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EQUIVALENT DOUBLE-LOOP NETWORKS

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Abstract. Hwang and Xu defined equivalent double-loop networks and gave one such result showing that the L-shapes of the two equivalent networks are recombinations of three rectangles. Recently, Rödseth gave an elegant algebraic theorem for equivalent multi-loop networks. We show that its double-loop version yields equivalent networks of the 3-rectangle version. We also show that other seemingly different geometric recombinations also all turn out to be special cases of the 3-rectangle version.

1. Introduction

A double loop $DL(n; a, b)$ has n nodes $0, 1, \cdots, n-1$ and $2n$ links of 2 types:

$$\text{a-links: } i \rightarrow i + a \pmod{n}, i = 0, 1, \cdots, n-1,$$
$$\text{b-links: } i \rightarrow i + b \pmod{n}, i = 0, 1, \cdots, n-1.$$

Double loops have been widely studied (see [4] for literature) as architecture for local area networks.

The minimum-distance diagram $L(n; a, b)$ of a double loop gives a shortest path from node u to node v for any u, v. Since a double loop is node-symmetric, it suffices to give a shortest path from node 0 to any other node. Let 0 occupy the (0,0)-cell. Then v occupies the (i, j)-cell if a shortest path from 0 to v consists of i a-links and j b-links. Wong and Coppersmith [6] proved that the diagram is always an L-shape (a rectangle is considered a degeneration). See Figure 1 for two examples.

Two double loops $DL(n; a, b)$ and $DL(n; a', b')$ are called *isomorphic* [2] if there exists an h prime to n such that $\{a', b'\} = \{ha, hb\}$. Let $d(k)$ denote the number of cells (i, j) in an L-shape such that $i + j = k$. Hwang and Xu [3] defined two double loops to be equivalent if they have the same $d(k)$ for every k. In

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Figure 1. Two examples of L-shapes

Figure 2. The 3-rectangle transformation

particular, two equivalent double loops have the same diameter and average distance. Trivially, isomorphic implies equivalent. Also note that if A and B are isomorphic, C and D are isomorphic, then B and C are equivalent implies A and D are.

Hwang and Xu proved that $DL(n;1,s)$ and $DL(n;1,n+1-s)$ are equivalent by showing that they correspond to different ways of piling up three rectangles. We call this the 3-rectangle transformation; see Figure 2.

Rödseth [5] considered the multi-loop $ML(n;S)$, where $S=s_1,\cdots,s_l$ and the type-j links are $i\to i+s_j \pmod n$, $j=1,\cdots,l$. Let $\bar{S}=\{S,0\}$. He proved that $ML(n;S)$ and $ML(n;S')$ are equivalent if $S'=\bar{S}-s_i$ for some $s_i\in S$ (the other part of Rödseth's theorem states that isomorphic double loops are equivalent). For $l=2$, we will write $(a,b,0)$ in the order $(a,0,b)$. Then $DL(n;a,b)$ is equivalent to $DL(n;n-a,b-a)$ and $DL(n;a-b,n-b)$. Since -1 is prime to n, $DL(n;(-1)(n-a),(-1)(b-a))=DL(n;a,a-b)$ is also equivalent to $DL(n;a,b)$. The Hwang-Xu result then corresponds to the special case $a=1$.

It is curious to know whether Rödseth's theorem on double loops yields transformations other than the 3-rectangle kind. We are also interested in the following two kinds of transformations (see Figure 3) proposed by Fiol, Yebra, Alegre and Valero [2], which clearly preserves equivalence:

In this paper we prove that Rödseth's theorem yields only the 3-rectangle mapping, and the top-turning mapping and the shadow-turning mapping are special cases of the 3-rectangle mapping.

(a) top-turning (b) shadow-turning

Figure 3. Two geometric transformations

Figure 4. L-shape with parameters

2. Rödseth's Theorem and the 3-rectangle Transformation

Let the segments of an L-shape be labeled as shown in Figure 4.

Fiol, Yebra, Alegre and Valero [2] showed that an L-shape always tessellates the plane. In such a tessellation, the length of a step is the usual Euclidean distance, not just one step-unit as shown in drawing an L-shape. By considering their relative positions of lattice points, Fiol *et al.* derived the following congruence:

$$\begin{aligned} la - nb &\equiv 0 \pmod{N}, \\ -pa + hb &\equiv 0 \pmod{N}. \end{aligned} \tag{1}$$

They also stated that the solution (a, b) of (1) is unique up to isomorphism.

Let $T(L)$ denote the L-shape obtained from L through a 3-rectangle transformation. Then the parameters of $T(L)$ can be expressed by the parameters of L as shown in Figure 5.

Theorem 1. *$L(N; N-a, b-a)$ can be obtained from $L(N; a, b)$ through the 3-rectangle transformation.*

Figure 5. The 3-rectangle transformation with parameters

Figure 6. The dual 3-rectangle transformation

Proof. It suffices to show that $(N-a, b-a)$ is a solution of (1) with the parameters of $T(L)$. Note that

$$(m+q)(N-a) - q(b-a) \equiv -(ma+qb) \equiv 0 \pmod{N},$$

since in L, m a-steps and q b-steps reach the cell at the upper corner of the L-shape which contains the element 0. Furthermore,

$$-(n-p+q)(N-a) + (n+q)(b-a) \equiv -pa + hb \equiv 0 \pmod{N}. \qquad \blacksquare$$

By symmetry, we can define a dual 3-rectangle transformation as shown in Figure 6, denoted by $T'(L)$.

By an argument analogous to the proof of theorem 1, we have

Theorem 2. *$L(N; a-b, N-b)$ can be obtained from $L(N; a, b)$ through the dual 3-rectangle transformation.*

Chen and Hwang [1] proved that $L(N; a, b)$ always satisfies $\ell \geq n$ and $h > p$. So L is well-defined, and $L(N; a, b)$ always has the 3-rectangle transformation as well as its dual.

Example 1.

Figure 7. An example

3. Special Cases

Three types of 2-rectangle transformations (see Figure 8) have been mentioned in [1].

Let us also denote the two L-shapes obtained by interchanging rows and columns of $T(L)$ and $T'(L)$ by $T^{-1}(L)$ and $T^{'-1}(L)$. By comparing Figure 8 (b), (c), (d) with $T(L), T'(L), T^{-1}(L)$ and $T^{'-1}(L)$, we obtain

Theorem 3.

(1) *If* $n = p$, *then* (d) $= T(L)$, (b) $= T'(L)$.

(2) *If* $p = q$, *then* (c) $= T(L)$.

(3) *If* $m = n$, *then* (c) $= T'(L)$.

(4) *If* $m + p = n + q$, *then* (d) $= T^{'-1}(L)$, (b) $= T^{-1}(L)$.

Example 2. Since $n = p = 2$ in $L(20; 3, 2)$, $L(20; 17, 19)$ is the side-turning transformation and $L(20; 1, 18)$ is the top-turning transformation.

Example 3. Since $p = q = 2$ in $L(14; 3, 4)$, $L(14, 11, 1)$ is the shadow-turning transformation.

Figure 8. Some 2-rectangle transformations

Figure 9. $p = q$

Example 4. Since $m = n = 3$ in $L(19; 1, 8)$, $L(19; 12, 11)$ is the shadow-turning transformation.

Example 5. Since $m + p = n + q = 5$ in $L(16; 1, 7)$, the inverse of $L(16; 15, 6)$, which is $L(16; 6, 15)$, is a top-turning transformation, and the inverse of $L(16; 10, 9)$, which is $L(16; 9, 10)$, is a side-turning transformation.

Figure 10. $m = n$

Figure 11. $m + p = n + p$

4. Conclusion

One of the main criteria in designing a local area network is its diameter (or sometimes average distance). Since equivalent double-loop networks have the same diameter and average distance, instead of searching over all networks for minimum diameter (or average distance), we could search just over the equivalent classes, a significant reduction of work.

In this paper, we showed that all equivalent transformations obtained from Rödseth's theorem are 3-rectangle transformations and their duals, a surprising relation between the algebraic analysis and the geometric interpretation. We also showed that other seemingly different geometric transformations are special cases of the 3-rectangle transformation. Our findings raise the interesting question whether two L-shapes are equivalent if and only if one is a 3-rectangle (or dual) transformation of the other. (This question was settled in the negative recently by a forthcoming paper of Chen and Hwang.)

References

1. C. Chen and F. K. Hwang, The minimum distance diagram of double-loop networks, *IEEE Trans. Comput.*, to appear.
2. M. A. Fiol, J. L. A. Yebra, I. Alegre and M. Valero, A discrete optimization problem in local networks and data alignment, *IEEE Trans. Comput.* **C-36** (1957), 702-713.
3. F. H. Hwang and Y. H. Xu, Double loop networks with minimum delay, *Discrete Math.* **66** (1987), 109-118.
4. J. M. Peha and F. A. Tobagi, Analyzing the fault tolerance of double-loop networks, *IEEE Trans. Network* **2** (1994), 363-373.
5. O. J. Rödseth, Weighted multi-connected loop networks, *Discrete Math.* **148** (1996), 161-173.
6. C. K. Wong and D. Coppersmith, A combinatorial problem related to multi-module memory organizations, *J. Assoc. Comput. Mach.* **21** (1974), 392-402.

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