TAIWANESE JOURNAL OF MATHEMATICS
Vol. 4, No. 3, pp. 365-383, September 2000

# ERGODIC THEOREMS AND APPROXIMATION THEOREMS WITH RATES

Sen-Yen Shaw

**Abstract.** $A$-ergodic nets and $A$-regularized approximation processes of operators are introduced and their convergence theorems are discussed. There are strong convergence theorems, uniform convergence theorems, theorems on optimal convergence, and theorems on non-optimal convergence and its sharpness. The general results provide unified approaches to investigation of convergence rates of ergodic limits and approximation of various operator families. In particular, we shall deduce some results for an $r$-times integrated resolvent family for a Volterra integral equation. The latter contains integrated semigroups and integrated cosine functions as special cases.

## 1. Introduction

In 1931, von Neumann published the first mean ergodic theorem. It states that if $\tau$ is a measure preserving transformation on a measure space $(\Omega, \mu)$, then for every $f \in L_2(\mu)$, $(1/n)\sum_{k=0}^{n-1} f\circ\tau^k$ converges in $\|\cdot\|_2$ norm to some $\overline{f} \in L_2(\mu)$ such that $\overline{f}\circ\tau = \overline{f}$. Since then, generalizations to more general operators on abstract spaces have been proved by numerous authors, among whom are Riesz (1938), Yosida (1938), Lorch (1939), and Dunford (1939), to mention only a few earlier ones. There was also Eberlein's (1949) abstract ergodic

[0] Received March 2, 2000.
Communicated by P. Y. Wu.
2000 *Mathematics Subject Classification*: 47A35, 47A58, 41A25, 47D06, 47D09.
*Key words and phrases*: $A$-ergodic net, $A$-regularized approximation process, saturation property, non-optimal convergence, $K$-functional, Grothendieck space, the Dunford-Pettis property, $r$-times integrated solution family, Volterra integral equation.
Addresses based on this paper were given at the 15th Southeastern Analysis Meeting, Vanderbilt University, May 20-23, 1999, and at the 1999 Mathematics Conference and Annual Meeting of the Mathematical Society of the R.O.C., Fu Jen Catholic University, December 10-12, 1999.
Research supported in part by the National Science Council of Taiwan.

theorem for $S$-ergodic nets. The first continuous version of mean ergodic theorem, which deals with convergence of Cesáro means of strongly continuous semigroups of operators, was proved by Dunford (1957). One can also consider convergence of Abel means, which leads to Yosida's (1961) ergodic theorem for pseudo-resolvents.

The above results are concerned with the strong convergence of the involved objects, so they are called strong ergodic theorems. It is also interesting to consider criteria for their convergence in operator norm. Theorems of this kind are called uniform ergodic theorems. Among them are theorems due to Yosida and Kakutani (1941), M. Lin (1974), Lotz (1985) [16], and Shaw (1986 [22], 1988 [23]).

The first result which considers the rates of convergence of ergodic limits is the saturation theorem of Butzer and Westphal (1971) [8] which describes optimal convergence rates of ergodic limits of discrete semigroups. Later, Butzer and Dickmeis (1981) [5] investigated not only optimal convergence but also non-optimal convergence for strongly continuous semigroups. Recently, Nasri-Roudsari, Nessel and Zeler (1995) [18] established the sharpness of non-optimal convergence. For a more detailed account of the developement of ergodic theorems, the readers are referred to [30].

Eberlein's abstract frame work, i.e., $S$-ergodic net, provides a unified approach to deduce many strong ergodic theorems, but it cannot deal with uniform ergodic theorems or convergence rates of ergodic limits. To improve this weakness, in recent years I have introduced an abstract frame work, called *$A$-ergodic nets*. One of the purposes of this paper is to review (in Section 2) some results in connection with $A$-ergodic nets.

While a Cesáro ergodic theorem with rate for a strongly continuous semigroup (usually called $C_0$-semigroup) $\{T(t); t \geq 0\}$ of operators is concerned with the convergence rates of the Cesáro mean $C_t x := t^{-1} \int_0^t T(s)xds$ as $t \to \infty$ for various $x$, a local ergodic theorem with rate deals with the convergence rate of $C_t x$ as $t \to 0^+$. Since $T(t)$ converges strongly to the identity operator $I$ as $t \to 0^+$, both $\{T(t)\}$ and $\{C_t\}$ are approximation processes. The first result in the direction of the convergence rate of an approximation process seems to be the saturation theorem of Butzer and Berens (1967) about optimal convergence of $C_0$-semigroups. In 1968, they also characterized non-optimal convergence rate (i.e., those $x$ for which $\|T(t)x - x\| = O(t^\beta)$ with $0 < \beta < 1$) in terms of the convergence rate of a $K$-functional. The first result concerning the sharpness of non-optimal convergence of $C_0$-semigroups was proved by Butzer and Dickmeis (1985), and later improved by Davydov (1993) [11].

To deal with optimal convergence rates for various approximation processes, Butzer and Nessel (1971) [7] introduced an abstract frame work, called *approximation process with a regularization process*. Unfortunately, it is not useful for investigating non-optimal convergence rates.

In Section 3, I shall introduce an improved abstract frame work, namely *A-regularized approximation process*, which can deal with optimal convergence, non- optimal convergence, and its sharpness.

As applications of the general results to be discussed in Sections 2 and 3, we shall deduce in Section 4 uniform ergodic theorem and strong ergodic theorem with rates for $r$-times integrated solution families for Volterra integral equations, and in Section 5 their approximation and local ergodic theorems with rates.

## 2. Abstract Ergodic Theorems for $A$-ergodic Nets

In [24-26] and [29] we considered the following framework for discussing general strong ergodic theorems, uniform ergodic theorems, and ergodic theorems with rates.

Let $A : D(A) \subset X \to X$ be a closed linear operator, and let $\{A_\alpha\}$ and $\{B_\alpha\}$ be two nets in $B(X)$ satisfying:

(C1) $\|A_\alpha\| \le M \quad$ for all $\alpha$;

(C2) $R(B_\alpha) \subset D(A)$ and $B_\alpha A \subset AB_\alpha = I - A_\alpha$ for all $\alpha$;

(C3) $R(A_\alpha) \subset D(A)$ for all $\alpha$, and $\|AA_\alpha\| = O(e(\alpha))$;

(C4) $B_\alpha^* x^* = \varphi(\alpha)x^*$ for all $x^* \in R(A)^\perp$, and $|\varphi(\alpha)| \to \infty$;

(C5) $\|A_\alpha x\| = O(f(\alpha))$ (resp. $o(f(\alpha))$) implies $\|B_\alpha y\| = O(f(\alpha)/e(\alpha))$ (resp. $o(f(\alpha)/e(\alpha))$),

where $e$ and $f$ are positive functions satisfying $0 < e(\alpha) \le f(\alpha) \to 0$. We call $\{A_\alpha\}$ an *$A$-ergodic net* and $\{B_\alpha\}$ its *companion net*.

The functions $e$ and $f$ are to act as estimators of the convergence rates of $\{A_\alpha x\}$ and $\{B_\alpha y\}$, which, in practical applications, approximate the ergodic limit and the solution $x$ of $Ax = y$, respectively.

Let $P$ and $B_1$ be the operators defined respectively by

$$\begin{cases} D(P) := \{x \in X; \lim_\alpha A_\alpha x \text{ exists}\}; \\ Px := \lim_\alpha A_\alpha x \text{ for } x \in D(P), \end{cases} \qquad \begin{cases} D(B_1) := \{y \in X; \lim_\alpha B_\alpha y \text{ exists}\}; \\ B_1 x := \lim_\alpha B_\alpha y \text{ for } y \in D(B_1). \end{cases}$$

$\{A_\alpha\}$ is said to be *strongly* (resp. *uniformly*) *ergodic* if $D(P) = X$ and $A_\alpha x \to Px$ for all $x \in X$ (resp. $\|A_\alpha - P\| \to 0$).

In [24, Theorem 1.1, Corollary 1.4 and Remark 1.7], we proved the following theorem.

**Theorem 2.1** (Strong Ergodic Theorem). *Under conditions* (C1) - (C4), *the following are true.*

(i) $P$ *is a bounded linear projection with range* $R(P) = N(A)$, *null space* $N(P) = \overline{R(A)}$, *and domain* $D(P) = N(A) \oplus \overline{R(A)} = \{x \in X; \{A_\alpha x\}$ *has a weak cluster point*$\}$.

(ii) $B_1$ *is the inverse operator* $A_1^{-1}$ *of the restriction* $A_1 := A|\overline{R(A)}$ *of* $A$ *to* $\overline{R(A)}$; *it has range* $R(B_1) = D(A_1) = D(A) \cap \overline{R(A)}$ *and domain* $D(B_1) = R(A_1) = A(D(A) \cap \overline{R(A)})$. *Moreover, for each* $y \in D(B_1)$, $B_1 y$ *is the unique solution of the functional equation* $Ax = y$ *in* $\overline{R(A)}$.

(iii) $\{A_\alpha\}$ *is strongly ergodic if and only if* $N(A)$ *separates* $R(A)^\perp$, *if and only if* $\{A_\alpha x\}$ *has a weak cluster point for each* $x \in X$. *In this case, we have* $R(A) = R(A_1)$. *These conditions are satisfied in particular when* $X$ *is reflexive.*

Note that when $A$ is densely defined, the condition $R(A) = R(A_1)$ is also equivalent to the strong ergodicity. The next theorem is proved in [25] under the assumption that $A$ is densely defined. It can be shown that the conclusion still holds without this assumption.

**Theorem 2.2** (Uniform Ergodic Theorem). *Under conditions* (C1) - (C3), *we have*: $D(P) = X$ *and* $\|A_\alpha - P\| \to 0$ *if and only if* $\|B_\alpha|_{R(A)}\| = O(1)$, *if and only if* $B_1$ *is bounded and* $\|B_\alpha|_{R(A)} - B_1\| \to 0$, *if and only if* $R(A)$ (*or* $R(A_1)$) *is closed, if and only if* $R(A^2)$ (*or* $R(A_1^2)$) *is closed, if and only if* $X = N(A) \oplus R(A)$. *Moreover, the convergence of these limits has order* $O(e(\alpha))$.

A Banach space $X$ is called a *Grothendieck space* if every weakly$^*$ convergent sequence in $X^*$ is weakly convergent (see, e.g., [28] for equivalent definitions), and is said to have the *Dunford-Pettis property* if every weakly compact operator from $X$ to any Banach space maps weakly compact sets into norm compact sets or, equivalently, if $\langle x_n, x_n^* \rangle \to 0$ whenever $x_n \to 0$ weakly in $X$ and $x_n^* \to 0$ weakly in $X^*$. The spaces $L^\infty$, $H^\infty$, and $B(S, \Sigma)$ are particular examples of Grothendieck spaces with the Dunford-Pettis property. An interesting phenomenon in such spaces is that strong operator convergence often implies uniform operator convergence (see, e.g., [15, 23, 27]). The following theorem slightly generalizes Theorem 2 in [25], which deals only with the case that $A$ has dense domain.

**Theorem 2.3.** *Let* $\{\{A_\alpha\}, \{B_\alpha\}, A\}$ *satisfy conditions* (C1) - (C3), *and suppose* $\{x \in D(A); Ax \in Y\}$ *is dense in* $Y := \overline{D(A)}$. *When* $Y$ *is a Grothendieck* *space with the Dunford-Pettis property,* $\{A_\alpha\}$ *is uniformly ergodic on* $Y$ *if and only if it is strongly convergent on* $Y$.

*Proof.* We first see that $A_\alpha Ax = AA_\alpha x$ for all $x \in D(A)$. Indeed, if $x \in D(A)$, then by (C2) we have $A_\alpha x = x - B_\alpha Ax \in D(A)$ and $AA_\alpha x = Ax - AB_\alpha Ax = Ax - (I - A_\alpha)Ax = A_\alpha Ax$. Hence $A_\alpha A \subset AA_\alpha$. Note that $Y$ is an invariant subspace for $A_\alpha$ and $B_\alpha$. Let $A_\alpha^\circ$ and $B_\alpha^\circ$ denote their restrictions to $Y$, and let $A^\circ$ denote the part of $A$ in $Y$. Being the intersection of the closed graph of $A$ and $Y \times Y$, the graph of $A^\circ$ is closed. Using this and the fact that $R(A_\alpha) \subset D(A)$ and $A_\alpha A \subset AA_\alpha$, we easily see that $R(A_\alpha^\circ) \subset A^\circ$, $A_\alpha^\circ A^\circ \subset A^\circ A_\alpha^\circ$ and $\|A^\circ A_\alpha^\circ\|_Y = O(e(\alpha))$, i.e., (C3) holds with $A_\alpha$ and $A$ replaced by $A_\alpha^\circ$ and $A^\circ$, respectively. Similarly, (C2) with the closedness of $A^\circ$ implies that it holds with $A_\alpha$, $B_\alpha$, and $A$ replaced by $A_\alpha^\circ$ $B_\alpha^\circ$, and $A^\circ$, respectively. Hence $\{A_\alpha^\circ\}$ is an $A^\circ$-ergodic net on $Y$ and $\{B_\alpha^\circ\}$ is its companion net. Since $D(A^\circ) = \{x \in D(A);\ Ax \in Y\}$ is assumed to be dense in $Y$, and $Y$ is assumed to be a Grothendieck space with the Dunford-Pettis property, it follows from Theorem 2 of [25] that the strong convergence of $\{A_\alpha^\circ\}$ on $Y$ implies its operator-norm convergence. ∎

The rates of covergence of ergodic limits are characterized by means of *K-functional* and *relative completion*, which we recall as below.

Let $X$ be a Banach space with norm $\|\cdot\|_X$, and $Y$ a submanifold with seminorm $\|\cdot\|_Y$. The *K-functional* is defined by

$$K(t,x) := K(t,x,X,Y,\|\cdot\|_Y) = \inf_{y\in Y}\{\|x-y\|_X + t\|y\|_Y\}.$$

If $Y$ is a Banach space with norm $\|\cdot\|_Y$, the *completion of* $Y$ *relative to* $X$ is defined as

$$Y\tilde{}_X := \{x \in X : \exists\{x_m\} \subset Y \text{ such that } \lim_{m\to\infty}\|x_m - x\|_X = 0 \text{ and } \sup\|x_m\|_Y < \infty\}.$$

It is known [4] that $K(t,x)$ is a bounded, continuous, monotone increasing and subadditive function of $t$ for each $x \in X$, and $K(t,x,X,Y,\|\cdot\|_Y) = O(t)$ $(t \to 0^+)$ if and only if $x \in Y\tilde{}_X$.

We next specify the required notations. Let $X_1 := \overline{R(A)}$ and $X_0 := D(P) = N(A) \oplus X_1$. Since the operator $B_1 : D(B_1) \subset X_1 \to X_1$ is closed, its domain $D(B_1)\,(= R(A_1))$ is a Banach space with respect to the norm $\|x\|_{B_1} := \|x\| + \|B_1 x\|$. Let $B_0 : D(B_0) \subset X_0 \to X_0$ be the operator $B_0 := 0 \oplus B_1$. Then its domain

$$D(B_0)\,(= N(A) \oplus D(B_1) = N(A) \oplus A(D(A) \cap \overline{R(A)}))$$

is a Banach space with norm $\|x\|_{B_0} := \|x\| + \|B_0 x\|$, and $[D(B_0)]\tilde{}_{X_0} = N(A) \oplus [D(B_1)]\tilde{}_{X_1}$.

Now we can state the following theorem (see [26] and [29, Theorem 1]), which is concerned with optimal convergence and non-optimal convergence rates of ergodic limits and approximate solutions.

**Theorem 2.4.** *Under conditions* (C1) - (C5), *the following statements hold.*

(i) *For* $x \in X_0$, *one has*: $\|A_\alpha x - Px\| = o(e(\alpha)) \Leftrightarrow x \in N(A)$.

(ii) *For* $x \in X_0 = N(A) \oplus \overline{R(A)}$, *one has*:

$$\begin{aligned}\|A_\alpha x - Px\| = O(f(\alpha)) &\Leftrightarrow K(e(\alpha), x, X_0, D(B_0), \|\cdot\|_{B_0}) = O(f(\alpha)) \\ &\Leftrightarrow x \in [D(B_0)]\tilde{}_{X_0} (\textit{in case } f = e).\end{aligned}$$

(iii) *For* $y \in D(B_1) = R(A_1)$, *one has*: $\|B_\alpha y - B_1 y\| = o(e(\alpha)) \Leftrightarrow y = 0$.

(iv) *For* $y \in D(B_1) = R(A_1)$, *one has*:

$$\begin{aligned}\|B_\alpha y - B_1 y\| = O(f(\alpha)) &\Leftrightarrow K(e(\alpha), B_1 y, X_1, D(B_1), \|\cdot\|_{B_1}) = O(f(\alpha)) \\ &\Leftrightarrow y \in A(D(A) \cap [D(B_1)]\tilde{}_{X_1}) \ (\textit{in case } f = e).\end{aligned}$$

Thus, when $A \neq 0$, the rate of optimal convergence of $\|A_\alpha y\| = O(e(\alpha))$ is sharp everywhere on $[D(B_1)]\tilde{}_{X_1} \setminus \{0\}$. The following theorem [29, Theorem 2] shows that the non-optimal convergence rate: $\|A_\alpha y\| = O(f(\alpha))$ with $f$ satisfying $f(\alpha)/e(\alpha) \to \infty$ is sharp.

**Theorem 2.5.** *Suppose that* $A$, $\{A_\alpha\}$, *and* $\{B_\alpha\}$ *satisfy conditions* (C1) - (C5), *with* $f(\alpha)/e(\alpha) \to \infty$. *Then* $R(A)$ *is not closed if and only if there exists an element* $y_f \in X_1$ *such that*

$$\|A_\alpha y_f\| \begin{cases} = O(f(\alpha)); \\ \neq o(f(\alpha)). \end{cases}$$

*Moreover*,

$$\|A_\alpha(x + y_f) - P(x + y_f)\| \begin{cases} = O(f(\alpha)); \\ \neq o(f(\alpha)) \end{cases}$$

*for all* $x \in N(A)$.

## 3. Regularized Approximation Processes

A net $\{T_\alpha\}$ of bounded linear operators on a Banach space $X$ is called an *approximation process* on $X$ if

$$\|T_\alpha x - x\| \to 0 \text{ for all } x \in X.$$

The process $\{T_\alpha\}$ is said to possess the *saturation property* if there exists a positive function $e(\alpha)$ tending to 0 such that every $x \in X$ for which

$$\|T_\alpha x - x\| = o(e(\alpha))$$

is an invariant element of $\{T_\alpha\}$, i.e., $T_\alpha x = x$ for all $\alpha$, and if the set

$$F[X; T_\alpha] = \{x \in X; \|T_\alpha x - x\| = O(e(\alpha))\}$$

contains at least one noninvariant element. In this case, the approximation process $\{T_\alpha\}$ is said to have *optimal approximation order* $O(e(\alpha))$ or to be *saturated in* $X$ *with order* $O(e(\alpha))$, and $F[X; T_\alpha]$ is called its *Favard class* or *saturation class*.

Let $e(\alpha)$ be a positive function tending to 0. A net $\{T_\alpha\}$ of bounded linear operators on $X$ is called an $A$-*regularized approximation process of order* $O(e(\alpha))$ on $X$ if it is uniformly bounded, i.e., $\|T_\alpha\| \le M$ for some $M > 0$ and all $\alpha$, and satisfies

(A1) there are a (necessarily densely defined) closed linear operator $A$ and a uniformly bounded approximation process $\{S_\alpha\}$ on $X$ such that $R(S_\alpha) \subset D(A)$ and

$$S_\alpha A \subset A S_\alpha = (e(\alpha))^{-1}(T_\alpha - I) \text{ for all } \alpha.$$

In this case, the process $\{S_\alpha\}$ is called a *regularization process* associated with $\{T_\alpha\}$.

The convergence rates of $\{T_\alpha\}$ were studied in [32] recently. In the following, we quote some of the general results.

**Lemma 3.1.** (i) *$x \in D(A)$ and $y = Ax$ if and only if $y = \lim_\alpha (e(\alpha))^{-1}(T_\alpha - I)x$.*

(ii) *$D(A)$ is dense in $X$, and $\|T_\alpha x - x\| \to 0$ for all $x \in X$.*

The next is a uniform convergence theorem.

**Theorem 3.2.** *Let $\{T_\alpha\}$ be an $A$-regularized approximation process of order $O(e(\alpha))$.*

(i) *If $A$ is bounded, then $\|T_\alpha - I\| = O(e(\alpha)) \to 0$.*

(ii) $\|T_\alpha - I\| \to 0$ *implies* $A \in B(X)$ *if either* $R(T_\alpha) \subset D(A)$ *for all* $\alpha$, *or* $S_\alpha$ *and* $T_\alpha$ *satisfy the following condition*:

(A2) $\|T_\alpha - I\| \to 0$ *implies* $\|S_\alpha - I\| \to 0$.

(iii) *If the space* $X$ *is a Grothendieck space with the Dunford-Pettis property and if* $R(T_\alpha) \subset D(A)$ *for all* $\alpha$, *then* $A \in B(X)$ *and* $\|T_\alpha - \| = O(e(\alpha))$.

As usual, the rates of covergence will be characterized by means of $K$-*functional* and *relative completion*. The following is an optimal convergence (saturation) theorem.

**Theorem 3.3.** *Let* $\{T_\alpha\}$ *be an* $A$*-regularized approximation process of order* $O(e(\alpha))$, *and let* $D(A)$ *be equipped with the graph norm* $\|\cdot\|_{D(A)}$. *For* $x \in X$, *we have*:

(i) $\|T_\alpha x - x\| = o(e(\alpha))$ *if and only if* $x \in N(A)$, *if and only if* $T_\alpha x = x$ *for all* $\alpha$.

(ii) $\|T_\alpha x - x\| = O(e(\alpha))$ *if and only if* $x \in [D(A)]\tilde{}_X$, *if and only if* $x \in D(A)$ *in the case that* $X$ *is reflexive.*

The next theorem is about non-optimal convergence.

**Theorem 3.4.** *Let* $0 \le e(\alpha) \le f(\alpha) \to 0$. *If* $K(e(\alpha), x, X, D(A), \|\cdot\|_{D(A)}) = O(f(\alpha))$, *then* $\|T_\alpha x - x\| = O(f(\alpha))$. *The converse statement is also true under the following assumption*:

(A3) $\|S_\alpha x - x\| = O(f(\alpha))$ *whenever* $\|T_\alpha x - x\| = O(f(\alpha))$.

The sharpness of non-optimal rate of convergence is shown by the following theorem.

**Theorem 3.5.** *Suppose an* $A$*-regularized approximation process* $\{T_\alpha\}$ *and its regularization process* $\{S_\alpha\}$ *satisfy condition* (A2). *Then* $A$ *is unbounded if and only if for each* $f(\alpha)$ *with* $0 \le e(\alpha) < f(\alpha) \to 0$ *and* $f(\alpha)/e(\alpha) \to \infty$ *there exists* $x_f \in X$ *such that*

$$\|T_\alpha x_f - x_f\| \begin{cases} = O(f(\alpha)); \\ \ne o(f(\alpha)). \end{cases}$$

The proof of Theorem 3.5 depends on the the following proposition which is a variation of a condensation theorem of Davydov [11, Theorem 1].

**Proposition 3.6.** *Let* $\{p_\alpha\}$ *be a net of continuous seminorms on a Banach space* $X$ *satisfying the conditions*:

(a) $\overline{\lim_\alpha} \|p_\alpha\| = \infty$, *where* $\|p_\alpha\| := \sup\{p_\alpha(x); x \in X, \|x\| \le 1\}$;

(b) *the set* $\{x \in X; \lim_{\alpha} p_\alpha(x) = 0\}$ *is dense in* $X$.
*Then there exists an element* $x_0 \in X$ *such that* $\sup_{\alpha} p_\alpha(x_0) \le 1$ *and* $\overline{\lim_{\alpha}}\, p_\alpha(x_0) = 1$.

*Proof of Theorem* 3.5. If $A$ is bounded, then by Theorem 3.2(i) we have $\|T_\alpha - I\| = O(e(\alpha))$ so that $\|T_\alpha - I\| = o(f(\alpha))$. This shows the sufficiency.

For the necessity, suppose $A$ is unbounded and define $p_\alpha(x) = (f(\alpha))^{-1}\|T_\alpha x - x\|$, $x \in X$. Note that $p_\alpha$ is a seminorm on $X$ with $\|p_\alpha\| \le (M+1)/f(\alpha)$. We show that $\{p_\alpha\}$ satisfies the hypothesis of Proposition 2.8.

By Theorem 3.2(i), we have $\overline{\lim}_\alpha \|T_\alpha - I\| > 0$, so that $\overline{\lim}_\alpha \|p_\alpha\| = \overline{\lim}_\alpha (f(\alpha))^{-1}\,\|T_\alpha - I\| = \infty$. Moreover, we have $p_\alpha(x) = (f(\alpha))^{-1} e(\alpha)\|S_\alpha A x\| \to 0$ for all $x \in D(A)$, by (A1) and the assumption $f(\alpha)/e(\alpha) \to \infty$. Hence the set $\{x \in X; \lim_\alpha p_\alpha(x) = 0\}$ contains $D(A)$, which is dense in $X$ by (A1).

The hypothesis of Proposition 3.6 being satisfied, it follows that there exists an $x_f \in X$ such that $\sup_\alpha p_\alpha(x_f) \le 1$ and $\overline{\lim}_\alpha p_\alpha(x_f) = 1$, i.e., $x_f$ satisfies $\|T_\alpha x_f - x_f\| = O(f(\alpha))$ and $\|T_\alpha x_f - x_f\| \ne o(f(\alpha))$. ■

## 4. Ergodic Theorems for $r$-times Integrated Solution Families

Let $A$ be a (not necessarily densely defined) closed linear operater in $X$ and $a \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ be a positive kernel. Consider the Volterra equation:

$$(VE, A, a, f) \qquad u(t) = f(t) + \int_0^t a(t-s)Au(s)ds,\ t \ge 0,$$

for $f \in C([0,\infty); X)$.

Let $r \in [0,\infty)$. A family $\{S(t); t \ge 0\}$ in $B(X)$ is called an *$r$-times integrated solution family* for $(VE, A, a, f)$ (see [2, 19] for the case $r = n \in \mathbb{N}$) if

(S1) $S(\cdot)$ is strongly continuous on $[0,\infty)$, and $S(0) = I$ if $r = 0$ and $0$ if $r > 0$;
(S2) $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \ge 0$;
(S3) for $x \in X$ and $t \ge 0$, $a * S(t)x \in D(A)$ and

$$S(t)x = \frac{t^r}{\Gamma(r+1)}x + A\int_0^t a(t-s)S(s)xds.$$

A 0-times integrated solution family is also called a *solution family* or *resolvent family* [3, 9, 14, 20].

The notion of an $r$-times integrated solution family is an extension of the concepts of *$r$-times integrated semigroups* (see [1, 12, 13, 17]) and *$n$-times integrated cosine functions* [31] (corresponding to the cases $a \equiv 1$ and $a(t) = t$, respectively). The existence of an $r$-times integrated solution family enables one to find the solution for the equation $(VE, A, a, f)$ (see [19]).

In this section, we deduce ergodic theorems for an $r$-times integrated solution family $S(\cdot)$. These are concerned with the convergence of some Cesáro type means $Q_m(t)$, $m \geq 1$, and Abel means of $S(\cdot)$ as $t \to \infty$. For the existence of the limits, the fulfilment of the condition $S(t) = O(t^r)$ as $t \to \infty$ and as $t \to 0$ is required. Thus, throughout the section we assume that

$$\|S(t)\| \leq Mt^r \text{ for all } t \geq 0. \tag{4.1}$$

Put $j_r(t) = t^r/\Gamma(r+1)$ for $t \geq 0$ and $r \geq 0$ and denote by $a_0$ the *Dirac measure* $\delta_0$ at 0. For each $m \geq 0$, let $a_{m+1}(t) = a * a_m(t)$ for $t \geq 0$, let $k_m(t) = a_{m+1} * j_r(t)/a_m * j_r(t)$ for $t > 0$, and define

$$Q_m(t)x = \frac{a_m * S(t)x}{a_m * j_r(t)} \quad \text{for} \quad x \in X \text{ and } \quad t > 0.$$

In particular, $k_0(t) = a*j_r(t)/j_r(t)$, $k_1(t) = a*a*j_r(t)/a*j_r(t)$, $Q_0(t) = (\Gamma(r+1)/t^r)\, S(t)$, and $Q_1(t) = a*S(t)/a*j_r(t)$, which are $\int_0^t a(s)ds$, $a*a*1(t)/a*1(t)$, $S(t)$, and $a * S(t)/a * 1(t)$, respectively, when $r = 0$.

Note that $a_m(t)$ and $a_m * j_r(t)$ are nondecreasing positive functions of $t$. Therefore,

$$k_m(t) = \frac{1}{a_m * j_r(t)} \int_0^t a(t-s)(a_m * j_r)(s)ds \leq \int_0^t a(s)ds \to 0 \tag{4.2}$$

as $t \to 0$.

By the assumption (4.1), we have

$$\begin{aligned} \|Q_m(t)x\| &\leq \frac{1}{a_m * j_r(t)} \int_0^t a_m(t-s)\|S(s)x\|ds \\ &\leq \frac{M\|x\|}{a_m * j_r(t)} \int_0^t a_m(t-s)s^r ds = M\Gamma(r+1)\|x\| \end{aligned}$$

for all $x \in X$, so that

$$\|Q_m(t)\| \leq M\Gamma(r+1) \qquad (m \geq 0,\ 0 < t \leq 1). \tag{4.3}$$

We shall need the next lemma [32], which relates $A$, $Q_m(t)$, and $Q_{m+1}(t)$.

**Lemma 4.1.** *Let $S(\cdot)$ be an $n$-times integrated solution family for $(VE, A, a, f)$ such that $\|S(t)\| \leq Mt^r$ for all $t \geq 0$, and let $A^\circ$ be the part of $A$ in $Y := \overline{D(A)}$. Then*

$$Q_0(t)D(A) \subset D(A) \textit{ and } Q_0(t)Ax = AQ_0(t)x \textit{ for } x \in D(A), \tag{4.4}$$

$$
\begin{aligned}
&Q_{m+1}(t)X \subset D(A) \textit{ and} \\
&Q_{m+1}(t)A \subset AQ_{m+1}(t) = \frac{1}{k_m(t)}(Q_m(t) - I),
\end{aligned} \tag{4.5}
$$

$$
Q_0(t)D(A^\circ) \subset D(A^\circ) \textit{ and } Q_0(t)A^\circ x = A^\circ Q_0(t)x \textit{ for } x \in D(A^\circ), \tag{4.6}
$$

$$
\begin{aligned}
&Q_{m+1}(t)Y \subset D(A^\circ) \textit{ and} \\
&Q_{m+1}(t)A^\circ \subset A^\circ Q_{m+1}(t)|_Y = \frac{1}{k_m(t)}(Q_m(t) - I)|_Y
\end{aligned} \tag{4.7}
$$

*for all* $m \geq 0$ *and* $t > 0$.

For $m \geq 0$ and $t > 0$, let $A_t$ and $B_t$ be operators defined respectively by

$$
A_t := Q_{m+1}(t) \quad \text{and} \quad B_t x := -k_{m+1}(t)Q_{m+2}(t) = -\frac{a_{m+2} * S(t)}{a_{m+1} * j_r(t)}.
$$

Then (2.5) becomes

$$
R(B_t) \subset D(A) \text{ and } B_t A \subset AB_t = I - A_t \text{ for } t > 0,
$$

that is, condition (C2) is satisfied. Moreover, $A_t A \subset AA_t = (k_m(t))^{-1}(Q_m(t) - I)$ for $t > 0$. Hence conditions (C1) and (C3) hold with $e(t) = 1/k_m(t)$.

We suppose $k_m(t) \to \infty$ and $k_{m+1}(t) \to \infty$ as $t \to \infty$. To check (C4), let $x^* \in R(A)^\perp$. Then, by (S3) we have

$$
\langle x, S^*(t)x^* \rangle = j_r(t)\langle x, x^* \rangle + \langle A \int_0^t a(t-s)S(s)x ds, x^* \rangle = j_r(t)\langle x, x^* \rangle
$$

for all $x \in X$, so that $S^*(t)x^* = j_r(t)x^*$ and hence

$$
(B_t)^*(t)x^* = -\frac{a_{m+2} * S^*(t)x^*}{a_{m+1} * j_r(t)} = -\frac{a_{m+2} * j_r(t)}{a_{m+1} * j_r(t)} x^* = -k_{m+1}(t)x^*
$$

for all $t \geq 0$. Thus condition (C4) is satisfied.

Finally, to see (C5) with $f(t) = e(t)^\beta = (k_m(t))^{-\beta}$, $0 < \beta \leq 1$, let $x$ be such that $\|A_t x\| \leq M_x (k_m(t))^{-\beta}$. Then

$$
\begin{aligned}
\|B_t x\| &= \left\|\frac{a_{m+2} * S(t)}{a_{m+1} * j_r(t)}\right\| = \left\|\frac{a * [a_{m+1} * j_r(t)A_t x]}{a_{m+1} * j_r(t)}\right\| \\
&\leq \frac{a * a_m * j_r(t)k_m(t)(k_m(t))^{-\beta}M_x}{a_{m+1} * j_r(t)} = M_x (k_m(t))^{1-\beta}
\end{aligned}
$$

for all $t \geq 0$. Similarly, $\|A_t x\| = o((k_m(t))^{-\beta})$ implies $\|B_t x\| = o((k_m(t))^{1-\beta})$.

Now, applying Theorems 2.1, 2.4, and 2.5, we can formulate the following strong ergodic theorem with rates.

**Theorem 4.2.** *Let $a \in L^1_{\text{loc}}(\mathbb{R}^+)$ be a positive function such that $k_m(t) \to \infty$ and $k_{m+1}(t) \to \infty$ as $t \to \infty$, and let $S(\cdot)$ be an $r$-times integrated solution family for $(VE, A, a, f)$ satisfying $\|S(t)\| \leq Mt^r$ for all $t \geq 0$. For $m \geq 0$ we have*:

(i) *The mapping $P : x \mapsto \lim_{t\to\infty} Q_{m+1}(t)x$ is a bounded linear projection with $R(P) = N(A)$, $N(P) = \overline{R(A)}$, and*

$$D(P) = N(A) \oplus \overline{R(A)} = \{x \in X; \{Q_{m+1}(t)x\} \text{ has a weak cluster point}\}.$$

*For $0 < \beta \leq 1$ and $x \in X_0 := D(P) = N(A) \oplus \overline{R(A)}$, one has*:

$$\begin{aligned}
&\|Q_{m+1}(t)x - Px\| = O((k_m(t))^{-\beta}) \\
&\Longleftrightarrow K((k_m(t))^{-1}, x, X_0, D(B_0), \|\cdot\|_{B_0}) = O((k_m(t))^{-\beta}) \\
&\Longleftrightarrow x \in [D(B_0)]^{\sim}_{X_0} (\textit{in case } \beta = 1).
\end{aligned}$$

(ii) *The mapping $B_1 : y \to -\lim_{t\to\infty} k_{m+1}(t)Q_{m+2}(t)y$ is the inverse operator $A_1^{-1}$ of the restriction $A_1 := A|\overline{R(A)}$ of $A$ to $\overline{R(A)}$; it has range $R(B_1) = D(A) \cap \overline{R(A)}$, and domain $D(B_1) = A(D(A) \cap \overline{R(A)})$. For each $y \in A(D(A) \cap \overline{R(A)})$, $B_1 y$ is the unique solution of the functional equation $Ax = y$ in $\overline{R(A)}$, and we have, for $0 < \beta \leq 1$,*

$$\begin{aligned}
&\|k_{m+1}(t)Q_{m+2}(t)y + A_1^{-1}y\| = O((k_m(t))^{-\beta})\,(t \to \infty) \\
&\Longleftrightarrow K((k_m(t))^{-1}, B_1 y, X_1, D(B_1), \|\cdot\|_{B_1}) = O((k_m(t))^{-\beta})\,(t \to \infty) \\
&\Longleftrightarrow y \in [D(B_1)]^{\sim}_{X_1} (\textit{in case } \beta = 1).
\end{aligned}$$

(iii) *$R(A)$ is not closed if and only if for every (some) $0 < \beta < 1$, there is a $y_\beta \in \overline{R(A)}$ such that*

$$\|Q_{m+1}(t)y_\beta ds\| \begin{cases} = O((k_m(t))^{-\beta}) \\ \neq o((k_m(t))^{-\beta}) \end{cases} (t \to \infty).$$

As will be seen in Section 5, $D(A^\circ)$ is dense in $Y = \overline{D(A)}$. We can apply Theorems 2.2 and 2.3 to deduce the following uniform ergodic theorem.

**Theorem 4.3.** *Under the hypothesis in Theorem 4.2, we have*:

(i) *$\|Q_{m+1}(t) - P\| \to 0$ if and only if $\|k_{m+1}(t)Q_{m+2}(t) + A_1^{-1}\|_{R(A)} \to 0$, if and only if $R(A)$ is closed, if and only if $R(A^2)$ is closed, if and only if*

$X = N(A) \oplus R(A)$. *In this case, we have* $\|Q_{m+1}(t) - P\| = O((k_m(t))^{-1})$ *and* $\|k_{m+1}(t)Q_{m+2}(t) + A_1^{-1}\|_{R(A)} = O((k_m(t))^{-1})$.

(ii) *If* $Y := \overline{D(A)}$ *is a Grothendieck space with the Dunford-Pettis property, and if* $Y \subset D(P)$, *then* $\|Q_{m+1}(t)|_Y - P|_Y\| = O((k_m(t))^{-1})$.

Next, we consider the case that the kernel $a \in L^1_{\text{loc}}(\mathbb{R}^+)$ is Laplace transformable, i.e., there is $\omega \geq 0$ such that $\hat{a}(\lambda) = \int_0^\infty e^{-\lambda t} a(t)dt < \infty$ for all $\lambda > \omega$. Under this and the assumption (4.1), we can take Laplace transform of the equation in (S3) to obtain

$$\hat{S}(\lambda)x = \begin{cases} \frac{1}{\lambda^{n+1}}x + \hat{a}(\lambda)\hat{S}(\lambda)Ax, & x \in D(A), \\ \frac{1}{\lambda^{n+1}}x + A\hat{a}(\lambda)\hat{S}(\lambda)x, & x \in X \end{cases}$$

for $\lambda > \omega$. Thus

$$\begin{aligned} &\lambda^{n+1}\hat{a}(\lambda)\hat{S}(\lambda)\left((\hat{a}(\lambda))^{-1} - A\right) \\ &\subset \left((\hat{a}(\lambda))^{-1} - A\right)\lambda^{n+1}\hat{a}(\lambda)\hat{S}(\lambda) = I, \end{aligned} \tag{4.8}$$

that is, $(\hat{a}(\lambda))^{-1} \in \rho(A)$ and $((\hat{a}(\lambda))^{-1} - A)^{-1} = \lambda^{n+1}\hat{a}(\lambda)\hat{S}(\lambda)$ for $\lambda > \omega$. Moreover, (4.1) implies

$$\begin{aligned} &\|(\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1} - A)^{-1}\| = \|\lambda^{n+1}\hat{S}(\lambda)\| \\ &= \|\lambda^{n+1}\int_0^\infty e^{-\lambda t}S(t)dt\| \leq M\Gamma(r+1). \end{aligned} \tag{4.9}$$

If $a$ satisfies the condition $\int_0^\infty a(t)dt = \infty$, we have $\hat{a}(\lambda) \to \infty$ and $(\hat{a}(\lambda))^{-1} \to 0^+$ as $\lambda \to 0^+$. It is easy to see from (4.8) and (4.9) that $A_\lambda := (\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1} - A)^{-1}$ is an $A$-ergodic net and $B_\lambda := -((\hat{a}(\lambda))^{-1} - A)^{-1}$ is a companion net, with $e(\lambda) = (\hat{a}(\lambda))^{-1} \to 0^+$ and $\phi(\lambda) = \hat{a}(\lambda) \to \infty$ as $\lambda \to 0^+$. Hence we can apply Theorems 2.1, 2.4, and 2.5 to formulate the following strong ergodic theorem with rates.

**Theorem 4.4.** *Soppose that* $\int_0^\infty a(t)dt = \infty$ *and* $\hat{a}(\lambda) = \int_0^\infty e^{-\lambda t}a(t)dt < \infty$ *for all* $\lambda > 0$, *and suppose* $\|S(t)\| \leq Mt^r$ *for all* $t \geq 0$. *Then the following are true for* $0 < \beta \leq 1$:

(i) *For* $x \in X_0$, *one has*

$$\begin{aligned} &\|(\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1} - A)^{-1}x - Px\| = O((\hat{a}(\lambda))^{-\beta})(\lambda \to 0^+) \\ &\Longleftrightarrow K(\lambda, x, X_0, D(B_0), \|\cdot\|_{B_0}) = O(\lambda^\beta)\,(\lambda \to 0^+) \\ &\Longleftrightarrow x \in [D(B_0)]^\sim_{X_0} \ (\textit{in case } \beta = 1). \end{aligned}$$

(ii) *For* $y \in D(B_1) = R(A_1)$, *one has*

$$\begin{aligned}&\|((\hat{a}(\lambda))^{-1} - A)^{-1}y + B_1 y\| = O((\hat{a}(\lambda))^{-\beta})\,(\lambda \to 0^+)\\ &\Longleftrightarrow K(\lambda, B_1 y, X_1, D(B_1), \|\cdot\|_{B_1}) = O(\lambda^\beta)\,(\lambda \to 0^+)\\ &\Longleftrightarrow x \in [D(B_1)]^{\sim}_{X_1} (\text{in case } \beta = 1).\end{aligned}$$

(iii) $R(A)$ *is not closed if and only if for each* (*some*) $0 < \beta < 1$ *there exists an element* $y_\beta \in \overline{R(A)}$ *such that*

$$\|(\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1} - A)^{-1}y_\beta\| \begin{cases} = O((\hat{a}(\lambda))^{-\beta}) \\ \neq o((\hat{a}(\lambda))^{-\beta}) \end{cases} (\lambda \to 0^+).$$

From Theorems 2.2 and 2.3, one can deduce the following uniform ergodic theorem.

**Theorem 4.5.** *Under the hypothesis in Theorem* 4.4, *we have*:

(i) $\|(\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1} - A)^{-1} - P\| \to 0$ *if and only if* $\|((\hat{a}(\lambda))^{-1} - A)^{-1} + A_1^{-1}\|_{R(A)} \to 0$, *if and only if* $R(A)$ *is closed, if and only if* $R(A^2)$ *is closed, if and only if* $X = N(A) \oplus R(A)$. *In this case, we have* $\|(\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1} - A)^{-1} - P\| = O((\hat{a}(\lambda))^{-\beta})\,(\lambda \to 0^+)$ *and* $\|((\hat{a}(\lambda))^{-1} - A)^{-1} + A_1^{-1}\|_{R(A)} = O((\hat{a}(\lambda))^{-\beta})\,(\lambda \to 0^+)$.

(ii) *If* $Y := \overline{D(A)}$ *is a Grothendieck space with the Dunford-Pettis property, and if* $Y \subset D(P)$, *then* $\|(\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1} - A)^{-1}|_Y - P|_Y\| = O((\hat{a}(\lambda))^{-1})\,(\lambda \to 0^+)$.

## 5. Approximation Properties of $r$-times Integrated Solution Families

In this section, we consider approximation properties of $r$-times integrated solution families. Of concern are the convergence of $Q_m(t)$ as $t \to 0^+$ and that of $(\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1} - A)^{-1}$ as $\lambda \to \infty$.

**Lemma 5.1.** *Let* $a \in L^1_{\text{loc}}(\mathbb{R}^+)$ *and let* $S(\cdot)$ *be an* $r$*-times integrated solution family for* $(VE, A, a, f)$ *such that* $\|S(t)\| \le Mt^n$ *for all* $t \ge 0$.

(i) *For* $m \ge 0$, $\|Q_m(t)x - x\| \to 0$ *as* $t \to 0^+$ *if and only if* $Q_m(t)x \to x$ *weakly as* $t \to 0^+$, *if and only if there is a sequence* $\{t_n\}$ *such that* $Q_m(t_n)x \to x$ *weakly for the case* $m \ge 1$, *if and only if* $x \in X_1$.

(ii) *If* $r = 0$, *then* $A$ *is densely defined in* $X$.

*Proof.* (i) It follows from (4.2), (4.3), and (4.5) that for all $m \ge 0$

$$\begin{aligned}\|Q_m(t)x - x\| &\le k_m(t)\|Q_{m+1}(t)\|\|Ax\|\\ &\le k_m(t)Mn!\|Ax\| \to 0\end{aligned}$$

as $t \to 0^+$ for all $x \in D(A)$, and hence $Q_m(t)x \to x$ for all $x \in X_1$, by (4.3). Conversely, from the estimate:

$$
\begin{aligned}
&|\langle Q_{m+1}(t)x - x, x^*\rangle| \\
&= \frac{1}{a_{m+1} * j_n(t)} |\langle \int_0^t a(t-s)(a_m * S(s)x)ds \\
&\quad - \int_0^t a(t-s)(a_m * j_n)(s)x ds, x^*\rangle| \qquad (5.1)\\
&\le \frac{1}{a_{m+1} * j_n(t)} \int_0^t a(t-s)(a_m * j_n)(s)|\langle Q_m(s)x - x, x^*\rangle| ds \\
&\le \sup\{|\langle Q_m(s)x - x, x^*\rangle|; 0 \le s \le t\},\ x \in X, x^* \in X^*,
\end{aligned}
$$

one sees that if $Q_m(t)x \to x$ weakly, then $Q_{m+1}(t)x \to x$ weakly, which and the fact that $R(Q_{m+1}(t)) \subset D(A)$ show that $x \in X_1$. When $m \ge 1$, $R(Q_m(t_n)) \subset D(A)$, and so $x = \text{w-lim } Q_m(t_n)x \in X_1$.

(ii) When $r = 0$, since $Q_0(t) = S(t) \to I$ strongly as $t \to 0^+$, (5.1) implies that

$$\|Q_1(t)x - x\| \le \sup\{\|S(s)x - x\|; 0 \le s \le t\} \to 0$$

for all $x \in X$. Then we have $X_1 = X$, by the fact that $Q_1(t)X \subset D(A)$. That is, $A$ is densely defined for the case $r = 0$. ■

Thus, from (4.2), (4.4), (4.7), and Lemma 5.1(i), we see that $Y$ is invariant under $Q_m(t)$ for each $m \ge 0$, and $\{T_t := Q_m(t)|_Y\}$ is an $A^\circ$-regularized approximation process on $Y$ with the regularization process $\{S_t := Q_{m+1}(t)|_Y\}$ and with the optimal order $O(k_m(t))(t \to 0^+)$. In particular, $D(A^\circ)$ is dense in $Y$, by Lemma 3.1(ii). Moreover, we have $T_t D(A^\circ) \subset D(A^\circ)$ if $m = 0$ and $R(T_t) \subset D(A^\circ)$ if $m \ge 1$.

**Lemma 5.2.** *The above pair* $(\{T_t\}, \{S_t\})$ *satisfies* (A2). *If* $k_m(t)$ *is non-decreasing for* $t$ *near* 0, *then* (A3) *with* $f(t) = (k_m(t))^\beta$ $(0 < \beta \le 1)$ *also holds.*

*Proof.* From (5.1) one can see that $\|S_t - I\|_Y \le \sup\{\|T_s - I\|_Y; 0 \le s \le t\}$, which shows (A2). Moreover, if $\|T_t x - x\| \le M(k_m(t))^\beta$ for all $t \in [0,1]$, then $\|S_t x - x\| \le M \sup\{(k_m(s))^\beta; 0 \le s \le t\} \le M(k_m(t))^\beta$ for all $t \in [0,1]$, showing (A3). ■

From Theorem 3.2 and Lemma 5.2 we deduce the following uniform convergence theorem.

**Theorem 5.3.** *Let $a \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$, and let $S(\cdot)$ be an $r$-times integrated solution family for $(VE, A, a, f)$ such that $\|S(t)\| \le Mt^r$ for all $t \ge 0$.*

(i) *For $m \ge 0$, $\|Q_m(t) - I\| \to 0$ as $t \to 0^+$ if and only if $A \in B(X)$. In this case, $\|Q_m(t) - I\| = O(k_m(t))(t \to 0^+)$.*

(ii) *When $Y$ is a Grothendieck space with the Dunford-Pettis property, $A$ must be bounded on $X$, and consequently $\|S(t) - j_r(t)I\| = O(a * j_r(t))$ $(t \to 0^+)$.*

*Proof.* (i) follows from Theorem 3.2 and Lemma 5.2.

(ii) Applying Theorem 3.2 to $\{T_t := Q_1(t)|_Y\}$ yields that $A^\circ$ is bounded on $Y$, so that $\|Q_1(t)|_Y - I|_Y\| \le k_1(t)\|A^\circ\|\|Q_2(t)\| \le k_1(t)\|A^\circ\|M\Gamma(r+1) \to 0$ as $t \to 0^+$. Hence $Q_1(t)|_Y$ is invertible on $Y$ for small $t$. Then by (4.5) we have $Y = R(Q_1(t)|_Y) \subset R(Q_1(t)) \subset D(A)$, which shows that $D(A)$ is closed and $A$ is bounded. Due to Lemma 5.2, (i) and (ii) of Theorem 3.2 together imply that $A \in B(X)$. By (i), $\|Q_m(t) - I\| = O(k_m(t))(t \to 0^+)$, and in particular, $\|S(t) - j_r(t)I\| = O(a * j_r(t))$ $(t \to 0^+)$. ∎

From Theorems 3.3, 3.4, 3.5 and Lemma 5.2, we deduce the next theorem.

**Theorem 5.4.** *Let $S(\cdot)$ be as assumed in Theorem 5.3 and let $m \ge 0$, $0 < \beta \le 1$, and $x \in \overline{D(A)}$.*

(i) *$\|Q_m(t)x - x\| = o(k_m(t))$ $(t \to 0^+)$ if and only if $x \in N(A^\circ) = N(A)$.*

(ii) *$\|Q_m(t)x - x\| = O(k_m(t))(t \to 0^+)$ if and only if $x \in [D(A^\circ)]\tilde{}_Y$ $(= D(A^\circ)$, if $X$ is reflexive$)$.*

(iii) *If $K(k_m(t), x, X, D(A), \|\cdot\|_{D(A)}) = O((k_m(t))^\beta)(t \to 0^+)$, then $\|Q_m(t)x - x\| = O((k_m(t))^\beta)(t \to 0^+)$. The converse is also true if $k_m(t)$ is nondecreasing for $t$ near $0$.*

(iv) *$A$ is unbounded if and only if for some (each) $0 < \beta < 1$ and $m \ge 0$, there exists $x^*_{m,\beta} \in Y = \overline{D(A)}$ such that*

$$\|Q_m(t)x^*_{m,\beta} - x^*_{m,\beta}\| \begin{cases} = O((k_m(t))^\beta) \\ \ne o((k_m(t))^\beta) \end{cases} \quad (t \to 0^+).$$

When $a$ is Laplace transformable, it is easy to see that $\hat{a}(\lambda) \to 0$ as $\lambda \to \infty$. Then it can be verified that $\{T_\lambda := (\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1} - A^\circ)^{-1}\}$ is an $A^\circ$-regularized approximation process on $Y$ with the regularization process $\{S_\lambda := (\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1} - A^\circ)^{-1}\}$ and with the optimal order $O(\hat{a}(\lambda))(\lambda \to \infty)$. They satisfy conditions (A1), (A2) and (A3). Hence we can deduce from the theorems in Section 3 the following theorems.

**Theorem 5.5.** *Let $a \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ be Laplace transformable and let $S(\cdot)$ be an $r$-times integrated solution family for $(VE, A, a, f)$ such that $\|S(t)\| \le Mt^r$ for all $t \ge 0$.*

(i) $\|(\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1} - A)^{-1}x - x\| \to 0$ *as* $\lambda \to \infty$ *if and only if* $x \in Y$.

(ii) $\|(\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1} - A)^{-1} - I\| \to 0$ *as* $\lambda \to \infty$ *if and only if* $A \in B(X)$. *In this case,* $\|(\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1} - A)^{-1} - I\| = O(\hat{a}(\lambda))(\lambda \to \infty)$.

(iii) *For* $x \in Y$, $\|(\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1} - A)^{-1}x - x\| = o(\hat{a}(\lambda))(\lambda \to \infty)$ *if and only if* $x \in N(A)$.

(iv) *For* $0 < \beta \leq 1$ *and* $x \in Y$, $\|(\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1} - A)^{-1}x - x\| = O((\hat{a}(\lambda))^{\beta})$ *as* $\lambda \to \infty$ *if and only if* $K(t, x, X, D(A), \|\cdot\|_{D(A)}) = O(t^{\beta})(t \to 0^{+})$, *if and only if* $x \in [D(A^{\circ})]\tilde{}_{Y}$ *in the case that* $\beta = 1$, *if and only if* $x \in D(A^{\circ})$ *in the case that* $\beta = 1$ *and* $X$ *is reflexive.*

(v) $A$ *is unbounded if and only if for each* $0 < \beta < 1$, *there exists* $x^*_{\beta} \in Y$ *such that*

$$\|(\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1} - A)^{-1}x^*_{\beta} - x^*_{\beta}\| \begin{cases} = O((\hat{a}(\lambda))^{\beta}) \\ \neq o((\hat{a}(\lambda))^{\beta}) \end{cases} \quad (\lambda \to \infty).$$

**Remarks.** (i) For the case $r = 0, m = 0, 1$, direct proofs for Theorems 5.3(i) and 5.4 have been given in [9].

(ii) Theorem 5.3(ii) implies in particular that every resolvent family $S(\cdot)$ (i.e., the case $n = 0$) on a Grothendieck space with the Dunford-Pettis property satisfies $\|S(t) - I\| = O(\int_0^t a(s)ds)$ $(t \to 0^+)$. Specialization for the cases $a \equiv 1$ and $a(t) = t$ yields the same assertion for $C_0$-semigroups [16] and cosine operator functions [21].

(iii) If one takes $a \equiv 1$ and $a(t) = t$, then $S(\cdot)$ becomes respectively an $r$-times integrated semigroup and an $r$-times integrated cosine function, and the theorems in Sections 4 and 5 reduce to their ergodic theorems [29] and approximation theorems [10], respectively. When $r = 0$, they reduce further to results in [6].

## References


1. W. Arendt, Vector-valued Laplace transforms and Cauchy problems, *Israel J. Math.* **59** (1987), 327-352.

2. W. Arendt and H. Kellerman, Integrated solutions of Volterra integrodifferential equations and applications, in: *Volterra Integrodifferential Equations in Banach Spaces*, Pitman Res. Notes in Math., Vol. 190, 1989, pp. 21-51.

3. W. Arendt and J. Prüss, Vector-valued Tauberian theorems and asymptotic behavior of linear Volterra equations, *SIAM J. Math. Anal.* **23** (1992), 412-448.

4. H. Berens, *Interpolationsmethoden zur Behandlung von Approximationsprozessen auf Banachräumen*, Lecture Notes in Mathematics **64**, Springer-Verlag, Berlin, 1968.

5. P. L. Butzer and W. Dickmeis, Direct and inverse mean ergodic theorems with rates for semigroups of operators, in: *Approximation and Function Spaces* (Proc. Conf. Gdansk (Poland), August 27-31, 1979), North-Holland, Amsterdam, 1981, xiv +897, pp. 191-206.

6. P. L. Butzer and A. Gessinger, Ergodic theorems for semigroups and cosine operator functions at zero and infinity with rates; applications to partial differential equations. A survey, *Contemp. Math.* **190** (1995), 67-93.

7. P. L. Butzer and R. J. Nessel, *Fourior Analysis and Approximation*, Vol. 1, Birkhauser, Basel/Academic Press, New York, 1971.

8. P. L. Butzer and U. Westphal, The mean ergodic theorem and saturation, *Indiana Univ. Math. J.* **20** (1971), 1163-1174.

9. J.-C. Chang and S.-Y. Shaw, Rates of approximation and ergodic limits of resolvent families, *Arch. Math.* **66** (1996), 320-330.

10. J.-C. Chang and S.-Y. Shaw, Optimal and non-optimal rates of approximation for integrated semigroups and cosine functions, *J. Approx. Theory* **90** (1997), 200-223.

11. O. V. Davydov, Some condensation theorems for semigroup operators, *Manuscripta Math.* **79** (1993), 435-446.

12. M. Hieber, Laplace transforms and $\alpha$-times integrated semigroups, *Forum Math.* **3** (1991), 595-612.

13. Y.-C. Li and S.-Y. Shaw, $N$-times integrated $C$-semigroups and the abstract Cauchy problem, *Taiwanese J. Math.* **1** (1997), 75-102.

14. C. Lizama, A mean ergodic theorem for resolvent operators, *Semigroup Forum* **47** (1993), 227-230.

15. H. P. Lotz, Tauberian theorems for operators on $L^\infty$ and similar spaces, in: *Functional Analysis*: *Surveys and Recent Results III*, Noth-Holland, Amesterdam 1984, pp. 117-133.

16. H. P. Lotz, Uniform convergence of operators on $L^\infty$ and similar spaces, *Math. Z.* **190** (1985), 207-220.

17. I. Miyadera, M. Okubo and N. Tanaka, $\alpha$-times integrated semigroups and abstract Cauchy problems, *Mem. School Sci. Engrg. Waseda Univ.* **57** (1993), 256-289.

18. D. Nasri-Roudsari, R. J. Nessel and R. Zeler, Resonance principles with applications to mean ergodic theorems and projection operators, *Acta Math. Hungar.* (3) **68** (1995).

19. H. Oka, Linear Volterra equations and integrated solution families, *Semigroup Forum* **53** (1996), 278-297.

20. J. Prüss, *Evolutionary Integral Equations and Applications*, Birkhäuser Verlag, Basel, 1993.

21. S.-Y. Shaw, On $w^*$-continuous cosine operator functions, *J. Funct. Anal.* **66** (1986), 73-95.

22. S.-Y. Shaw, Uniform ergodic theorems for locally integrable semigroups and pseudo-resolvents, *Proc. Amer. Math. Soc.* **98** (1986), 61-67.

23. S.-Y. Shaw, Asymptotic behavior of pseudo-resolvents on some Grothendieck spaces, *Publ. Res. Inst. Math. Sci.* **24** (1988), 277-282.

24. S.-Y. Shaw, Mean ergodic theorems and linear functional equations, *J. Funct. Anal.* **87** (1989), 428-441.

25. S.-Y. Shaw, Uniform convergence of ergodic limits and approximate solutions, *Proc. Amer. Math. Soc.* **114** (1992), 405-411.

26. S.-Y. Shaw, Convergence rates of ergodic limits and approximate solutions, *J. Approx. Theory* **75** (1993), 157-166.

27. S.-Y. Shaw, On uniform ergodic theorems in Grothendieck spaces with the Dunford-Pettis property, *Houston J. Math.* **20** (1994), 313-320.

28. S.-Y. Shaw, Grothendieck space, in: *Encyclopaedia*: *Mathematics*, Supplement Vol. 1, Kluwer Academic Publishers, Dordrecht, 1997, p. 278.

29. S.-Y. Shaw, Non-optimal rates of ergodic limits and approximate solutions, *J. Approx. Theory* **94** (1998), 285-299.

30. S.-Y. Shaw, Abstract ergodic theorems, Proceedings of the 3rd International Conference on Functional Analysis and Approximation Theory, Acquafredda di Maratea, Italy, September 23-28, 1996; *Rendiconti del Circolo Matematico di Palermo, Serie II, Suppl.* **52** (1998), 141-155.

31. S.-Y. Shaw and Y.-C. Li, On $n$-times integrated $C$-cosine functions, in: *Evolution Equations*, Marcel Dekkar, New York, 1995, pp. 393-406.

32. S.-Y. Shaw and H. Liu, Convergence rates of regularized approximation processes, preprint.

Department of Mathematics, National Central University, Chung-Li, Taiwan
E-mail: shaw@math.ncu.edu.tw