TAIWANESE JOURNAL OF MATHEMATICS
Vol. 4, No. 3, pp. 427-437, September 2000

ON SOME CHARACTERIZATIONS OF POPULATION DISTRIBUTIONS

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Abstract. In this paper, earlier works of the present authors and a method due to Anosov for solving certain intego-functional equations are combined to show that the independence of the sample mean $\bar{X}_n$ and the Z_n-statistic characterizes the normal population, when the random samples are iid from a population having a continuous density function on $\mathbb{R}$, and the sample size $n \geq 3$; obviously the sample standard deviation is a Z_n-statistic. Further, an important subclass of Z_n-statistic with the form of a linear combination $\sum_{i=1}^n a_i X_{(i)}$ of order statistics is found, where $a_1 \leq \cdots \leq a_n$, not all equal and $\sum_{i=1}^n a_i = 0$, which includes Gini′s mean difference and the sample range but not the sample standard deviation.

Similar approach can be applied to prove that the independence of $\bar{X}_n$ and $Z_n/\bar{X}_n$ characterizes the gamma distribution; obviously the independence of sample mean and sample coefficient of variation characterizes the gamma distribution.

The study of identifying Z_n to more known statistics will be the future work.

1. Introduction and Main Result

The problem of characterizing population distributions through various properties of statistics has long been a subject for study; see Kagan *et al.* [9], Galambos-Kotz [3], Kakosyan *et al.* [10], and the references therein. To characterize normality through the independence of a tube statistic with finite basis and the sample mean, the corresponding integro-functional equation can

[0] Received October 7, 1998; revised March 15, 1999.
Communicated by P. Y. Wu.
2000 *Mathematics Subject Classification*: 62E10.
Key words and phrases: Gini′s mean difference, normal density, range, studentized order statistics.

be found if Anosov′s approach [1] is applied. Note that a statistic is called a *tube statistic*, if it is continuous, nonnegative, homogeneous with a positive degree of homogeneity (cf. [9, pp. 101-102]); the sample variance is the most well-known.

The motivation of this paper is to characterize some population distributions through the independence of more known statistics with the sample mean under weaker supplementary conditions than the classical approach. Our approach is to modify the corresponding integro-functional equation by using the earlier results of Hwang and Hu [4, 7] and the tube statistics to more general statistics.

Section 2 gives the definition of the modified statistics, Z_n-statistics, say. Section 3 shows that the independence of the sample mean $\bar{X}_n$ and the Z_n-statistics characterizes the normal population; an important subclass of Z_n-statistics with the form of a linear function $\Sigma_{i=1}^n a_i X_{(i)}$ of order statistics is presented, where $a_1 \leq \cdots \leq a_n$ are not all equal and $\Sigma_{i=1}^n a_i = 0$. Note that the sample standard deviation not included in the subclass is also a Z_n-statistic. Since similar approaches can be applied to prove that the independence of $\bar{X}_n$ and $Z_n/\bar{X}_n$ characterizes the gamma distribution, thus the corresponding main theorem and corollaries will be presented in Section 4. Corollary 4.1 has been found by Hwang and Hu [8].

This paper obtains more results than those classical approaches that need some suplementary conditions, e.g., the existence of moments of $X_i{}'$s (cf. [9, Theorem 6.2.9]). Since more known Z_n's are still not found, the study of identifying Z_n to more known statistics will be the future work.

2. Definition of Z_n-Statistics for Normal Population

Let $X_1, \cdots, X_n$ be iid random samples of size $n \geq 3$ from an absolutely continuous distribution, $X_{(1)} \leq \cdots \leq X_{(n)}$ be their order statistics, and $\bar{X}_n$ and S_n be the sample mean and the sample standard deviation, respectively, that is,

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2. \tag{2.1}$$

The studentized order statistics are denoted by

$$\lambda_i = \frac{X_{(i)} - \bar{X}_n}{S_n}, \quad 1 \leq i \leq n, \tag{2.2}$$

and hence $\lambda_1 \leq \cdots \leq \lambda_n$,

$$\sum_{i=1}^{n} \lambda_i = 0$$

and

$$\sum_{i=1}^{n} \lambda_i^2 = n - 1. \tag{2.3}$$

Define statistics t_i as follows:

$$t_i = \left[\frac{n-i+1}{(n-1)(n-i)}\right]^{\frac{1}{2}} \cdot \left(\lambda_i + \frac{1}{n-i+1}\sum_{k=1}^{i-1}\lambda_k\right), \quad 1 \le i \le n-1, \tag{2.4}$$

where the summation in (2.4) is taken as zero for $i = 1$. It can be shown that these statistics satisfy $t_1^2 + \cdots + t_{n-1}^2 = 1$. Let $\underset{\sim}{t} = (t_1, \cdots, t_{n-1})$. Then from (2.4), the inverse relationship of $\underset{\sim}{t} = (t_1, \cdots, t_{n-1})$ with respect to $\underset{\sim}{\lambda} = (\lambda_1, \cdots, \lambda_{n-1})$ can be derived as follows:

$$\lambda_i(\underset{\sim}{t}) = \left[\frac{(n-i)(n-1)}{n-i+1}\right]^{\frac{1}{2}} \cdot t_i - \sum_{k=1}^{i-1}\left[\frac{n-1}{(n-k)(n-k+1)}\right]^{\frac{1}{2}} \cdot t_k, \quad 1 \le i \le n-1,$$

and $\sum\limits_{i=1}^{n} \lambda_i = 0$ gives

$$\lambda_n(\underset{\sim}{t}) = -\sum_{k=1}^{n-1}\left[\frac{n-1}{(n-k)(n-k+1)}\right]^{\frac{1}{2}} \cdot t_k, \tag{2.5}$$

where the summation will be taken as zero for $i = 1$. In the following, we set $\lambda_i = \lambda_i(\underset{\sim}{t})$, that is, λ_i is a function of the $n-1$ variables $t_1, \cdots, t_{n-1}$, and $\underset{\sim}{\lambda}(\underset{\sim}{t}) = (\lambda_1(\underset{\sim}{t}), \cdots, \lambda_n(\underset{\sim}{t}))$ is a vector function of the vector $\underset{\sim}{t} = (t_1, \cdots, t_{n-1})$.

In view of the relationships (2.4) and (2.5), the vector function $\underset{\sim}{\lambda}(\underset{\sim}{t}) = (\lambda_1(\underset{\sim}{t}), \cdots, \lambda_n(\underset{\sim}{t}))$ establishes a one-to-one correspondence between the domain B_n and the range A_n (see [5]), where

$$B_n = \left\{ \underset{\sim}{t} : \begin{array}{l} t_1^2 + \cdots + t_{n-1}^2 = 1, \\ \left(\frac{n-k+2}{n-k}\right)^{\frac{1}{2}} \cdot t_{k-1} \le t_k \le 0, \quad 2 \le k \le n-1 \end{array} \right\} \tag{2.6}$$

and

$$A_n = \left\{ \underset{\sim}{\lambda} : \begin{array}{l} \lambda_1 \le \cdots \le \lambda_n, \\ \sum\limits_{i=1}^{n} \lambda_i = 0 \quad \text{and} \quad \sum\limits_{i=1}^{n} \lambda_i^2 = n-1 \end{array} \right\}. \tag{2.7}$$

In the following, we will define a class of functions of order statistics by using the studentized order statistics λ_i, given as in (2.2), or equivalently by the special linear combinations of the statistics t_i as given in (2.4).

First, let A_n be defined as in (2.7) and let M be an open subset of $\mathbb{R}^n$ such that $A_n \subset M$. Furthermore, let ψ be a real-valued function defined over M and assume that there exist two constants a and b depending upon n such that

$$-\infty < b = \inf_{t \in B_n} \psi(\lambda(t)) \leq \sup_{t \in B_n} \psi(\lambda(t)) = a < +\infty, \tag{2.8}$$

where B_n is as given in (2.6). Thus we have the following

Definition 2.1. Let $\underset{\sim}{t} = (t_1, \cdots, t_{n-1})$, $\underset{\sim}{\lambda}(\underset{\sim}{t}) = (\lambda_1(\underset{\sim}{t}), \cdots, \lambda_n(\underset{\sim}{t}))$ and ψ be as defined in (2.4), (2.5) and (2.8), respectively, and let ϕ be the composite function of ψ and λ, that is, $\phi(\underset{\sim}{t}) = \psi(\underset{\sim}{\lambda}(\underset{\sim}{t}))$. Define a function of $t_1, t_2, \cdots, t_{n-1}$ as follows:

$$Z_n = S_n \cdot e^{\phi(t)}, \tag{2.9}$$

where S_n is as given in (2.1).

For convenience, the statistic (2.9) will be called Z_n-statistics. Note that since $\phi = \psi \circ \lambda$, it follows from the definitions of ψ and λ that the function ϕ is continuous and satisfies

$$-\infty < b = \inf_{t \in B_n} \phi(\underset{\sim}{t}) \leq \sup_{t \in B_n} \phi(\underset{\sim}{t}) = a < +\infty, \tag{2.10}$$

where the constants a and b depend on n, as given in (2.8).

In this paper, we establish the characterization theorem as follows:

Main Theorem. *Let $X_1, \cdots, X_n$ be iid random samples from a population having a continuous density function $f(x)$ on $\mathbb{R}$. Then, the independence of the sample mean $\bar{X}_n$ and the Z_n-statistics, as given in* (2.9), *is equivalent to*

$$f(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot e^{-(x-u)^2/2\sigma^2}, \quad -\infty < x < +\infty,$$

where $\sigma > 0$ and u are parameters.

The detailed proof will be presented in Section 3. Sometimes the results derived from the main theorem might be misunderstood as some well-known characterization results. As a matter of fact, it holds without any supplementary conditions on the moments of the X_i's (cf. [9, Theorem 6.2.9]).

Four particular Z_n's will be shown in the following four corollaries.

Corollary 2.1. *Under the conditions of Main Theorem, the independence of the sample mean* $\bar{X}_n$ *and the sample standard deviaton* S_n *is quivalent to the population having the normal density.*

Proof. This corollary is established by letting $\phi(\underset{\sim}{t}) = \Sigma_{i=1}^{n-1} t_i^2 = 1$. ■

Note that Corollary 2.1 still holds without the condition of continuity imposed on $f(x)$; see Kawata-Sakamoto [1], [9, Theorem 4.2.1].

Corollary 2.2. *Under the conditions of the Main Theorem, the independence of* $\bar{X}_n$ *and* Z_n *is equivalent to the population having the normal density, where* $Z_n = \Sigma_{i=1}^n a_i X_{(i)}$ *with* $a_1 \leq \cdots \leq a_n$ *not all equal and* $\Sigma_{i=1}^n a_i = 0$.

Proof. Let

$$Z_n = \sum_{i=1}^n a_i X_{(i)}, \tag{2.11}$$

where $a_1 \leq \cdots \leq a_n$ are not all equal and $\Sigma_{i=1}^n a_i = 0$. The statistics (2.11) can be rewritten as the form (2.9) if we define

$$\psi\left(\lambda_1, \cdots, \lambda_n\right) = \mathrm{Ln}\left(\sum_{i=1}^n a_i \lambda_i\right), \tag{2.12}$$

where Ln is the natural logarithmic function and λ_i the studentized order variables. In view of the relationship $\phi(t) = \psi(\lambda(t))$ and

$$\sum_{i=1}^n a_i X_{(i)} = \sum_{i=1}^n \left(a_i - \bar{a}_n\right)\left(X_{(i)} - \bar{X}_n\right)$$

with $a_1 \leq \cdots \leq a_n$ not all equal and $\Sigma_{i=1}^n a_i = 0$, it follows from [5] and the Cauchy-Schwarz inequality that the real-valued function ψ as given in (2.8) is well-defined and

$$\inf_{t \in B_n} \psi(\lambda(t)) \geq \frac{1}{2}\mathrm{Ln}\left(\frac{1}{n-1}\sum_{i=1}^n a_i^2\right) > -\infty$$

and

$$\sup_{t \in B_n} \psi(\lambda(t)) \leq \frac{1}{2}\mathrm{Ln}\left((n-1)\sum_{i=1}^n a_i^2\right) < +\infty,$$

that is, the condition (2.8) is satisfied. Other conditions of Definition 2.1 are clear, and thus the function of order statistics (2.11) is a Z_n-statistic. ■

Corollary 2.3. *Under the conditions of Main theorem, let G_n be Gini′s mean difference*

$$G_n = \frac{1}{n(n-1)} \sum_{i,j=1}^{n} |X_i - X_j| .$$

Then the independence of $\bar{X}_n$ and G_n is equivalent to the population having the normal density.

Proof. Gini′s mean difference can be rewritten as (cf. [2, p.216])

$$G_n = \frac{1}{n(n-1)} \sum_{i=1}^{n} [2i - n - 1] X_{(i)}$$

and this corollary can be established by Corollary 2.2. ■

Corollary 2.4. *Under the conditions of Main theorem, let $W_n = X_{(n)} - X_{(1)}$ be the sample range. Then the independence of $\bar{X}_n$ and W_n is equivalent to the population having the normal density.*

Proof. Taking $a_1 = -1$, $a_i = 0$, $i = 2, \cdots, n-1$, and $a_n = 1$, this corollary is established by Corollary 2.2. ■

Note that Corollaries 2.3 and 2.4 are new results. Since Gini′s mean difference and the sample range are in terms of order statistics which are not tube statistic, so they cannot be dealt with by the classical approach such as Anosov′s approach.

3. Proof of Main Theorem

For representing the integro-functional equation, we need the following notation:

Let B_n be as given in (2.6), and let $\alpha_i(t)$, $1 \le i \le n$, be n continuous functions on B_n with

$$\sum_{i=1}^{n} \alpha_i(t) = 0 \quad \text{and} \quad \sum_{i=1}^{n} \alpha_i^2(t) = g_n(t) > 0 \tag{3.1}$$

for all $t \in B_n$, where the continuous function g_n is uniformly bounded over B_n, that is, there exists a constant K_n such that

$$\sup_{t \in B_n} |g_n(t)| = K_n < +\infty. \tag{3.2}$$

Thus, we have

Definition 3.1. Let f be a continuous density function on $(-\infty, +\infty)$, and let the integro-functional equation be defined by

$$\int_{B_n} \prod_{i=1}^{n} f(x + z \cdot \alpha_i(t)) dH_n(t) = c_n \cdot [f(x)]^n \cdot \int_{B_n} \prod_{i=1}^{n} f(z \cdot \alpha_i(t)) dH_n(t) \tag{3.3}$$

for all $x \in \mathbb{R}$ and $z > 0$, where c_n is a constant, $H_n(t)$ the distribution function over B_n and $\alpha_i(t)$ and B_n are as given in (2.6) and (3.1).

In order to prove the main theorem, we need the following lemmas. The proofs of Lemmas 3.1 and 3.2 use some facts in [4].

Lemma 3.1 [4, Theorem 2.1].
Define a nonlinear transformation $(x_1, \cdots, x_n) \longmapsto (\bar{x}_n, s_n, t_1, \cdots, t_{n-2})$, *where* $\bar{x}_n$ *and* s_n *are as given in* (2.1), *and* $t_i's$ *are as defined in* (2.4). *Then, the nonlinear transformation establishes a one-to-one correspondence between the domain* $\{x : x_1 \le \cdots \le x_n\}$ *and the set*

$$\left\{ \begin{array}{c} -1 \le t_1 \le \frac{-1}{n-1}, \\ (\bar{x}_n, s_n t_1, \cdots, t_{n-2}) : \max\left\{ \left(\frac{n-k+2}{n-k}\right)^{\frac{1}{2}} \cdot t_{k-1}, -f_{k-1}^{\frac{1}{2}} \right\} \le t_k \le -\frac{f_{k-1}^{\frac{1}{2}}}{n-k}, \\ 2 \le k \le n-2, \bar{x}_n \in \mathbb{R}, s_n \ge 0 \end{array} \right\} \tag{3.4}$$

except for a set of n*-dimensional Lebesgue measure zero. The absolute value of its Jacobian is*

$$|J| = \sqrt{n} \cdot (n-1)^{(n-1)/2} \cdot s_n^{n-2} \cdot f_{n-2}^{\frac{1}{2}}, \tag{3.5}$$

where $f_i = 1 - t_1^2 - \cdots - t_i^2, 1 \le i \le n-2$.

Lemma 3.2 [4, Formula 1.2].
The inverse transformation $(\bar{x}_n, s_n, t_1, \cdots, t_{n-2}) \longmapsto \left(x_{(1)}, \cdots, x_{(n)}\right)$ *of Lemma* 3.1 *is given by*

$$x_{(i)} = \bar{x}_n + s_n \cdot \lambda_i(t), \quad 1 \le i \le n, \tag{3.6}$$

where $t = (t_1, \ldots, t_{n-1})$ with $t_1^2 + \cdots + t_{n-1}^2 = 1$ and $\lambda_1(t), \cdots, \lambda_n(t)$ are defined in (2.2) and satisfy (2.3).

Lemma 3.3. *Under the conditions of Main Theorem, let* $Z_n = S_n e^{\phi(t)}$ *be as defined in* (2.9) *and assume that the continuous function* ϕ *satisfies* (2.10) *such that the transformation* $(x_{(1)}, \cdots, x_{(x)}) \longmapsto (\bar{x}_n, z_n t_1, \cdots, t_{n-2})$ *from the domain* $\{x : x_{(1)} \leq \cdots \leq x_{(n)}\}$ *into its range is a one-to-one correspondence except for a set of* n*-dimensional Lebesgue measure zero. Then the joint density* $f(\bar{x}, z)$ *of* $\bar{X}_n$ *and* Z_n *exists and satisfies*

(3.7)
$$f(\bar{x}, z) = c_n \cdot z^{n-2} \cdot \int_{B_n} e^{-(n-1)\phi(t)} \cdot \prod_{i=1}^{n} f\left(\bar{x} + ze^{-\phi(t)} \cdot \lambda_i(t)\right) d\sigma_n(t)$$

for all real $\bar{x} > 0$ *and* $z > 0$, *and zero otherwise, where* $\lambda_i(t)$ *and* $\alpha_n(t)$ *are as defined in* (2.5), (2.6) *and* (3.1), *respectively,* $\sigma_n(t)$ *is the uniform distribution over* B_n *and the normalizing constant is given by*

$$c_n = 2\sqrt{n} \cdot [\pi \cdot (n-1)]^{(n-1)/2} / \Gamma\left(\frac{n-1}{2}\right). \tag{3.8}$$

Proof. Kagan-Linnik-Rao [9, Theorems 3.1, 3.2 and pp.139-141] establishes the proof. ∎

Lemma 3.4. *Under the conditions of Lemma* 3.3, *assume that* $\bar{X}_n$ *and* Z_n *are independent. Then, the marginal pdf's of* $\bar{X}_n$ *and* Z_n *are respectively given by*

$$f_{\bar{X}_n}(x) = a_n \cdot [f(x)]^n, \quad -\infty < x < +\infty, \tag{3.9}$$

where a_n *is the normalizing constant, and*

$$f_{Z_n}(z) = b_n \cdot z^{n-2} \int_{B_{n,z}} e^{-(n-1)\phi(t)} \cdot \prod_{i=1}^{n} f(ze^{-\phi(t)} \cdot \lambda_i(t)) d\sigma_n(t) \tag{3.10}$$

for $z > 0$ *and zero otherwise, where* b_n *is the normalizing constant, and* $\lambda_i(t)$, B_n *and* $\sigma_n(t)$ *are as defined in Lemma* 3.3.

Proof. Hwang-Hu [6, Lemma 2.2] uses the similar approach to establish this lemma. ∎

Lemma 3.5. *Under the conditions of Lemma* 3.4, *the following integro-functional equation holds*:

$$\begin{aligned} &\int_{B_n} e^{-(n-1)\phi(t)} \prod_{i=1}^{n} f\left(x + ze^{-\phi(t)} \cdot \lambda_i(t)\right) d\sigma_n(t) \\ &= c_n' \cdot [f(x)]^n \cdot \int_{B_n} e^{-(n-1)\phi(t)} \cdot \prod_{i=1}^{n} f\left(ze^{-\phi(t)} \cdot \lambda_i(t)\right) d\sigma_n(t) \end{aligned} \tag{3.11}$$

for all x and $z > 0$, where $c_n' > 0$ is a constant, and $\lambda_i(t)$, B_n and $\sigma_n(t)$ are as defined in Lemma 3.3.

Proof. This lemma follows immediately from Lemmas 3.3, 3.4, and the independence of $\bar{X}_n$ and Z_n. ∎

Lemma 3.6 (Anosov [1]). *The normal density is the only solution of the integro-functional equation* (3.3).

Proof. The proof of this lemma is essentially the same as the one given in [1] provided that Lemma 3.5 holds, and hence the detailed proof is omitted here. ∎

Finally, we are in a position to prove the main theorem by using these lemmas.

Proof of Main Theorem. First, define a probability measure u_n on B_n by

$$u_n(A) = a_n \cdot \int_A e^{-(n-1)\phi(t)} \cdot d\sigma_n(t)$$

for all measurable subsets A of B_n, where a_n is the normalizing constant, $\alpha_n(t)$ is as defined in Definition 3.1, $\phi = \psi \circ \lambda$ in Definition 2.1, and B_n in (2.6). In view of the condition (2.10), this probability measure on B_n is well-defined. For convenience, let $H_n^*(t)$ be the corresponding distribution function of u_n, and define

$$\alpha_i^*(t) = e^{-\phi(t)} \cdot \lambda_i(t), \quad 1 \leq i \leq n,$$

for $t \in B_n$, where $\lambda_i(t)$ are as defined in (2.5). It follows from (2.3) that $\alpha_i^*(t)$ satisfy the conditions (3.2) and (3.3) with $g_n(t) = (n-1) \cdot e^{-2\phi(t)}$ and $K_n = (n-1) \cdot e^{-2b}$, where b is as given in (2.10). And hence, the integro-function equation (3.10) can be rewritten as the form (3.3) with $H_n(t) = H_n^*(t)$ and $\alpha_i(t) = \alpha_i^*(t)$. Therefore, the sufficiency of main theorem is now easily established in a few words. Suppose that the function ϕ also satisfies the conditions of Lemma 3.1. Then the sufficiency follows immediately from Lemma 3.7. Note that if ϕ is a multivalue function, then the proof of Lemma 3.3 is unchanged, but we obtain an expression of the joint density of $(\bar{X}_n, Z_n)$, which is a finite sum of the form (3.7), and hence (3.10), (3.11) will be a finite sum of such integrals. However, the domain of these integrations is independent upon the variable z. Thus Lemma 3.6 is also true and can be applied to obtain the same conclusion. The necessity of main theorem follows immediately from Lemma 3.3, taking the relationship (2.3) into account, that is, $\Sigma_{i=1}^n \lambda_i(t) = 0$. Note that the necessity also follows from the fact that

$\bar{X}_n$ and $(X_{(1)} - \bar{X}_n, \cdots, X_{(n)} - \bar{X}_n)$ are independent when the population is normal. Thus, the proof of our main theorem is completed. ■

4. Gamma Population

Since similar approaches as presented in Section 2, 3 and in [7] can be applied to prove that the independence of $\bar{X}_n$ and $Z_n/\bar{X}_n$ characterizes the gamma distribution, thus we have the corresponding main theorem and four corollaries as follows:

Theorem 4.1. *Let $X_1, \cdots, X_n$ be iid random samples from a population having a probability density function $f(x)$, where $f(x)$ is continuous on $(0, +\infty)$. Then, the independence of the sample mean $\bar{X}_n$ and $Z_n/\bar{X}_n$-statistic is equivalent to*

$$f(x) = \frac{1}{\Gamma(\alpha) \cdot \beta^{\alpha}} x^{\alpha_1} \cdot e^{-x/\beta}, \quad x > 0, \tag{4.1}$$

and zero otherwise, where $\alpha, \beta > 0$ are parameters, and Z_n-statistic is as defined in (2.9).

Corresponding to Corollaries 2.1, 2.2, 2.3 and 2.4, we have the following results.

Corollary 4.1. *Under the conditions of Theorem* 4.1, *let $V_n = S_n/\bar{X}_n$ be the sample coefficient of variation. Then the independence of $\bar{X}_n$ and V_n is equivalent to the gamma density* (4.1).

Corollary 4.2. *Under the conditions of Theorem* 4.1, *the independence of $\bar{X}_n$ and $Z_n/\bar{X}_n$ is equivalent to the gamma density* (4.1), *where Z_n is as defined in Corollary* 2.2.

Corollary 4.3. *Under the conditions of Theorem* 4.1, *the independence of $\bar{X}_n$ and $G_n/\bar{X}_n$ is equivalent to the gamma density* (4.1), *where G_n is as defined in Corollary* 2.3.

Note that Corollary 4.2 still holds in the corresponding regression problem (cf. [9, Theorem 6.2.9]); however, it is only necessary to assume the existence of the first and second moments.

Corollary 4.4. *Under the conditions of Theorem* 4.1, *let $W_n = X_{(n)} - X_{(1)}$ be the sample range. Then the independence of $\bar{X}_n$ and $W_n/\bar{X}_n$ is equivalent to the gamma density* (4.1).

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