rho(A)}(A) = 1$. So, by Theorem 3.4, A^T has an eigenvector in int K^*. It is not difficult to show that $A \in \mathrm{Aut}(K)$ if and only if $A^T \in \mathrm{Aut}(K^*)$. Hence, we have $A^T \in \mathrm{Aut}(K^*)$ and A^T has an eigenvector in int K^*. By another nontrivial result (see [130, Lemma 5.6]), the latter condition is equivalent to that $A \in \mathrm{Aut}(K)$ and A has an eigenvector in int K. Thus, condition (d) follows.

4.8. More About the Core

When $\mathrm{core}_K(A)$ need not be polyhedral, the best description about the elements of $\mathrm{core}_K(A)$, as given in [130], is the following:

Theorem 4.9. *Let* $A \in \pi(K)$. *For each nonzero distinguished eigenvalue* λ *of* A, *denote by* W_λ *the intersection of* $\mathbb{R}^n$ *with the direct sum of all eigenspaces of* A *corresponding to eigenvalues with moduli equal to* λ. *For each positive integer* k, *let* $D_k(A)$ *be the cone generated by the distinguished eigenvectors of* A^k *corresponding to its nonzero distinguished eigenvalues for* K. *Then* $\bigcup_{i=1}^{\infty} D_i(A) \subseteq \bigoplus(W_\lambda \bigcap K) \subseteq \mathrm{core}_K(A)$, *where the direct sum is taken over all nonzero distinguished eigenvalues* λ *of* A. *When* $\mathrm{core}_K(A)$ *is polyhedral*, *the inclusions become equalities.*

4.8.1. The Core of a K-irreducible Matrix

We would like to take this opportunity to add the following new result:

Theorem 4.10. *Let* $A \in \pi(K)$ *be nonnilpotent. If* $\rho(A)$ *is the only distinguished eigenvalue of* A *for* K *and* $\nu_{\rho(A)} = 1$ (*which is the case if* A *is* K*-irreducible*), *then* $\mathrm{core}_K(A)$ *is precisely the convex cone* $W \bigcap K$*, where* W *is the intersection of* $\mathbb{R}^n$ *with the direct sum of all eigenspaces of* A *corresponding to eigenvalues in the peripheral spectrum of* A*.*

Proof. Since $\rho(A)$ is the only distinguished eigenvalue of A and $\nu_{\rho(A)}(A) = 1$, by Theorem 3.4, A^T has an eigenvector in int K^* corresponding to $\rho(A)$, say w. Normalizing A, we may assume that $\rho(A) = 1$. Let C denote the set $\{y \in K : \langle w, y\rangle = 1\}$. Then C is a complete cross-section of A, which is compact convex and is invariant under A. Choose an eigenvector u of A corresponding to $\rho(A)$ that lies in C. Then $C-u$ is a compact convex body of (span $\{w\})^\perp$, which is also invariant under A. By Theorem 4.5 (iii), we have $W \bigcap K \subseteq \mathrm{core}_K(A)$, and as a consequence we also have $(W \bigcap C) - u \subseteq \bigcap_{i=0}^{\infty} A^i(C-u)$. Now (span $\{w\})^\perp$ is the direct sum of two A-invariant subspaces, namely, $W \bigcap (\mathrm{span}\ \{w\})^\perp$ and the intersection of $\mathbb{R}^n$ with the direct sum of all generalized eigenspaces of A corresponding to eigenvalues with moduli less than 1 $(= \rho(A))$. Since the kth power of the restriction of A to the latter subspace tends to the zero operator as k tends to infinity, it is not difficult to show that $\bigcap_{i=0}^{\infty} A^i(C-u) \subseteq W \bigcap (C-u)$ $(= (W \bigcap C) - u)$, from which it follows that $\bigcap_{i=1}^{\infty} A^i C = W \bigcap C$, and hence $\mathrm{core}_K(A) = W \bigcap K$. ■

By Theorem 4.10, if $A \in \pi(K)$ satisfies the hypothesis of the theorem, then we must have span($\mathrm{core}_K(A)) \bigcap K = \mathrm{core}_K(A)$. Note that the latter property is not shared by the core of a cone-preserving map in general, not even for a general nonnegative matrix A. As a simple example, consider $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$. We have $\mathrm{core}_{\mathbb{R}^2_+}(A) = \mathrm{pos}\left\{\binom{1}{1}, \binom{0}{1}\right\} \neq \mathbb{R}^2_+ = \mathrm{span}\ \mathrm{core}_{\mathbb{R}^2_+}(A) \bigcap \mathbb{R}^2_+$.

4.8.2. A Characterization of K-primitivity

If A is K-primitive, then A is K-irreducible and $\rho(A)$ is a simple eigenvalue of A, which is also the only eigenvalue in the peripheral spectrum of A; so by Theorem 4.10, $\mathrm{core}_K(A)$ is a single ray (which is generated by a vector in int K, namely the Perron vector of A). Conversely, if $\mathrm{core}_K(A)$ is a single ray generated by a vector in int K and if, in addition, A satisfies the condition $\mathcal{N}(A) \bigcap K = \{0\}$, then by the characterization of K-irreducibility of A in terms of $\mathrm{core}_K(A)$ (see Subsection 4.7.1), A is K-irreducible. Then by Theorem 4.10 again, the fact that $\mathrm{core}_K(A)$ is a single ray implies that $\rho(A)$ is the only eigenvalue in the periperal spectrum of A, and hence A is K-primitive.

In the above, we have provided another way to recover the characterization of K-primitivity of A in terms of $\mathrm{core}_K(A)$ as given in [130, Theorem 5.6].

4.8.3. Special Linear Subspaces

According to Theorem 4.10, if A is K-irreducible, then $\text{core}_K(A)$ must be of the form $W \bigcap K$ for some linear subspace W that meets the interior of K. In below we characterize all possible candidates for such W. Our proof also suggests a method to construct examples of K-irreducible matrices.

Theorem 4.11. *Let K be a proper cone in $\mathbb{R}^n$. For any nonzero linear subspace W of $\mathbb{R}^n$, the following conditions are equivalent* :

(a) *There exists a K-irreducible matrix A such that W is equal to the intersection of $\mathbb{R}^n$ with the direct sum of all eigenspaces of A corresponding to eigenvalues that lie in the peripheral spectrum of A.*

(b) *There exists a K-irreducible matrix A such that $\text{core}_K(A) = W \bigcap K$.*

(c) *W satisfies all of the following* :

  (i) $W \bigcap \text{int } K \neq \emptyset$;

  (ii) *there exists an idempotent matrix $P \in \pi(K)$ such that $\mathfrak{R}(P) = W$ and $\mathcal{N}(P) \bigcap K = \{0\}$;*

  (iii) Aut$(W \bigcap K)$ *contains an operator which is irreducible with respect to $W \bigcap K$.*

Proof. The equivalence of conditions (a) and (b) follows from Theorem 4.10.

Suppose that conditions (a) and (b) both hold, and let A be a K-irreducible matrix with the properties as described in (a) and (b). Since W contains the Perron vector of A, which lies in int K, clearly we have $W \bigcap \text{int } K \neq \emptyset$, which is condition (c)(i). Note that the latter condition implies that span $\text{core}_K(A) = \text{span}(W \bigcap K) = W$. Let P denote the projection of $\mathbb{R}^n$ onto W along the intersection of $\mathbb{R}^n$ with the direct sum of all generalized eigenspaces of A corresponding to eigenvalues with moduli less than $\rho(A)$. Since $\nu_{\rho(A)}(A) = 1$ (as A is K-irreducible), by Theorem 4.5 we have $P \in \pi(K)$. Moreover, $\mathcal{N}(P) \bigcap K$ cannot be a nonzero cone; otherwise, since $\mathcal{N}(P)$ is an A-invariant subspace, by applying the Perron-Frobenius theorem to $A|_{\mathcal{N}(P) \bigcap K} \in \pi(\mathcal{N}(P) \bigcap K)$, we infer that A has a distinguished eigenvalue other than $\rho(A)$, which is a contradiction. So we have condition (c)(ii). Since A is K-irreducible, $A|_{\text{span}(\text{core}_K(A))}$ is also irreducible with respect to $\text{core}_K(A)$ $(= W \bigcap K)$. But $A|_{\text{span}(\text{core}_K(A))}$ is also an automorphism of $\text{core}_K(A)$, hence condition (c)(iii) follows. This shows that (a) implies (c).

Suppose that condition (c) holds. Choose any $Q \in \text{Aut}(W \bigcap K)$ which is irreducible with respect to $W \bigcap K$, and let A be the operator P followed by Q. Then it is clear that $A \in \pi(K)$ and $\text{core}_K(A) = W \bigcap K$. Furthermore, we have $\mathcal{N}(A) \bigcap K = \mathcal{N}(P) \bigcap K = \{0\}$, $\text{core}_K(A) \bigcap \text{int } K \neq \emptyset$ (as $W \bigcap \text{int } K \neq \emptyset$), and $A|_{\mathrm{span}(\mathrm{core}_K(A))} = Q$ is irreducible with respect to $\mathrm{core}_K(A)$. So A is K-irreducible. This establishes the implication (c)$\Longrightarrow$(b), thus completing the proof. ■

Remark 4.12. According to Gritzmann, Klee and Tam [52, Theorem 5.5], when $n \geq 3$, for "almost all" proper cones K in $\mathbb{R}^n$ (in the sense of Baire category), $\mathrm{Aut}(K)$ consists of scalar matrices only. So, in general, condition (c)(iii) of Theorem 4.11 is hard to satisfy.

We call a proper cone K *self-dual* if $K = K^*$.

Corollary 4.13. *For any linear subspace W of $\mathbb{R}^n$, the following conditions are equivalent* :

(a) *There exists an $n\times n$ irreducible nonnegative matrix A such that* $\mathrm{core}_{\mathbb{R}^n_+}(A) = W \bigcap \mathbb{R}^n_+$.

(b) *$W \bigcap \mathrm{int}\ \mathbb{R}^n_+ \neq \emptyset$ and $W \bigcap \mathbb{R}^n_+$ is a simplicial self-dual cone.*

(c) *$W \bigcap \mathrm{int}\ \mathbb{R}^n_+ \neq \emptyset$ and $P_W \in \pi(\mathbb{R}^n_+)$, where P_W denotes the orthogonal projection of $\mathbb{R}^n$ onto W.*

Proof. (a)$\Longrightarrow$(b): By Theorem 4.11, we have $W \bigcap \mathrm{int}\ \mathbb{R}^n_+ \neq \emptyset$, and there exists a nonnegative idempotent matrix P such that $\mathfrak{R}(P) = W$ and $\mathcal{N}(P) \bigcap \mathbb{R}^n_+ = \{0\}$. As a nonnegative idempotent matrix, P is expressible as $x_1 y_1^T + \cdots + x_r y_r^T$, where the x_i's and y_i's are nonnegative vectors of $\mathbb{R}^n$ that satisfy $y_i^T x_j = \delta_{ij}$ for all $i,\ j$, where δ_{ij} denotes the Kronecker delta symbol (see Tam [117, Corollary 4.7]). It is ready to see that we have $W \bigcap \mathbb{R}^n_+ = \mathfrak{R}(P) \bigcap \mathbb{R}^n_+ = P(\mathbb{R}^n_+) = \mathrm{pos}\{x_1, \ldots, x_r\}$, and $x_1, \ldots, x_r$ are the distinct extreme vectors of this cone. But W satisfies $\mathrm{span}(W \bigcap \mathbb{R}^n_+) = W$ and is the range space of the nonnegative idempotent matrix P. By Tam [117, Corollary 4.6] it follows that $\mathrm{pos}\{x_1, \ldots, x_r\}$ is a simplicial cone. Note that if the supports of the vectors $x_1, \ldots, x_r$ are not pairwise disjoint, say $k \in \mathrm{supp}(x_1) \bigcap \mathrm{supp}(x_2)$, then in view of the assumptions $y_i^T x_j = \delta_{ij}$ for all $i,\ j$, the kth components of the vectors $y_1, \ldots, y_r$ must all be zero; but then the kth standard unit vector e_k of $\mathbb{R}^n$ will be a semipositive vector that belongs to $\mathcal{N}(P)$, which contradicts our assumption on P. Hence, the vectors $x_1, \ldots, x_r$ have disjoint supports and so they are mutually orthogonal. But $x_1, \ldots, x_r$ are the extreme vectors of the simplicial cone $W \bigcap \mathbb{R}^n_+$, so the cone must be self-dual.

(b)$\Longrightarrow$(c): Let $x_1, \ldots, x_r$ be the distinct extreme vectors of the simplicial self-dual cone $W \bigcap \mathbb{R}^n_+$. After normalizing the vectors, we may assume that $x_j^T x_i = \delta_{ij}$ for all $i,\ j$. Then $x_1 x_1^T + \cdots + x_r x_r^T$ is a nonnegative, symmetric, idempotent matrix with $\mathfrak{R}(P) = \mathrm{span}\,\{x_1, \ldots, x_r\} = W$. Hence, we have $P_W = x_1 x_1^T + \cdots + x_r x_r^T \in \pi(\mathbb{R}^n_+)$.

(c)$\Longrightarrow$(a): It suffices to show that conditions (c)(i)–(iii) of Theorem 4.11 are all satisfied. We already have (c)(i). Since $W \bigcap \mathrm{int}\ \mathbb{R}^n_+ \neq \{0\}$, $W^\perp$ cannot contain a semipositive vector; so we have $\mathcal{N}(P_W) \bigcap \mathbb{R}^n_+ = \{0\}$ and hence condition (c)(ii). Since W is the range space of the nonnegative idempotent matrix P_W, by Tam [117, Corollary 4.6], $W \bigcap \mathbb{R}^n_+$ is a simplicial cone. But then condition (c)(iii) is trivially satisfied. ■

In the proof of Corollary 4.13, (a)$\Longrightarrow$(b), we could have deduced the pairwise disjointness between the supports of the vectors $x_1, \ldots, x_r$ by invoking Pullman's geometric condition that $\mathbb{R}^n$ is a direct sum of the r coordinate subspaces each containing one of the vectors $x_1, \ldots, x_r$ in its positive orthant. That we can do otherwise means that our above sequence of results (Theorems 4.10, 4.11, and Corollary 4.13, together with other supporting results) provide yet another way to arrive at Pullman's geometric condition.

4.9. The Core and the Peripheral Spectrum

In view of the results we have obtained, given $A \in \pi(K)$, it is natural to ask the question of when $\text{core}_K(A)$ is polyhedral, and when it is simplicial. In general, these problems seem intractable. Even for the question of when $\text{core}_K(A)$ is a single ray, there is no complete satisfactory answer (see [130, Section 5]). The point is, a matrix can leave invariant two different proper cones such that the cores of the matrix relative to these cones are quite different. As a simple example, take A to be the 2×2 matrix diag(2,1). Then $A \in \pi(\mathbb{R}^2_+)$ and $\text{core}_{\mathbb{R}^2_+}(A) = \mathbb{R}^2_+$. On the other hand, if we take $K = \text{pos}\,\{e_1,\ e_1 + e_2\}$, where $e_1,\ e_2$ denote the standard unit vectors of $\mathbb{R}^2$, then we also have $A \in \pi(K)$, but in this case $\text{core}_K(A)$ is the single ray generated by the vector e_1.

In order to obtain further fruitful results, we modify our problems as follow:

Given an $n \times n$ real matrix A that satisfies the Perron-Schaefer condition, find an equivalent condition on A so that there exists a proper cone K such that $A \in \pi(K)$ and $\text{core}_K(A)$ is polyhedral, a single ray, or simplicial.

Below are the answers to the above problems, as given in Section 7 of [130]. As the reader will see, the answers are all given in terms of the peripheral spectrum of the matrix under consideration.

Theorem 4.14. *Let A be an $n \times n$ real matrix. Then there exists a proper cone K such that $A \in \pi(K)$ and* $\text{core}_K(A)$ *is polyhedral if and only if A satisfies the Perron-Schaefer condition, and every eigenvalue in the peripheral spectrum of A with the same index as that of $\rho(A)$ is equal to $\rho(A)$ times a root of unity.*

Theorem 4.15. *Let A be an $n \times n$ matrix. Then there exists a polyhedral cone K such that $A \in \pi(K)$ if and only if A satisfies the Perron-Schaefer condition, and every eigenvalue in the peripheral spectrum of A is equal to $\rho(A)$ times a root of unity.*

Theorem 4.16. *Let A be an $n \times n$ real matrix. Then there exists a proper cone K such that $A \in \pi(K)$ and $\mathrm{core}_K(A)$ is a single ray if and only if $\rho(A) > 0$, and the Jordan form of A has exactly one block of maximal order corresponding to $\rho(A)$, and the index of $\rho(A)$ is greater than that of every other eigenvalue in the peripheral spectrum of A.*

We would like to mention that the problem of determining an invariant polyhedral cone (with certain specific properties) of a given matrix arises in the study of nonnegative realization problems (the discrete or continuous version). We refer our reader to Förster and Nagy [40, 41], and van den Hof [135] for the details.

For convenience, to answer the question of when there exists a proper cone K such that $A \in \pi(K)$ and $\mathrm{core}_K(A)$ is a nonzero simplicial cone, we normalize the given matrix A and assume that $\rho(A) = 1$.

Theorem 4.17. *Let $A \in \mathcal{M}_n(\mathbb{R})$ with $\rho(A) = 1$ that satisfies the Perron-Schaefer condition. Let S denote the multi-set of eigenvalues in the peripheral spectrum of A with maximal index (i.e., $\nu_1(A)$), the multiplicity of each element being equal to the number of corresponding blocks in the Jordan form of A of order $\nu_1(A)$. Let T be the multi-set of eigenvalues in the peripheral spectrum of A for which there are corresponding blocks in the Jordan form of A of order less than $\nu_1(A)$, the multiplicity of each element being equal to the number of such corresponding blocks. Also let Z_m denote the set $\{e^{2\pi t i/m} : \ t = 0, \ldots, m-1\}$. Then there exists a proper cone K such that $A \in \pi(K)$ and $\mathrm{core}_K(A)$ is simplicial if and only if there exists a multi-set $\widetilde{T}$ of T such that $S \bigcup \widetilde{T}$ is the multi-subset union of certain Z_m's.*

To illustrate the condition given in Theorem 4.17, consider the 24×24 real matrix A with Jordan form $J_3(1) \oplus J_2(1) \oplus J_1(1) \oplus J_3(-1) \oplus J_3(-1) \oplus J_3(e^{2\pi i/3}) \oplus J_3(e^{4\pi i/3}) \oplus J_2(-1) \oplus J_2(e^{\pi i\theta}) \oplus J_2(e^{-\pi i\theta})$, where θ is irrational. Then $S = \{1, -1, -1, e^{2\pi i/3}, e^{4\pi i/3}\}$ and $T = \{1, 1, -1, e^{\pi i\theta}, e^{-\pi i\theta}\}$. Take $\widetilde{T} = \{1, 1\}$. Then $S \bigcup \widetilde{T} = Z_2 \bigcup Z_2 \bigcup Z_3$; i.e., the condition of Theorem 4.17 is satisfied.

To prove the "only if" parts of Theorems 4.14–4.17, we need the following result about the peripheral spectrum of a linear mapping preserving a polyhedral cone:

Remark 4.18. If $A \in \pi(K)$, where K is a polyhedral cone, then each eigenvalue in the peripheral spectrum of A is equal to $\rho(A)$ times a root of unity.

The result first appeared in Barker and Turner [13, Theorem 2]. But their proof is invalid, as pointed out by this author in [121]. Let us see why.

In the paper, the result is already established for a K-irreducible matrix A. To complete the proof by a limiting argument (as suggested by these authors), we need to show that if λ is any eigenvalue in the peripheral spectrum of a K-reducible matrix A, different from $\rho(A)$, then given any $\varepsilon > 0$, we can always find a K-irreducible matrix B_ε, together with an eigenvalue μ in the peripheral spectrum of B_ε, such that $|\lambda - \mu| < \varepsilon$. But given the general nature of K, A and λ, and our present rather inadequate knowledge of the elements of $\pi(K)$, the construction of the desired B_ε and μ is beyond our capability. In the paper, it is suggested that B_ε can be chosen to be A plus a small positive multiple of a rank-one K-positive matrix. But then the matrix B_ε is also K-positive and cannot serve our purpose, since the peripheral spectrum of a K-positive matrix contains only one element, namely, its spectral radius. In fact, even we now know that λ equals $\rho(A)$ times a root of unity, we still cannot see any way to construct the desired B_ε and μ.

A correct proof of Remark 4.18 was given in [121, Theorem 7.6]; it depends on the concept of a minimal generating matrix for a polyhedral cone, together with a couple of other results (in particular, Theorem 3.5 of this paper).

Now back to the proof of the "only if" parts of Theorems 4.14–4.17. We consider only the "only if" part of Theorem 4.14. Let us denote by S the set $\{\lambda \in \sigma(A) : |\lambda| = \rho(A),\ \nu_\lambda(A) = \nu_{\rho(A)}(A)\}$. It suffices to show that $S \subseteq \sigma(A|_{\mathrm{span}(\mathrm{core}_K(A))})$, as $A|_{\mathrm{span}(\mathrm{core}_K(A))} \in \pi(\mathrm{core}_K(A))$, $\rho(A|_{\mathrm{span}(\mathrm{core}_K(A))}) = \rho(A)$, and $\mathrm{core}_K(A)$ is polyhedral. Let $M = \bigoplus_\lambda[(\lambda I_n - A)^{\nu-1}\mathcal{N}((\lambda I_n - A)^\nu)]$, where $\nu = \nu_{\rho(A)}(A)$ and the direct sum is taken over all eigenvalues $\lambda \in S$. It is clear that M is an A-invariant subspace and $\sigma(A|_M) = S$. Hence, it reduces to proving the following result [130, Theorem 7.1], which is an extension of Theorem 4.5, (i)–(iii).

Theorem 4.19. *Let $A \in \pi(K)$ with $\rho(A) > 0$. Denote $\nu_{\rho(A)}(A)$ by ν. For any eigenvalue λ of A and any nonnegative integer r, let $E_\lambda^{(r)}$ denote the component of A given by $E_\lambda^{(r)} = (A - \lambda I)^r E_\lambda^{(0)}$, where $E_\lambda^{(0)}$ is the projection of $\mathbb{C}^n$ onto the generalized eigenspace of A corresponding to λ along the direct sum of generalized eigenspaces of A corresponding to eigenva Then we have the following* :

(i) *There is a subsequence of $((\nu - 1)!A^k/[\rho(A)^{k-\nu+1}k^{\nu-1}])_{k\in\mathbb{N}}$ which converges to $\sum_\lambda E_\lambda^{(\nu-1)}$, where the summation runs through all eigenvalues λ in the peripheral spectrum of A with the same index as that of $\rho(A)$. Hence $(\sum_\lambda E_\lambda^{(\nu-1)})|_{\mathbb{R}^n} \in \pi(K)$.*

(ii) *Let M denote the intersection of $\mathbb{R}^n$ with $\bigoplus_\lambda[(\lambda I_n - A)^{\nu-1}\mathcal{N}((\lambda I_n - A)^\nu)]$, where λ runs through the same set of eigenvalues as that described in the sum appearing in part* (i). *Then $M = \mathrm{span}(M \bigcap K)$ and $M \bigcap K \subseteq \mathrm{core}_K(A)$.*

To prove the "if" parts of Theorems 4.14–4.17, we need to construct the invariant proper cone K with the desired properties. The constructions are fairly long and technical. We omit the details.

4.10. The Nonnegative Inverse Elementary Divisor Problem

Before we end this section, we would like to mention that Theorem 4.17, a climax of the paper [131], also sheds light on the Nonnegative Inverse Elementary Divisor Problem. Recall first the famous Nonnegative Inverse Eigenvalue Problem:

Determine a necessary and sufficient condition for a complex n-tuple to be the spectrum of an $n \times n$ nonnegative matrix.

The latter problem has aroused a lot of research activities and a major breakthrough was made in 1991 by Boyle and Handelman [20] using symbolic dynamics: they settled completely the question of when an n-tuple of nonzero complex numbers is the nonzero part of the spectrum of a primitive matrix, and hence also the question of when a given n-tuple can be the nonzero part of the spectrum of a nonnegative matrix. On the other hand, only some initial work has been done on the more difficult Nonnegative Inverse Elementary Divisors Problem of determining a necessary and sufficient condition for a complex matrix to be similar to a nonnegative matrix. (For more information, see Berman and Plemmons [17, Chapter 4, Section 2 and Chapter 11, Section 2] and Minc [79, Chapter VII].) Since the core of a nonnegative matrix (relative to the corresponding nonnegative orthant) is always simplicial, Theorem 4.17 has the following unexpected, highly nontrivial consequence:

If A is an $n \times n$ nonnilpotent nonnegative matrix normalized so that $\rho(A) = 1$, then the Jordan blocks of A corresponding to eigenvalues that lie in its peripheral spectrum must satisfy the condition given in Theorem 4.17.

It would be interesting to find a direct proof of the above result, one that does not involve the concept of the core.

Recently, with B.G. Zaslavsky, this author also treated the related Inverse Elementary Divisor Problem for an eventually nonnegative matrix. In terms of the new concept of a Frobenius collection of Jordan blocks, we characterize the collection of Jordan blocks that appear in the Jordan form of an irreducible m-cyclic eventually nonnegative matrix whose mth power is permutationally similar to a direct sum of m eventually positive matrices. For the details, see Zaslavsky and Tam [144].

5. The Invariant Faces

5.1. Motivation

The A-invariant faces of the cone K are the focus of interest in Tam and Schneider [131]. The motivation is clear: as shown in the preceding paper [130], the set $\mathrm{core}_K(A)$ does not capture all the important information about the spectral properties of A. In particular, if $\mathrm{core}_K(A)$ is a polyhedral cone, then it does not contain any distinguished generalized eigenvectors of A other than eigenvectors (see our Remark 4.3). This means that the index of the spectral radius of A cannot be determined from a knowledge of its core. On the other hand, in the nonnegative matrix case, by the Rothblum Index Theorem, the index of the spectral radius can be described in terms of its classes. Also, the work of [130] shows that there is a close connection between the distinguished classes of a nonnegative matrix and its distinguished invariant faces. This suggests that a study of the invariant faces associated with a cone-preserving map may be worthwhile.

5.2. Invariant Faces of a Nonnegative Matrix

For a proper cone K, we use $\mathcal{F}(K)$ to denote the set of all faces of K. Under inclusion as the partial ordering, $\mathcal{F}(K)$ forms a lattice with meet and join given by: $F \wedge G = F \bigcap G$ and $F \vee G = \Phi(F \bigcup G)$. If $A \in \pi(K)$, then the set of all A-invariant faces of K, which we denote by $\mathcal{F}_A$, forms a sublattice of $\mathcal{F}(K)$.

The paper [131] begins by examining the nonnegative matrix setting thoroughly in order to see clearly the ideas for treating deeper spectral questions for matrices leaving invariant a proper cone in $\mathbb{R}^n$.

It is well-known that each face of $\mathbb{R}^n_+$ is of the form

$$F_I = \{x \in \mathbb{R}^n_+ : \ \mathrm{supp}(x) \subseteq I\},$$

where $I \subseteq \langle n \rangle$, and $\mathrm{supp}(x)$ is the support of x. Indeed, the association $I \mapsto F_I$ gives an isomorphism between the lattice $2^{\langle n \rangle}$ of all subsets of $\langle n \rangle$ and the face lattice $\mathcal{F}(\mathbb{R}^n_+)$ of $\mathbb{R}^n_+$, both under inclusion as the partial ordering. It turns out that if P is an $n \times n$ nonnegative matrix, then the P-invariant faces of $\mathbb{R}^n_+$ are all of the form F_I, where I is an initial subset for P. Here we call a subset I of $\langle n \rangle$ an *initial subset for* P if either I is empty, or I is nonempty and $P_{I'I} = 0$, where $I' = \langle n \rangle \backslash I$ and $P_{I'I}$ denotes the submatrix of P with rows indexed by I' and columns indexed by I; or equivalently, for every $j \in \langle n \rangle$, I contains j whenever j has access to I. It is not difficult to see that a nonempty subset I of $\langle n \rangle$ is an initial subset for P if and only if I is the union of an *initial collection of classes* of P, where a nonempty collection of classes of P is said to be initial if whenever it contains a class α, it also contains all classes having access to α. If $\mathcal{K}$ is a nonempty collection of classes of P, we call a class $\alpha \in \mathcal{K}$ *final in* $\mathcal{K}$ if it has no access to other classes of $\mathcal{K}$. An initial subset I for P is said to be *determined by a class* α of P if it is the union of all classes of P having access to α (or, in other words, α is the only class final in the initial collection corresponding to I).

Let $A \in \pi(K)$. We call an A-invariant face F of K *A-invariant join-reducible* if F is join-reducible in the lattice $\mathcal{F}_A$ in the usual lattice-theoretic sense; or, in other words, F is the join of two A-invariant faces of K that are properly included in F. An A-invariant face which is not A-invariant join-reducible is said to be *A-invariant join-irreducible*.

The following characterizations [131, Theorems 3.1 and 3.6] of various types of P-invariant faces associated with a nonnegative matrix P motivate the subsequent work in [131]:

Theorem 5.1. *Let P be an $n \times n$ nonnegative matrix. Denote by $\mathcal{I}$ the lattice of all initial subsets for P and by $\mathcal{F}_P$ the lattice of all P-invariant faces of $\mathbb{R}^n_+$. Then the association $I \longmapsto F_I$ induces an isomorphism from the lattice $\mathcal{I}$ onto the lattice $\mathcal{F}_P$. Furthermore, for any initial subset I for P, we have*

(i) *F_I is a minimal nonzero P-invariant face if and only if I is an initial class of P.*

(ii) *F_I is a nonzero P-invariant join-irreducible face if and only if I is an initial subset determined by a single class.*

(iii) *F_I is a P-invariant face which contains in its relative interior a generalized eigenvector (respectively, an eigenvector) of P corresponding to λ if and only if I is a nonempty initial subset such that each class final in the initial collection of classes corresponding to I is a semi-distinguished (respectively, distinguished) class associated with λ.*

(iv) *F_I is a P-invariant join-irreducible face which contains in its relative interior a generalized eigenvector (respectively, an eigenvector) of P corresponding to λ if and only if I is an initial subset determined by a semi-distinguished (respectively, distinguished) class associated with λ.*

Suggested by Theorem 5.1 (iv), we call a face F of K a *semi-distinguished A-invariant face* (associated with λ) if F is an A-invariant join-irreducible face which contains in its relative interior a generalized eigenvector of A (corresponding to λ). According to an earlier result of [131, Theorem 4.11], F is a distinguished A-invariant face of K if and only if F is an A-invariant join-irreducible face which contains in its relative interior an eigenvector of A. Thus, in the one-to-one correspondence $I \mapsto F_I$ between $\mathcal{I}$ and $\mathcal{F}_P$, an initial subset determined by a distinguished (respectively, semi-distinguished) class corresponds to a distinguished (respectively, semi-distinguished) A-invariant face. (The term "semi-distinguished class" and the important concept of semi-distinguished A-invariant faces are both introduced in [131].)

In the course of establishing Theorem 5.1, cone-theoretic proofs for the Frobenius-Victory theorem and an extension of the Nonnegative Basis Theorem (to the case of a distinguished eigenvalue, due to Hershkowitz and Schneider) are also provided (see [131, Theorems 3.3 and 3.4]).

5.3. An Extension of the Rothblum Index Theorem

The following extension of the Rothblum Index Theorem to the case of a linear mapping preserving a polyhedral cone is obtained in [131, Theorem 5.1]:

Theorem 5.2. *Let K be a polyhedral cone, and let $A \in \pi(K)$. Let λ be a distinguished eigenvalue of A for K. Denote by m_λ the maximal order of distinguished generalized eigenvectors of A corresponding to λ. Then there is a chain $F_1 \subset F_2 \subset \cdots \subset F_{m_\lambda}$ of m_λ distinct semi-distinguished A-invariant faces of K associated with λ, but there is no such chain with more than m_λ members.*

When K is a nonpolyhedral cone, the maximum cardinality of a chain of semi-distinguished A-invariant faces of K associated with a distinguished eigenvalue λ can be less than, equal to, or greater than m_λ, where m_λ has the same meaning as before (see [131, Examples 5.3–5.5]).

In order to give the reader the flavor of cone-theoretic methods, we are going to describe in some detail the fairly long proof of Theorem 5.2. (The same kind of arguments, but with much more elaboration, is used in the later parts of the paper to derive other results.) As can be readily seen, the proof should consist of two parts: to show the existence of the desired chain, and to establish the maximality of the chain. In the course of the proof, we introduce as a machinery the concept of the spectral pair of a face relative to a cone-preserving map. For the purpose, we begin with the concept of the spectral pair of a vector relative to a complex square matrix first.

5.3.1. The Spectral Pair

If A is an $n \times n$ complex matrix and x is a nonzero vector of $\mathbb{C}^n$, we can write $x = x_1 + \cdots + x_m$, where $x_1, \ldots, x_m$ are generalized eigenvectors of A corresponding respectively to the distinct eigenvalues $\lambda_1, \ldots, \lambda_m$. As mentioned before, one way to define $\rho_x(A)$, the local spectral radius of A at x, is to set $\rho_x(A) = \max_{1 \le i \le m} |\lambda_i|$. Now we also set $\mathrm{ord}_A(x) = \max\{\mathrm{ord}_A(x_i) : |\lambda_i| = \rho_x(A)\}$, where $\mathrm{ord}_A(x_i)$ denotes the order of the generalized eigenvector x_i, i.e., the least positive integer k such that $(\lambda_i I_n - A)^k x_i = 0$. We refer to $\mathrm{ord}_A(x)$ as the *order of x relative to A*; clearly, it extends the usual concept of the order of a generalized eigenvector to an arbitrary vector. Now, also set $\mathrm{sp}_A(x) = (\rho_x(A), \mathrm{ord}_A(x))$ and call it the *spectral pair of x relative to A*. (For the zero vector, set $\mathrm{sp}_A(\mathbf{0}) = (0, 0)$.) The following important observation is made in [131, Lemma 4.3]:

Remark 5.3. Let $A \in \pi(K)$. For any $x \in \mathrm{int}\ K$, we have

$$\mathrm{sp}_A(x) = (\rho(A), \nu_{\rho(A)}(A)).$$

Note that, according to Remark 5.3, we have $\rho_x(A) = \rho(A)$ for any $x \in \text{int } K$. This clearly extends the known fact that if x is an eigenvector or a generalized eigenvector of A that lies in int K, then necessarily x corresponds to the eigenvalue $\rho(A)$. The proof of the remark depends on the Perron-Schaefer condition for a cone-preserving map together with the following known result (see Schneider [102, Theorem 5.2]):

Theorem 5.4. *Let $A \in \pi(K)$. Let $E_\rho^{(0)}$ denote the projection of $\mathbb{C}^n$ onto the generalized eigenspace of A corresponding to $\rho(A)$ along the direct sum of other generalized eigenspaces of A. For each positive integer k, let $E_\rho^{(k)}$ denote the component of A given by $E_\rho^{(k)} = (A - \rho(A)I_n)^k E_\rho^{(0)}$. Then the restriction of $E_\rho^{(\nu-1)}$ to $\mathbb{R}^n$, where $\nu = \nu_{\rho(A)}(A)$, belongs to $\pi(K)$, and* $\text{rank}(E_\rho^{(\nu-1)})$ *is equal to the number of maximal Jordan blocks of A corresponding to $\rho(A)$.*

To show Remark 5.3, consider any $x \in \text{int } K$. According to Theorem 5.4, $E_\rho^{(\nu-1)}x$ belongs to K and must be a nonzero vector; otherwise, $E_\rho^{(\nu-1)}$ equals the zero operator, which is a contradiction. By the definition of $E_\rho^{(\nu-1)}$, it follows that in the representation of x as a sum of generalized eigenvectors of A, there must be a term which is a generalized eigenvector of A corresponding to $\rho(A)$ of order ν. On the other hand, by the Perron-Schaefer condition for A, ν cannot be less than the order of any generalized eigenvector that appears in the representation and corresponds to an eigenvalue with modulus $\rho(A)$. Thus, we have $\text{sp}_A(x) = (\rho(A), \nu)$.

If F is an A-invariant face of K, by applying Remark 5.3 to $A|_{\text{span } F}$, we see that $\text{sp}_A(x)$ is independent of the choice of x from the relative interior of F. In fact, the same remark also holds for any face F of K; the proof depends on the basic fact [131, Lemma 2.1] that for any $x \in K$, $\Phi((I_n + A)^{n-1}x)$ is the smallest A-invariant face of K containing x. So, for any face F of K, we use $\text{sp}_A(F)$ to denote $\text{sp}_A(x)$, where x is any vector chosen from relint F, and refer to it as the *spectral pair of F relative to A.* (Here a relevant basic fact is that, for any face F of K and any vector $x \in K$, we have $F = \Phi(x)$ if and only if $x \in \text{relint } F$.)

Note that the spectral pair of a face is not an extension of an existing concept for nonnegative matrices. However, it has proved to be a useful concept. This probably is due to the fact that its definition implicitly involves the Perron-Schaefer condition, a characterizing property for a cone-preserving map.

5.3.2. The Local Perron-Schaefer Condition

Now we make a digression and take note of the following useful observation [131, Theorem 4.7], which we exploit much in our future paper [132]:

Remark 5.5. If $A \in \pi(K)$, then for any $0 \neq x \in K$, the following condition is always satisfied:

There is a generalized eigenvector y of A corresponding to $\rho_x(A)$ that appears as a term in the representation of x as a sum of generalized eigenvectors of A. Furthermore, we have $\mathrm{ord}_A(x) = \mathrm{ord}_A(y)$.

The condition mentioned in Remark 5.5 is now called the *local Perron-Schaefer condition at x*. This author has proved the following interesting result, which will appear in another future paper Tam [128]:

Theorem 5.6. *Let A be an $n \times n$ real matrix, and let x be a given nonzero vector of $\mathbb{R}^n$. The following conditions are equivalent* :

(a) *A satisfies the local Perron-Schaefer condition at x.*
(b) *$A|_{W_x}$ satisfies the Perron-Schaefer condition.*
(c) *The convex cone* $\mathrm{cl}(\mathrm{pos}\ \{A^i x: \ i = 0, 1, \dots\})$ *is pointed.*
(d) *There is a closed, pointed convex cone C containing x such that $AC \subseteq C$.*

Making use of Theorem 5.6, this author [128] also gives an elementary alternative proof for the following intrinsic Perron-Frobenius theorem, which was derived by Schneider [102, Theorem 1.4] (in its complex version) by an analytic argument:

Corollary 5.7. *Let A be an $n \times n$ real matrix. Then A satisfies the Perron-Schaefer condition if and only if for any (or, for some) nonnegative integer k, the cone* $\mathrm{cl}(\mathrm{pos}\ \{A^i: \ i = k, k+1, \dots\})$ *is pointed.*

5.3.3. Proof of Theorem 5.2

Now back to the proof of Theorem 5.2. After introducing the concepts of spectral pairs of a vector and of a face, we can derive the following useful properties involving them [131, Theorem 4.9]. In below we use $\preceq$ to denote the lexicographic ordering between ordered pairs of real numbers, i.e., $(a, b) \preceq (c, d)$ if either $a < c$, or $a = c$ and $b \leq d$. In case $(a, b) \preceq (c, d)$ but $(a, b) \neq (c, d)$, we write $(a, b) \prec (c, d)$.

Theorem 5.8. *Let $A \in \pi(K)$.*

(i) *For any faces F, G of K, we have*

  (a) $\mathrm{sp}_A(F) = \mathrm{sp}_A(\hat{F})$, *where $\hat{F}$ is the smallest A-invariant face of K including F*;

  (b) *if $F \subseteq G$, then* $\mathrm{sp}_A(F) \preceq \mathrm{sp}_A(G)$;

  (c) $\mathrm{sp}_A(F \vee G) = \max\{\mathrm{sp}_A(F), \mathrm{sp}_A(G)\}$, *where the maximum is taken with respect to the lexicographic ordering.*

(ii) *For any vectors x, $y \in K$, we have*

(a) $\mathrm{sp}_A(x) = \mathrm{sp}_A((I_n + A)^{n-1}x)$;

(b) *if* $x \in \Phi(y)$, *then* $\mathrm{sp}_A(x) \preceq \mathrm{sp}_A(y)$;

(c) $\mathrm{sp}_A(x+y) = \max\{\mathrm{sp}_A(x), \mathrm{sp}_A(y)\}$, *where the maximum is taken with respect to the lexicographic ordering.*

As the reader may now easily guess, to construct the desired chain of semi-distinguished A-invariant faces associated with λ, we require the ith face F_i in the chain has the property that $\mathrm{sp}_A(F_i) = (\lambda, i)$. Of course, if F is a face that contains in its relative interior a generalized eigenvector of A of order i corresponding to λ, then F satisfies this property.

To begin the construction, take a distinguished generalized eigenvector x of A corresponding to λ of order m_λ; this exists by the definition of m_λ. If the face $\Phi(x)$ is not A-invariant, replace x by $(I_n + A)^{n-1}x$. Then $\Phi(x)$ is an A-invariant face which contains in its relative interior a generalized eigenvector of A corresponding to λ of order m_λ. Choose F_{m_λ} to be an A-invariant face of K minimal with respect to the property that F_{m_λ} contains in its relative interior a generalized eigenvector of A corresponding to λ of order m_λ. (Such F_{m_λ} must exist, because our underlying space is finite-dimensional.) Then clearly we have $\mathrm{sp}_A(F_{m_\lambda}) = (\lambda, m_\lambda)$. Furthermore, the face F_{m_λ} must be A-invariant join-irreducible, and hence is semi-distinguished A-invariant. If not, we can find A-invariant faces G_1, G_2 properly included in F_{m_λ} such that $F_{m_\lambda} = G_1 \vee G_2$. By Theorem 5.8(i)(c), $\mathrm{sp}_A(F_{m_\lambda})$ is equal to $\mathrm{sp}_A(G_1)$ or $\mathrm{sp}_A(G_2)$, say $\mathrm{sp}_A(G_1)$. Then $\rho_{G_1} = \lambda$ and $\nu_{\rho_{G_1}}(A|_{\mathrm{span}\ G_1}) = m_\lambda$. As a face of the polyhedral cone K, G_1 is itself a polyhedral cone. So we can find in G_1 a generalized eigenvector of A corresponding to λ of order m_λ (see Theorem 3.6 (ii)). Then by repeating the above argument, we obtain an A-invariant face of G_1 that contains in its relative interior a generalized eigenvector of A corresponding to λ of order m_λ, in contradiction with the minimality property of F_{m_λ}.

Next, by the polyhedrality of F_{m_λ} and Theorem 3.6 (ii) again, we can find in F_{m_λ} a generalized eigenvector of A corresponding to λ of order $m_\lambda - 1$. Then take $F_{m_\lambda - 1}$ to be an A-invariant face of F_{m_λ} minimal with respect to the property that it contains in its relative interior a generalized eigenvector of A corresponding to λ of order $m_\lambda - 1$, and repeat the argument. Continuing in this way, after a finite number of steps, we can eventually construct the desired chain of semi-distinguished A-invariant faces.

5.3.4. A Crucial Lemma

To complete the proof of Theorem 5.2, it remains to establish the maximality of the length of our chain. Since the faces of a polyhedral cone are still polyhedral, it suffices to establish the following [131, Lemma 5.2]:

Lemma 5.9. *Let K be a polyhedral cone, and let $A \in \pi(K)$. If K is a semi-distinguished A-invariant face of itself, then* $\mathrm{sp}_A(K) \succ \mathrm{sp}_A(F)$ *for any A-invariant face F properly included in K.*

To prove the above lemma, first note that by the definition of a semi-distinguished A-invariant face, A has a generalized eigenvector lying in $\mathrm{int}\, K$. So by Theorem 3.3, $\rho(A)$ is the only distinguished eigenvalue of A^T for K^*. We contend that (up to multiples) A^T has only one distinguished eigenvector corresponding to $\rho(A)$. Suppose otherwise. Choose two distinct (up to multiples) extreme vectors $z_1,\ z_2$ of the cone $\mathcal{N}(\rho(A)I_n - A^T) \bigcap K^*$. If $\Phi(z_1) \bigcap \Phi(z_2) \neq \{0\}$, then by the Perron-Frobenius theorem, this nonzero A^T-invariant face of K^* will contain an eigenvector w of A^T. But since $\rho(A)$ is the only distinguished eigenvalue of A^T, w must correspond to the eigenvalue $\rho(A)$. This contradicts the extremality assumption on z_1 and z_2. So $\Phi(z_1) \wedge \Phi(z_2) = \{0\}$. Let d_K denote the duality operator of K, i.e., the mapping from the face lattice $\mathcal{F}(K)$ to the face lattice $\mathcal{F}(K^*)$ given by: $d_K(F) = (\mathrm{span}\ F)^\perp \bigcap K^*$. Using the basic properties of the duality operator (see, for instance, Tam [119]), we have

$$K = d_{K^*}(\{0\}) = d_{K^*}(\Phi(z_1) \wedge \Phi(z_2)) = d_{K^*}(\Phi(z_1)) \vee d_{K^*}(\Phi(z_2)),$$

where the last equality depends on the fact that K, and hence K^*, is polyhedral. Hence, K is the join of the proper A-invariant faces $d_{K^*}(\Phi(z_1))$ and $d_{K^*}(\Phi(z_2))$, which violates the A-invariant join-irreducibility of K. This establishes our contention.

By the last part of Theorem 5.4, a consequence of our contention is that A^T has precisely one Jordan block of maximal order corresponding to $\rho(A)$, and, furthermore, the unique distinguished eigenvector of A^T, say w, must correspond to this unique maximal block, in the sense that $w = (A^T - \rho(A)I_n)^{\nu-1}z$ for some generalized eigenvector z of A^T of order ν, where $\nu = \nu_{\rho(A)}(A)$. Then we have

$$(\mathrm{span}\ \{w\})^\perp = \bigoplus_{\lambda \in \sigma(A)\setminus\{\rho(A)\}} \mathcal{N}((\lambda I - A)^n) \oplus U,$$

where U is the space of generalized eigenvectors of A corresponding to $\rho(A)$ of order less than or equal to $\nu - 1$, together with the zero vector.

Let F be any A-invariant face properly included in K. Then $d_K(F)$ is an A^T-invariant face of K^*, and by the Perron-Frobenius theorem it must contain an eigenvector of A^T, which necessarily equals w, the only distinguished eigenvector of A^T. So we must have span $F \subseteq (\mathrm{span}\{w\})^\perp$. In view of the above direct decomposition for (span $\{w\})^\perp$, we readily see that $\mathrm{sp}_A(F) \prec \mathrm{sp}_A(K)$. This completes the proof of Lemma 5.9, and hence also that of Theorem 5.2.

5.3.5. A Class of A-invariant Faces

As a consequence of the basic properties of spectral pairs, we also obtain in the following a class of A-invariant faces of K which extends and refines in the finite-dimensional case the class of invariant ideals discovered by Meyer-Nieberg for a positive linear operator defined on a Banach lattice (see Meyer-Nieberg [78, pp. 293]).

Corollary 5.10. [131, Corollary 4.10]. *Let $A \in \pi(K)$. For any nonnegative real number λ and any positive integer k, the set*

$$F_{\lambda,k} = \{x \in K : \ \mathrm{sp}_A(x) \preceq (\lambda, k)\}$$

is an A-invariant face of K. Furthermore, for any nonnegative real numbers λ_1, λ_2, and positive integers k_1, k_2, if $(\lambda_1, k_1) \preceq (\lambda_2, k_2)$, then $F_{\lambda_1,k_1} \subseteq F_{\lambda_2,k_2}$.

5.4. Semi-distinguished A-invariant Faces

One may propose to define a semi-distinguished A-invariant face by the following property:

F is nonzero A-invariant, and $\mathrm{sp}_A(G) \prec \mathrm{sp}_A(F)$ for any A-invariant face G properly included in F.

It is not difficult to show that when A is a nonnegative matrix, a face F of the nonnegative orthant is semi-distinguished A-invariant if and only if it has the preceding property. But the same is not true for a general cone-preserving map A. Nevertheless, in [131, Theorem 6.6] we identify an interesting class of proper cones whose semi-distinguished invariant faces are characterized by the above property.

Theorem 5.11. (i) *Let K be a proper cone with the property that the dual cone of each of its faces is a facially exposed cone, and let $A \in \pi(K)$. Then for any nonzero A-invariant face F of K, F is semi-distinguished A-invariant if and only if $\mathrm{sp}_A(G) \prec \mathrm{sp}_A(F)$ for all A-invariant faces G properly included in F.*

(ii) *A proper cone K has the property given in the hypothesis of part* (i), *if it fulfills one of the following* :

(a) *K^* is a facially exposed cone, and all nontrivial faces of K are polyhedral (which is the case if K is polyhedral, or is a strictly convex smooth cone).*

(b) *K is a perfect cone.*

(c) *K equals $P(n)$ for some nonnegative integer n.*

We call a proper cone K *facially exposed* if each of its faces is exposed (i.e., is of the form $d_{K^*}(G)$ for some face G of K^*; or, equivalently, each of its nontrivial faces equals the intersection of K with a supporting hyperplane). A proper cone K is said to be *perfect* if each of its faces is self-dual in its own linear span. The term was suggested by Raphael Loewy and first appeared in Barker [6]. (See also Barker and Tam [10, Theorems 3.3 and 4.7] for equivalent conditions.) Examples of perfect cones include, the nonnegative orthant $\mathbb{R}^n_+$, the n-dimensional ice-cream cone $K_n := \{(\xi_1, \ldots, \xi_n)^T : \ \xi_1 \geq (\xi_2^2 + \cdots + \xi_n^2)^{1/2}\}$, and the cone of $n \times n$ positive semi-definite hermitian (or real symmetric) matrices. Perhaps, the more interesting fact is that, the positive cone $\mathbb{A}_+$ of a finite-dimensional C^*-algebra $\mathbb{A}$ can always be regarded as a perfect cone in the real space of hermitian elements of $\mathbb{A}$. This is because, as is well-known, every finite-dimensional C^*-algebra is $*$-isomorphic with $\mathcal{M}_{k_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{k_p}(\mathbb{C})$ for some positive integers $k_1, \ldots, k_p$.

For each nonnegative integer n, we use $P(n)$ to denote the cone of all real polynomials of degree not exceeding n that are nonnegative on the closed interval [0, 1] (see Barker and Thompson [12]).

In Section 6 of [131], further properties of invariant faces associated with a linear mapping A preserving a polyhedral cone K are also given. Here are some of them:

(a) Any A-invariant face of K which contains in its relative interior a generalized eigenvector of A corresponding to λ can be expressed as a join of semi-distinguished A-invariant faces associated with λ.

(b) For each nonzero A-invariant face F of K, there exists a semi-distinguished A-invariant face $G \subseteq F$ such that $\mathrm{sp}_A(G) = \mathrm{sp}_A(F)$.

(c) For each nonzero A-invariant face F of K, there exists in F a generalized eigenvector of A corresponding to ρ_F of order $\nu_{\rho_F}(A|_{\mathrm{span}\ F})$.

In fact, in that paper, the logical relations between the above conditions and other conditions are examined in the setting when A preserves a general proper cone K (see [131, Theorems 6.4 and 6.7]).

The paper [131] contains a lot more results. Two theorems extending respectively Theorems 2.2 and 2.3 (two early results on the combinatorial spectral theory of a nonnegative matrix) are obtained in Section 7, and cleverly devised counterexamples for various natural questions are also given in Section 8. We refer the interested reader to the paper for the details.

5.5. Open Problems

At the end of [131], the following two open questions are posed:

Question 5.12. *Let K be a proper cone whose dual cone K^* is a facially exposed cone. Is it true that for any $A \in \pi(K)$, we have the following ?*

(i) *For any nonzero A-invariant face F of K, F is semi-distinguished A-invariant if and only if $\mathrm{sp}_A(G) \prec \mathrm{sp}_A(F)$ for all A-invariant faces G properly included in F.*

(ii) *There exists in* K *a generalized eigenvector of* A *corresponding to* $\rho(A)$ *of order* $\nu_{\rho(A)}(A)$.

(iii) *For any* A*-invariant face* F *which contains in its relative interior a generalized eigenvector of* A, *there exists a semi-distinguished* A*-invariant face* G *included in* F *such that* $\mathrm{sp}_A(G) = \mathrm{sp}_A(F)$.

Question 5.13. *Let* K *be a proper cone with the property that the dual cone of each of its faces is a facially exposed cone. Is it true that, for any* $A \in \pi(K)$ *and any distinguished eigenvalue* λ *of* A, *any* A*-invariant face of* K *which contains in its relative interior a generalized eigenvector of* A *corresponding to* λ *can be expressed as a join of semi-distinguished* A*-invariant faces associated with* λ ?

6. Further Results and Remarks

6.1. Matrices Preserving a Polyhedral Cone: Another Look

A useful way to derive results on a linear mapping preserving a polyhedral cone from known results on a nonnegative matrix is to use the "minimal generating matrix" as a tool. In Subsection 3.5 we have demonstrated how the method can be used to show the existence of a K-semipositive basis and also that of a K-semipositive Jordan chain of maximum length for the Perron generalized eigenspace of a linear mapping preserving a polyhedral cone. The key point is that, if A preserves a polyhedral cone K with an $n \times m$ minimal generating matrix P, then P intertwines some $m \times m$ nonnegative matrix B with A (i.e., $PB = AP$), and the spectral properties of the cone-preserving maps A ($\in \pi(K)$) and B ($\in \pi(\mathbb{R}^m_+)$) are closely related, as given by Theorem 3.5. Of course, the method is not all powerful; because, the nonnegative matrix B is in general not unique, the spectrum of A is usually properly included in that of B, and in the hypotheses of Theorem 3.5 the definition of P as a minimal generating matrix for K (for instance, the fact that P cannot contain in its nullspace a vector which has one component positive and the remaining components nonpositive) is not fully taken into account. Indeed, we have attempted to use the method to give another proof of Theorem 5.2 (an extension of the Rothblum Index Theorem to the polyhedral cone case), but in vain, because of technical difficulties. (The point is, the mapping P is not one-to-one; it does not give a nice correspondence between the semi-distinguished A-invariant faces of K and the semi-distinguished B-invariant faces of $\mathbb{R}^m_+$.) Nevertheless, in the process of our investigation, we found the following interesting results as supplements to Theorem 3.5.

6.1.1. The Minimal Generating Matrix Approach: Further Results

Theorem 6.1. *Let $K_1,\ K_2$ be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Let $A \in \pi(K_1),\ B \in \pi(K_2)$, and $P \in \pi(K_2,\ K_1)$ be such that $AP = PB$, $PK_2 = K_1$, and $\mathcal{N}(P) \bigcap K_2 = \{0\}$. Then* :

(i) *If F is a B-invariant face of K_2, then $\Phi(PF)$ is an A-invariant face of K_1.*

(ii) *If G is a face (respectively, A-invariant face) of K_1, then $P^{-1}G$ is a face (respectively, B-invariant face) of K_2.*

(iii) *For any $x \in \mathbb{R}^m$, $\mathrm{sp}_B(x) \succeq \mathrm{sp}_A(Px)$.*

(iv) *For any $x \in K_2$, $\mathrm{sp}_B(x) = \mathrm{sp}_A(Px)$.*

If, in addition, K_2 is polyhedral, then we also have the following :

(v) *For any distinguished eigenvalue λ of A, A and B have the same maximal order for the distinguished generalized eigenvectors corresponding to λ.*

(vi) *For any semi-distinguished A-invariant face G of K_1, there exists a semi-distinguished B-invariant face F of K_2 such that $\Phi(PF) = G$.*

(vii) *If F is a semi-distinguished B-invariant face of K_2 with the property that the face $\Phi(PE)$ of K_1 is minimal (with respect to inclusion) among all faces of the form $\Phi(PE)$, where E is a semi-distinguished B-invariant face of K_2 such that $\mathrm{sp}_B(E) = \mathrm{sp}_B(F)$, then $\Phi(PF)$ is a semi-distinguished A-invariant face of K_1.*

Proof. The verification of (i) and (ii) is straightforward.

It is easy to show that if u is a generalized eigenvector of B corresponding to λ and if $u \notin \mathcal{N}(P)$, then Pu is a generalized eigenvector of A corresponding to λ and $\mathrm{ord}_A(Pu) \leq \mathrm{ord}_B(u)$.

To prove (iii), let $x = x_1 + \cdots + x_k$ be the representation of x as a sum of generalized eigenvectors of B, with x_i corresponding to the eigenvalue λ_i. Then $Px = Px_1 + \cdots + Px_k$, where each Px_i is either the zero vector or is, by the above observation, a generalized eigenvector of A corresponding to λ_i of order less than or equal to that of x_i. By the definition of spectral pair, it is clear that we have $\mathrm{sp}_A(Px) \preceq \mathrm{sp}_B(x)$.

To prove (iv), suppose that $x \in K_2$. We still use the above representation of x as a sum of generalized eigenvectors of B. By Remark 5.5, we may assume that x_1 is a generalized eigenvector of B corresponding to $\rho_x(B)$ and furthermore we have $\mathrm{ord}_{x_1}(B) = \mathrm{ord}_x(B) = p$, say. Moreover, by [131, Corollary 4.8], $(B - \rho_x(B)I_m)^{p-1}x_1$ is a distinguished eigenvector of B corresponding to $\rho_x(B)$, and hence must be nonzero. Since $\mathcal{N}(P) \bigcap K_2 = \{0\}$, it follows that we have

$$(A - \rho_x(B)I_n)^{p-1}Px_1 = P(B - \rho_x(B)I_m)^{p-1}x_1 \neq 0$$

and

$$(A - \rho_x(B)I_n)^{p}Px_1 = P(B - \rho_x(B)I_m)^{p}x_1 = 0,$$

i.e., Px_1 is a generalized eigenvector of A corresponding to $\rho_x(B)$ of order p. From the definition of spectral pair, now it follows that we have $\mathrm{sp}_A(Px) = \mathrm{sp}_B(x)$.

Hereafter, we assume that K_2 is a polyhedral cone.

To prove (v), let m_A (respectively, m_B) denote the maximal order for the distinguished generalized eigenvectors of A (respectively, of B) corresponding to λ. If w is a distinguished generalized eigenvector of B for K_2 corresponding to λ of order m_B, then, in view of (iv), Pw is also a distinguished generalized eigenvector of A for K_1 corresponding to λ of order m_B. This shows that $m_B \leq m_A$. To prove the reverse inequality, let $x \in K_2$ be a vector such that Px is a generalized eigenvector of A corresponding to λ of order m_A. (Here we make use of the assumption that $PK_2 = K_1$. Note that x need not be a generalized eigenvector of B.) By (iv) again, we have $\mathrm{sp}_B(x) = \mathrm{sp}_A(Px) = (\lambda,\ m_A)$. By Corollary 5.10, $F_{\lambda,m_A} := \{y \in K_2 : \mathrm{sp}_B(y) \preceq (\lambda,\ m_A)\}$ is a B-invariant face of K_2. By definition, clearly $\mathrm{sp}_B(F_{\lambda,m_A}) \preceq (\lambda,\ m_A)$. Since F_{λ,m_A} contains x, we must have $\mathrm{sp}_B(F_{\lambda,m_A}) = (\lambda,\ m_A)$, and hence $\rho(B|_{F_{\lambda,m_A}}) = \lambda$ and $\nu_\lambda(B|_{F_{\lambda,m_A}}) = m_A$. But F_{λ,m_A} is a polyhedral cone (as K_2 is), so by Theorem 3.6 we can find in F_{λ,m_A}, and hence in K_2, a generalized eigenvector of B corresponding to λ of order m_A. This establishes the equality $m_A = m_B$.

To prove (vi), let G be a semi-distinguished A-invariant face of K_1. By (ii) and (iv), $P^{-1}G$ is a B-invariant face of K_2 such that $\mathrm{sp}_B(P^{-1}G) = \mathrm{sp}_A(G)$. Since K_2 is polyhedral, we can find a semi-distinguished B-invariant face $F \subseteq P^{-1}G$ such that $\mathrm{sp}_B(F) = \mathrm{sp}_B(P^{-1}G)$ (see the discussion following Theorem 5.11). Then $\Phi(PF)$ is an A-invariant face of K_1, included in the semi-distinguished A-invariant face G, such that $\mathrm{sp}_A(\Phi(PF)) = \mathrm{sp}_B(F) = \mathrm{sp}_A(G)$. Now the cone K_1 is also polyhedral, as $K_1 = PK_2$ and K_2 is polyhedral. So by Lemma 5.9, we have $G = \Phi(PF)$, i.e., G can be expressed in the desired form.

Now we are going to prove (vii). Since $\Phi(PF)$ is an A-invariant face of K_1 and K_1 is polyhedral, there exists a semi-distinguished A-invariant face G of K_1 such that $G \subseteq \Phi(PF)$ and $\mathrm{sp}_A(G) = \mathrm{sp}_A(\Phi(PF))$. By (vi), there exists a semi-distinguished B-invariant face E of K_2 such that $\Phi(PE) = G$. Then $\Phi(PE) \subseteq \Phi(PF)$, and we have

$$\mathrm{sp}_B(E) = \mathrm{sp}(\Phi(PE)) = \mathrm{sp}_A(G) = \mathrm{sp}_A(\Phi(PF)) = \mathrm{sp}_B(F).$$

By the minimality property of $\Phi(PF)$, we obtain $\Phi(PF) = \Phi(PE) = G$; hence $\Phi(PF)$ is semi-distinguished A-invariant. ■

Note that in the above proof for parts (vi), (vii) of Theorem 6.1, we are using Lemma 5.9 and also a nontrivial property of a linear mapping preserving a polyhedral cone.

By a duality argument, we can rephrase the hypotheses of Theorem 6.1 (also, Theorem 3.5) and obtain an interesting result.

Two cone-preserving maps $A \in \pi(K_1)$ and $B \in \pi(K_2)$ are said to be *equivalent* if there exists a linear isomorphism $P:\ \operatorname{span} K_2 \longrightarrow \operatorname{span} K_1$ such that $PK_2 = K_1$ and $P^{-1}AP = B$.

Theorem 6.2. *Let $K_1,\ K_2$ be proper cones in possibly different euclidean spaces.*

(i) *For any $P \in \pi(K_2,\ K_1)$, the following conditions are equivalent* :

(a) $PK_2 = K_1$ *and* $\mathcal{N}(P) \bigcap K_2 = \{0\}$.

(b) $P^T(K_1^*) = \Re(P^T) \bigcap K_2^*$, $\Re(P^T) \bigcap \operatorname{int} K_2^* \neq \emptyset$ *and P^T is one-to-one.*

(ii) *Let $P \in \pi(K_2,\ K_1)$ satisfy the equivalent conditions of* (i). *If $A \in \pi(K_1)$ and $B \in \pi(K_2)$ are such that $AP = PB$, then the cone-preserving maps $A^T \in \pi(K_1^*)$ and $B^T|_{\Re(P^T)} \in \pi(\Re(P^T) \bigcap K_2^*)$ are equivalent.*

Proof. The equivalence of conditions (a), (b) in part (i) follows readily from the following result (Tam [117, Proposition 5.1]):

If A is an $m \times n$ real matrix and K_1 and K_2 are closed cones in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, then we have $\operatorname{cl} AK_1 = \Re(A) \bigcap K_2$ *if and only if* $\operatorname{cl} A^T K_2^* = \Re\,(A^T) \bigcap K_1^*$.

When $P \in \pi(K_2,\ K_1)$ satisfies the equivalent conditions in (i), by condition (b) the cones K_1^* and $\Re(P^T) \bigcap K_2^*$ are clearly linearly isomorphic under P^T. If, in addition, $A \in \pi(K_1)$ and $B \in \pi(K_2)$ satisfy $AP = PB$, then it is ready to see that $A^T \in \pi(K_1^*)$ and $B^T|_{\Re(P^T)} \in \pi(\Re(P^T) \bigcap K_2^*)$ (noting that $\Re(P^T) = \operatorname{span}(\Re(P^T) \bigcap K_2^*)$), and we have the following commutative diagram:

$$\begin{array}{ccc} K_1^* & \xrightarrow{\quad A^T \quad} & K_1^* \\ {\scriptstyle P^T}\Big\downarrow & & \Big\downarrow{\scriptstyle P^T} \\ \Re(P^T) \bigcap K_2^* & \xrightarrow{B^T|_{\Re(P^T)}} & \Re(P^T) \bigcap K_2^* \end{array}$$

Then it is clear that the cone-preserving maps A^T and $B^T|_{\Re(P^T)}$ are equivalent. ■

6.1.2. A New Look

Applying Theorem 6.2 to the nonnegative matrix case, we obtain the following useful new result:

Theorem 6.3. *Let K be a polyhedral cone in $\mathbb{R}^n$ with m maximal faces. Then for any $A \in \pi(K)$, there exists an $m \times m$ nonnegative matrix B, and a B-invariant subspace W of $\mathbb{R}^m$, $W \bigcap \operatorname{int}\ \mathbb{R}^m_+ \neq \emptyset$, such that the cone-preserving maps $A \in \pi(K)$ and $B|_W \in \pi(W \bigcap \mathbb{R}^m_+)$ are equivalent.*

Proof. Consider any polyhedral cone K in $\mathbb{R}^n$ with m maximal faces. Let $A \in \pi(K)$. Choose a minimal generating matrix P for the polyhedral cone K^*. Clearly P is $n \times m$, as the number of extreme rays of K^* is equal to the number of maximal faces of K. Then we can write $A^T P = PB^T$ for some $m \times m$ nonnegative matrix B. Here we have $A^T \in \pi(K^*)$, $B^T \in \pi(\mathbb{R}^m_+)$ and $P \in \pi(\mathbb{R}^m_+,\ K^*)$ such that $P\mathbb{R}^m_+ = K^*$ and $\mathcal{N}(P) \bigcap \mathbb{R}^m_+ = \{0\}$. Now apply Theorem 6.2 with $K_1 = K^*,\ K_2 = \mathbb{R}^m_+$, and $A^T,\ B^T$ in place of A and B respectively. ∎

Theorem 6.3 yields an easy proof for Remark 4.18, as we are going to do below. Using Theorem 6.3 and the Preferred Basis Theorem (Theorem 2.4 (ii)), we shall also provide a new proof for Lemma 5.9, the crucial lemma used in the proof of Theorem 5.2. However, based on Theorem 6.3 but not on Theorem 3.5, we are unable to find a direct proof for Theorem 3.6.

6.1.3. An Alternative Proof for Remark 4.18

To show Remark 4.18, first, by Theorem 6.3 we have $\rho(A) = \rho(B|_W)$. Since $W \bigcap \text{int } \mathbb{R}^m_+ \neq \emptyset$, we also have $\rho(B|_W) = \rho(B)$ (by Remark 5.3 or Tam [121, Lemma 7.1]). Hence, the peripheral spectrum of A is included in that of the nonnegative matrix B. But it is well-known that every eigenvalue in the peripheral spectrum of a nonnegative matrix is equal to the spectral radius times a root of unity, of order not exceeding the size of the matrix, so Remark 4.18 follows. ∎

6.1.4. An Alternative Proof for Lemma 5.9

Let $A,\ K$ and B have the meanings as given in Theorem 6.3. Since the cone-preserving maps $A \in \pi(K)$ and $B|_W \in \pi(\mathbb{R}^m_+ \bigcap W)$ are equivalent, it suffices to show that if $\mathbb{R}^m_+ \bigcap W$ is a semi-distinguished $B|_W$-invariant face of itself, then $\text{sp}_{B|_W}(\mathbb{R}^m_+ \bigcap W) \succ \text{sp}_{B|_W}(F)$ for any $B|_W$-invariant face F properly included in $\mathbb{R}^m_+ \bigcap W$. To begin with, note that for any vector $x \in W$, $\text{sp}_{B|_W}(x) = \text{sp}_B(x)$; this is because, the representation of x as a sum of generalized eigenvectors of $B|_W$ is the same as its representation as a sum of generalized eigenvectors of B. As a consequence, for any face F of $\mathbb{R}^m_+ \bigcap W$, we have $\text{sp}_{B|_W}(F) = \text{sp}_B(\Phi(F))$, where $\Phi(F)$ denotes the face of $\mathbb{R}^m_+$ generated by F.

Since $\mathbb{R}^m_+ \bigcap W$ is a semi-distinguished $B|_W$-invariant face of itself and $W \bigcap$ int $\mathbb{R}^m_+ \neq \emptyset$, we can find in W a positive generalized eigenvector of B (corresponding to $\rho(B)$), say y. Let $\alpha_1, \ldots, \alpha_q$ be all the basic classes (of B), and let $\mathfrak{B} = \{x^{(\alpha_i)} :\ i = 1, \ldots, q\}$ be a preferred basis of the Perron generalized eigenspace of B, i.e., a semipositive basis that satisfies the reguirement of Theorem 2.4 (ii). Then we can write y as a linear combination of the vectors in $\mathfrak{B}$, say $y = \sum_{i=1}^{q} c_i x^{(\alpha_i)}$. Note that because B has a positive generalized eigenvector, every final class is necessarily basic. Following Rothblum [93], we say a class α has access to a class

β in k steps if k is the length of the longest chain from α to β. (Recall that the length of a chain is the number of basic classes it contains.) Denote $\nu_{\rho(B)}(B)$ by ν. For each $k = 1, \ldots, \nu$, let $I_k = \{i \in \langle q \rangle : \ \alpha_i$ has access to a final class in k steps$\}$. Then $I_1, \ldots, I_q$ are each nonempty, and $\langle q \rangle$ is equal to their disjoint union. Furthermore, the collection of final classes is precisely the set $\{x^{(\alpha_i)} : \ i \in I_1\}$. In order that y is a positive vector, it is clear that $c_i > 0$ for all $i \in I_1$; after normalizing the vectors $x^{(\alpha_i)}$, hereafter we assume that $c_i = 1$ for all $i \in I_1$.

Let F be a nontrivial $B|_W$-invariant face of $W \bigcap \mathbb{R}^m_+$, and suppose to the contrary that $\mathrm{sp}_{B|_W}(F) = \mathrm{sp}_{B|_W}(W \bigcap \mathbb{R}^m_+)$. Then their common spectral pair value is $(\rho(B), \nu)$. Since F is polyhedral, by Theorem 3.6 (ii), F contains a generalized eigenvector u of B corresponding to $\rho(B)$ of order ν. Certainly, we can also write u in terms of the vectors in $\mathfrak{B}$, say $u = \sum_{i=1}^q d_i x^{(\alpha_i)}$. By the Preferred Basis Theorem, we readily see that there exists at least one $i \in \langle q \rangle$ such that $d_i \neq 0$ and α_i is a basic class of height ν. Of course, any such i must belong to I_1. Since u is a nonnegative vector, it is also clear that $d_i \geq 0$ for all $i \in I_1$. Now let $\lambda = \max\{d_i : \ i \in I_1\}$. Then for all $i \in I_1$, we have $\lambda y_{\alpha_i} \geq u_{\alpha_i}$ with at least one equality, where we use y_β to denote the subvector of y determined by the class β. Indeed, for any nonbasic class β which has access to a final class in one step, we also have $\lambda y_\beta \geq u_\beta$. (However, we need not have $\lambda y \geq u$.)

Now, let z denote the vector $y + \sum_{k=1}^q \mu_k (B - \rho(B)I_m)^k y$. Since W is a B-invariant subspace, clearly $z \in W$. We contend that by choosing $\mu_k, \ k = 1, \ldots, q$, to be positive numbers sufficiently large, we would obtain a positive vector z such that $\lambda z \geq u$. Here we make use of the property of a Preferred Basis that, for each basic class α, $(B - \rho(B)I_m)x^{(\alpha)}$ is a positive linear combination of all of the $x^{(\beta)}$'s such that β is a basic class, $\beta >\!\!- \alpha$. Since $y = \sum_{i \in I_1} y^{(\alpha_i)} + \sum_{i \in \langle q \rangle \setminus I_1} c_i y^{(\alpha_i)}$, a consequence of the latter property is that $(B - \rho(B)I_m)y$ is expressible as $\sum_{i=1}^q e_i y^{(\alpha_i)}$ with $e_i = 0$ for all $i \in I_1$ and $e_i > 0$ for all $i \in I_2$. If β is a class (basic or nonbasic) that has access to a final class in one step, then since $((B - \rho(B)I_m)^k y)_\beta = 0$ for $k = 1, \ldots, q$, we have $(\lambda z - u)_\beta = \lambda y_\beta - u_\beta \geq 0$, where the last inequality is already mentioned above. If β is a class that has access to a final class in two steps, then $(\lambda z - u)_\beta = \lambda y_\beta + \lambda \mu_1 \sum_{i \in I_2} e_i (y^{(\alpha_i)})_\beta - u_\beta$; since $(y^{(\alpha_i)})_\beta$ is a positive vector for at least one $i \in I_2$ (as $\beta >= \alpha_i$ for at least one such i), by choosing $\mu_1 > 0$ sufficiently large, we have $z_\beta > 0$ and $(\lambda z - u)_\beta \geq 0$. Similarly, by choosing μ_2 sufficiently large, we also have $z_\beta > 0$ and $(\lambda z - u)_\beta \geq 0$ for all classes β that have access to a final class in three steps. Continuing in this way, we can choose positive numbers $\mu_1, \ldots, \mu_q$ with the desired property. This proves our contention.

By our choice of λ, we have $(\lambda z - u)_{\alpha_j} = \lambda (y)_{\alpha_j} - u_{\alpha_j} = 0$ for at least one $j \in I_1$. But $\{\alpha_i : \ i \in I_1\}$ is the collection of final classes, it follows that the smallest B-invariant face of $\mathbb{R}^m_+$ containing $\lambda z - u$ is not $\mathbb{R}^m_+$ itself, and hence the smallest $B|_W$-invariant face of $W\bigcap\mathbb{R}^m_+$ containing $\lambda z - u$ is also not $W\bigcap\mathbb{R}^m_+$ itself. Since $z \in \operatorname{relint}(W\bigcap\mathbb{R}^m_+)$ and $\lambda z = (\lambda z - u) + u$ with $\lambda > 0$, it follows that $W\bigcap\mathbb{R}^m_+$ is the join of the smallest $B|_W$-invariant faces of $W\bigcap\mathbb{R}^m_+$ containing $\lambda z - u$ and u respectively, where the latter two faces are both strictly included in $W\bigcap\mathbb{R}^m_+$ (the one containing u being included in F). This contradicts the $B|_W$-invariant join-irreducibility of $W\bigcap\mathbb{R}^m_+$. The proof is complete. ■

The reader may think that the above proof is not easier than the original proof as given in Subsection 5.3.4. (Both proofs depend on Theorem 3.6.) Indeed, this author prefers the original proof, as it is more conceptual and more readily adaptable to the non-polyhedral cone case. We include the above proof just to show how it can be done if one really wants to apply the nonnegative matrix theory (and maybe someone has a better idea).

6.2. K-semipositive Bases

According to Theorem 2.4, there always exists a semipositive basis for the Perron generalized eigenspace of a nonnegative matrix. Indeed, semipositive bases of various kinds have been introduced and studied by Rothblum [93], Richman and Schneider [92], and Hershkowitz and Schneider [59, 60], using matrix combinatorial methods. In [55], Hartwig, Neumann and Rose offer an algebraic-analytic proof for the existence of a semipositive basis, analytic in the sense that it utilizes the resolvent expansion but does not involve the Frobenius normal form. The connection between the combinatorial and the algebraic-analytic approaches is examined in detail by Neumann and Schneider [82, 83, 84].

As noted in Section 3 (Theorem 3.6), we now know that if A preserves a polyhedral cone K, then the Perron generalized eigenspace of A always contains a K-semipositive basis and there is a K-semipositive Jordan chain of maximal length. The proof of the latter results as given in Tam [121] relies on the corresponding results for a nonnegative matrix and uses the minimal generating matrix as a tool. By introducing cone-theoretic arguments into the method of Hartwig, Neumann and Rose, recently this author [127] also found a new approach to rederive these results, without assuming the corresponding nonnegative matrix results. Thus, our treatment can be kept independent of the known results on a nonnegative matrix.

First, the following is obtained:

Theorem 6.4. *Let $A \in \pi(K)$, and let $E^{(0)}_\rho$ denote the projection of $\mathbb{R}^n$ onto the Perron generalized eigenspace of A along the intersection of $\mathbb{R}^n$ with the direct sum of other generalized eigenspaces of A. Consider the following conditions* :

(a) $E^{(0)}_\rho \in \pi(K)$.

(b) $(\lambda I_n - A)^{-1}E^{(0)}_\rho \in \pi(K)$ *for all* $\lambda > \rho(A)$ (*or*, *for all* λ *sufficiently large*).

(c) $(\lambda I_n - A)^{-1}E_\rho^{(0)} \in \pi(K)$ *for all* $\lambda > \rho(A)$, *sufficiently close to* $\rho(A)$.

(d) $(\lambda I_n - A)^{-1}E_\rho^{(0)} \in \pi(K)$ *for at least one* $\lambda > \rho(A)$.

(e) *The Perron generalized eigenspace of* A *has a* K*-semipositive basis.*

Then conditions (a) *and* (b) *are equivalent*, *conditions* (c) *and* (d) *are also equivalent*, *and the following logical relations hold* : (a)$\Longrightarrow$(c)$\Longrightarrow$(e).

Then it is proved in [127] that, if the underlying cone K is polyhedral, condition (c) of Theorem 6.4 always holds; hence, condition (e) also follows. The proof is a modification of the method of Hartwig, Neumann and Rose, and depends on the fact that the polyhedrality of K implies that of $\pi(K)^*$, the dual cone of $\pi(K)$ (with respect to the inner product $\langle A,\ B\rangle = \mathrm{tr}(AB^T)$ of the underlying matrix space).

By a *K-semipositive Jordan basis* we mean a basis which is composed of K-semipositive Jordan chains. Following the usage of Hershkowitz and Schneider [59], we call a basis $\mathfrak{B}$ for $\mathcal{N}((\rho(A)I_n - A)^n)$ a *height basis for* A if the number of vectors in $\mathfrak{B}$ of order k equals η_k, where η_k is the kth height characteristic number of A (associated with $\rho(A)$). (The definition of η_k has already been introduced near the end of Section 2.)

The following is another main result of [127]:

Theorem 6.5. *Let* $A \in \pi(K)$. *Consider the following conditions* :

(a) *There exists a* K*-semipositive Jordan basis for* A.

(b) *There exists a* K*-semipositive height basis for* A.

(c) *For each* k, $k = 1, \ldots, \nu_{\rho(A)}(A)$, *the subspace* $\mathcal{N}((\rho(A)I_n - A)^k)$ *contains a* K*-semipositive basis.*

Conditions (b) *and* (c) *are always equivalent*, *and are implied by* (a). *When* K *is polyhedral*, (a) *is also another equivalent condition.*

In the nonnegative matrix case, the equivalence of conditions (a)–(c) of Theorem 6.5 is known. Another known equivalent condition is that $\lambda(A) = \eta(A)$, where $\lambda(A)$ is the level characteristic of A associated with $\rho(A)$. In fact, we have mentioned only four of the thirty-five known equivalent conditions (see Hershkowitz and Schneider [60, Theorem 6.6]).

As a preliminary step towards Theorem 6.5, the following result is obtained in [127], extending Theorem 3.6(ii):

If $A \in \pi(K)$, *where* K *is a polyhedral cone*, *then there always exists a Jordan basis for the Perron generalized eigenspace of* A *such that all chains of maximal length are* K*-semipositive.*

The proof is again a modification of the method of Harwig, Neumann and Rose.

6.3. Linear Equations over Cones

In the intended future paper [132], we treat the following linear equations over cones:

$$(6.1) \qquad (\lambda I_n - A)x = b, \; x \in K$$

and

$$(6.2) \qquad (A - \lambda I_n)x = b, \; x \in K,$$

where $A \in \pi(K),\ 0 \neq b \in K$ and $\lambda > 0$ are given. Equation (6.1) has been treated before by a number of people, in the finite-dimensional as well as infinite-dimensional settings. In particular, it is known that equation (6.1) is solvable if and only if $\rho_b(A) < \lambda$. Our contribution in [132] is to provide a more complete set of equivalent conditions for solvability, and to give simpler and more elementary proofs for the finite-dimensional case. The study of equation (1.2) is relatively new. A treatment of the equation (by graph-theoretic arguments) for the special case when $\lambda = \rho(A)$ and $K = \mathbb{R}^n_+$ can be found in Tam and Wu [133]. In [132], the following results are obtained in connection with equation (6.2):

Theorem 6.6. *Let $A \in \pi(K)$, let $0 \neq b \in K$, and let λ be a given positive real number such that $\lambda > \rho_b(A)$. Then the equation* (6.2) *is solvable if and only if λ is a distinguished eigenvalue of A for K and $b \in \Phi(\mathcal{N}(\lambda I_n - A) \bigcap K)$. In this case, for any solution x of* (6.2) *we have* $\mathrm{sp}_A(x) = (\lambda, 1)$.

Theorem 6.7. *Let $A \in \pi(K)$, and let $b \in K$. If the linear equation*

$$(A - \rho(A)I_n)x = b, \; x \in K$$

is solvable, then $b \in \Phi(\mathcal{N}((\rho(A)I_n - A)^n) \bigcap K)$.

Interestingly, the face $\Phi(\mathcal{N}(\lambda I_n - A) \bigcap K)$ [respectively, $\Phi(\mathcal{N}((\rho(A)I_n - A)^n) \bigcap K)$] involved in the condition of Theorem 6.6 [respectively, Theorem 6.7] is A-invariant. Similarly, the above-mentioned condition for solvability of equation (6.1) can also be reformulated as: $b \in F_\lambda$, where F_λ is the A-invariant face $\{y \in K : \rho_y(A) < \lambda\}$.

Specializing to the nonnegative matrix case, we have the following:

Corollary 6.8. *Let P be an $n \times n$ nonnegative matrix, let $b \in \mathbb{R}^n_+$, and let λ be a positive real number such that $\lambda > \rho_b(P)$. Then the equation*

$$(P - \lambda I_n)x = b, \; x \geq 0$$

is solvable if and only if λ is a distinguished eigenvalue of P such that for any class α of P, if $\alpha \bigcap \operatorname{supp}(b) \neq \emptyset$, then α has access to a distinguished class of P associated with λ.

Corollary 6.9. *Let A be an $n \times n$ singular M-matrix, and let c be a nonpositive vector. If there is a nonnegative vector x such that $Ax = c$, then for each class α for which $\alpha \bigcap \operatorname{supp}(c) \neq \emptyset$, α has access to some basic class.*

In [132], we also apply the results of Theorem 6.6 or 6.7 to determine when the inequalities in (3.1) (of Subsection 3.1.3) become equalities. It is proved that $R_A(x) = \rho_x(A)$ if and only if x can be written as $x_1 + x_2$, where x_1 is an eigenvector of A corresponding to $\rho_x(A)$ and x_2 satisfies $\rho_{x_2}(A) < \rho_x(A)$ and $R_A(x_2) \leq \rho_x(A)$. As for when the equality $r_A(x) = \rho_x(A)$ holds, only a partial result is obtained.

6.4. Geometric Spectral Theory

In [131, Lemma 6.10 and Theorem 6.7], it is shown that cone-preserving maps on a strictly convex cone (i.e., a proper cone each of whose boundary vectors is an extreme vector) and those on a polyhedral cone share the following common properties:

(i) If F is a semi-distinguished A-invariant face, then $\operatorname{sp}_A(F) \succ \operatorname{sp}_A(G)$ for all A-invariant faces G properly included in F.

(ii) Any A-invariant face which contains in its relative interior a generalized eigenvector of A corresponding to λ can be expressed as a join of semi-distinguished A-invariant faces associated with λ.

On the other hand, by the following result, due to Gritzmann, Klee and Tam [52, Corollary 3.2], cone-preserving maps on a strictly convex cone can have special properties of their own:

If $A \in \pi(K)$, where K is a strictly convex cone, then A cannot have more than two distinct distinguished eigenvalues for K.

Surely, the spectral theory of a cone-preserving map depends much on the geometry of the underlying cone. Since the class of perfect cones contains the nonnegative orthants and also the positive cones of finite-dimensional C^*-algebras (see Subsection 5.4), the spectral theory of cone-preserving maps on perfect cones seems worthy of study.

6.5. Use of $\pi(K)$ as a Tool

Before this author started his study on the spectral theory of positive linear operators, he had more than ten years of working experience with convex cones. He had successfully applied the theory of cones to the study of generalized inverses and semigroups of nonnegative matrices (see [117, 118]), and to the study of the Green's relations on the semigroup of nonnegative matrices (see [118, 120]). He had also studied the structure of $\pi(K)$ as a cone or as a semiring (see [115, 116, 124, 125]). Apparently, these studies are remote from the study of the spectral theory of a single positive linear operator. But this author has benefited much from such studies; at least, his geometric intuition and the abilities to construct delicate examples or counterexamples are greatly sharpened as a result. In Subsections 4.8.3 and 6.1.1–6.1.2, we have already seen how some old results from [117] play a key role in our latest investigations. We have also mentioned that the cone $\pi(K)$ plays a role in the proof of Theorem 6.4. Now we would like to give two other evidences which "explain" why a study of the cone $\pi(K)$ is somehow relevant to the study of individual operators in $\pi(K)$:

First, if A belongs to $\pi(K)$, then $\pi(K)$ includes the cone cl(pos $\{A^i : i = 0, 1, \ldots\}$); but the spectral property of A and the geometry of the latter cone are related (see Corollary 5.7).

Second, if $A \in \pi(K)$, then the linear operator L_A on span $\pi(K)$, defined by $L_A(X) = AX$, belongs to $\pi(\pi(K))$. Also, the spectral properties of A and those of L_A are closely related.

6.5.1. Automorphisms of Polyhedral Cones

Here is another example where a knowledge of the cone $\pi(K)$ comes into play. We call a proper cone K *decomposable* if it can be expressed as a direct sum of two nonzero subcones; otherwise, K is *indecomposable*. According to a result of Loewy and Schneider [74, Theorem 3.3] (see also Tam [125] for extensions), K is indecomposable if and only if the face $\Phi(I_n)$ is an extreme ray of the cone $\pi(K)$. Based on this result, very recently this author [128] obtained the following:

Theorem 6.10. *Let A be an $n \times n$ real matrix, where $n \geq 3$. Then there exists an indecomposable polyhedral cone K such that $A \in \mathrm{Aut}(K)$ if and only if A is nonzero, diagonalizable, $\rho(A)$ is an eigenvalue of A, and every eigenvalue of A is equal to $\rho(A)$ times a root of unity.*

The above theorem, in turn, leads to the following characterization of real matrices that are automorphisms of some polyhedral cones:

Theorem 6.11. *Let A be an $n \times n$ real matrix. In order that there exists a polyhedral cone K such that $A \in \mathrm{Aut}(K)$, it is necessary and sufficient that A is nonsingular, and for any eigenvalue λ of A, λ equals $|\lambda|$ times a root of unity and $|\lambda|$ is also an eigenvalue of A.*

In passing, we would like to point out that Theorem 4.10 yields immediately the following new result:

Theorem 6.12. *Let A be an $n \times n$ real matrix. Then there exists a proper cone K in $\mathbb{R}^n$ such that $A \in \operatorname{Aut}(K)$ and A is K-irreducible if and only if A is nonzero, diagonalizable, all eigenvalues of A are of the same modulus, and $\rho(A)$ is a simple eigenvalue.*

As far as we know, the following question is still open:

Question 6.13. *Find an equivalent condition on a given $n \times n$ real matrix A so that there exists a proper cone K in $\mathbb{R}^n$ such that $A \in \operatorname{Aut}(K)$.*

6.6. Matrices with a Fully Cyclic Peripheral Spectrum

In the study of positive operators on infinite-dimensional spaces, lots of work have been done on the cyclic properties of the peripheral spectrum (or spectrum) of the operator, extending part (ii) of the Frobenius theorem (Theorem 1.2). (For works in the setting of Banach lattices, see Schaefer [96] and Meyer-Nieberg [78]; in the setting of finite-dimensional C^*-algebras, see Groh [53]; and for Banach lattice algebras, see Burger and Grobler [22].) Besides, for the (point) peripheral spectrum of a positive operator, there is a stronger property, which has also attracted much attention; namely, its full cyclicity. We say the peripheral spectrum of an $n \times n$ matrix A is *fully cyclic* if whenever $\rho(A)\alpha x = Ax$, $\mathbf{0} \neq x \in \mathbb{C}^n$, $|\alpha| = 1$, then $|x|(\operatorname{sgn} x)^k$ is an eigenvector of A corresponding to $\rho(A)\alpha^k$ for all $k \in \mathbb{Z}$. Here for any $x = (\xi_1, \ldots, \xi_n)^T \in \mathbb{C}^n$ and $k \in \mathbb{Z}$, we use $|x|(\operatorname{sgn} x)^k$ to denote the vector $(|\xi_1|(\operatorname{sgn} \xi_1)^k, \ldots, |\xi_n|(\operatorname{sgn} \xi_n)^k)^T$, where sgn δ equals $\delta/|\delta|$ if $\delta \neq 0$ and equals 1 if $\delta = 0$. It is well-known that the peripheral spectrum of every irreducible nonnegative matrix is fully cylic (see Schaefer [96, Proposition 1.2.8], and also [96, Proposition 5.4.6] for an extension in the setting of a Banach function lattice). Ten years ago, in [122] this author has found some necessary conditions and a set of sufficient conditions for a nonnegative matrix to have a fully cyclic peripheral spectrum. The conditions are given in terms of the classes of the nonnegative matrix. Recently, these results were extended by Förster and Nagy [42] to the setting of a nonnegative linear operator A on a Banach lattice for which $\rho(A)$ is a pole of the resolvent of A.

6.7. Matrices with Cyclic Structure

Recently, this author [126] also obtained another extension of part (ii) of the Frobenius theorem by dropping the nonnegativity assumption. In the context of a general square complex matrix, he examined the logical relations among the conditions that appear in part (ii) of the Frobenius theorem, and some other conditions. Calling a square matrix *m-cyclic* if it is permutationally similar to a matrix of the form

$$\begin{bmatrix} \mathbf{0} & A_{12} & & & \\ & \mathbf{0} & A_{23} & & \\ & & \mathbf{0} & \ddots & \\ & & & \ddots & A_{m-1,m} \\ A_{m1} & & & & \mathbf{0} \end{bmatrix},$$

where the blocks along the diagonal are all square, he obtained the following result [126, Theorem 4.1]:

Theorem 6.14. *Let A be a square complex matrix, and let $m \geq 2$ be a positive integer. Consider the following conditions* :

(a) *A is m-cyclic.*

(b) *A is diagonally similar to $e^{2\pi i/m}A$.*

(c) *All cycles of $G(A)$ have signed length an integral multiple of m.*

(d) *All circuits of $G(A)$ have length an integral multiple of m.*

The implication (a)$\Longrightarrow$(b) *always holds, and conditions* (b), (c) *are always equivalent. When $G(A)$ has at least one cycle with nonzero signed length, conditions* (a)–(c) *are equivalent. When A is irreducible, condition* (d) *is also an equivalent condition. When A is irreducible, nonnegative, the following conditions are each an additional equivalent condition* :

(e) *A and $e^{2\pi i/m}A$ are similar.*

(f) *A and $e^{2\pi i/m}A$ have the same characteristic polynomial.*

(g) *A and $e^{2\pi i/m}A$ have the same peripheral spectrum.*

In the above theorem, by a circuit we mean as usual a simple closed directed walk. A cycle and its signed length are less common concepts. We refer the interested reader to the paper for the details.

6.8. Perron-Frobenius Type Results on the Numerical Range

By the *numerical range* of an $n \times n$ complex matrix A, we mean the set $W(A)$ given by: $W(A) = \{z^*Az : \ z \in \mathbb{C}^n, \ z^*z = 1\}$. It is well-known that $W(A)$ is always a compact convex set in the complex plane, which includes the spectrum of A.

If A is an irreducible nonnegative matrix with index of imprimitivity $h > 1$, then the numerical range of A possesses certain properties similar to those for its spectrum as given in part (ii) of the Frobenius theorem. Indeed, in this case, by Wielandt's Lemma (see, for instance, Berman and Plemmons [17, Theorem 2.2.14]), there exists a unitary diagonal matrix D such that $e^{2\pi i/h}A = D^{-1}AD$. Since the numerical range of a matrix is invariant under unitary similarity, it follows that we have $e^{2\pi i/h}W(A) = W(A)$, i.e., the numerical range of A, like its spectrum, is invariant under a rotation about the origin through an angle of $2\pi/h$. The following reformulated unpublished main result of Issos [66, Theorem 7] shows that the elements of maximum modulus in the numerical range of an irreducible nonnegative matrix are also equally spaced around a circle with center at the origin and with one element on the positive real axis, like its peripheral spectrum. By the *numerical radius* of A, denoted by $r(A)$, we mean the quantity max $\{|w| : w \in W(A)\}$.

Theorem 6.15. *Let A be an irreducible nonnegative matrix with index of imprimitivity h. For any complex number $\lambda \in W(A)$, $|\lambda| = r(A)$ if and only if λ equals $r(A)$ times a hth root of unity.*

The proof given by Issos for the above result depends on a number of auxillary results and is rather involved. Recently, Tam and Yang [134] offered a different, more conceptual proof. Other results on the numerical range of a nonnegative matrix can also be found in [134].

6.9. Other Perron-Frobenius Type Results

Due to the limitation of space, time and the knowledge of this author, this review is not intended to be comprehensive. Nowadays, in the literature, there are many research works that are given under the name "Perron-Frobenius". Below we give a list of some of them, together with some references. We leave to the interested reader to explore whether our geometric spectral theory of positive linear operators has any connections with these works.

(i) Nonlinear Perron-Frobenius theory (see Solow and Samuelson [111], Schaefer [94], Brualdi, Parter and Schneider [21], Morishima [80], Schneider and Turner [105], Morishima and Fujimoto [81], Fujimoto [49, 50], Menon and Schneider [77], Nussbaum [85, 86], Sine [110], and Hyers, Isac and Rassias [65, Chapter 2]);

(ii) Perron-Frobenius theorems in relative spectral theory (see Bidard and Zerner [18], R. Stern and H. Wolkowicz [112], and Bapat, Olesky and Van Den Driessche [3]);

(iii) Perron-Frobenius theory in operator polynomial (see Förster and Nagy [39]);

(iv) Perron-Frobenius theory over real closed ordered fields and fractional power series expansions (see Eaves, Rothblum and Schneider [32]);

(v) Perron-Frobenius type theorems for cross-positive matrices (see Schneider and Vidyasagar [106], Elsner [34], and Berman, Neumann and Stern [16]).

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Department of Mathematics, Tamkang University
Tamsui, Taiwan 25137, R.O.C.
E-mail: bsm01@mail.tku.edu.tw