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STABILITY OF OSCILLATORY SOLUTIONS OF DIFFERENCE EQUATIONS WITH DELAYS

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Abstract. Existence criteria for oscillatory solutions of difference equations have been obatined by many authors. It is therefore of interest to obtain additional conditions which are needed to yield stability of these solutions. In this paper, such conditions are derived for two functional difference equations.

1. Introduction

Criteria for the existence of oscillatory solutions of differential and difference equations have been derived by many authors. It is therefore of interest to know what additional conditions are needed to yield stability of oscillatory solutions. While such questions have been dealt with in the area of differential equations [4, 5, 9], to the best of our knowledge, there are only a limited number of studies which are related to difference equations. As an example, in [6], Ladas *et al.* established the following result: Let $\{p(n)\}_{n=0}^{\infty}$ be a positive sequence such that

$$\limsup_{n\to\infty} \sum_{i=n-\sigma}^{n} p(i) < 1.$$

Then every oscillatory solution $\{x_n\}$ of the following difference equation

$$x_{n+1} - x_n + p(n)x_{n-\sigma} = 0, \; n = 0, 1, 2, \ldots \tag{1}$$

tends to zero as $n \to \infty$.

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In this paper, we are concerned with two functional difference equations. The first is the following equation

$$x_{n+1} - x_n + p(n)f(x_{\sigma(n)}) = g(n),\ n = 0, 1, 2, \ldots, \tag{2}$$

where $\{p(n)\}_{n=0}^{\infty}$, $\{g(n)\}_{n=0}^{\infty}$ are real sequences, and $\sigma(n)$ is an integer-valued function defined for $n \geq 0$ such that $\sigma(n) \to \infty$ as $n \to \infty$. The function $f : \mathbb{R} \to \mathbb{R}$ is nondecreasing and satisfies $xf(x) > 0$ for $x \neq 0$.

The second is the equation

$$\Delta\left(h(n)\Delta x_n\right) + p(n)f\left(x_{\sigma(n)}\right) = g(n), \quad n = 0, 1, 2, ..., \tag{3}$$

where $\{h(n)\}_{n=0}^{\infty}$ is a positive sequence, and $\{p(n)\}_{n=0}^{\infty}$, $\{g(n)\}_{n=0}^{\infty}$, $\sigma(n)$ as well as well as f satisfy the same assumptions stated above. We will establish several stability criteria which are sufficient for every oscillatory solutions of (2) or (3) to be bounded or to converge to zero.

Since (2) or (3) are recurrence relations, given appropriate conditions on σ (e.g., $\sigma(n) \leq n$ for $n \geq 0$), existence and uniqueness theorems for their solutions are easily formulated and proved. Here, a solution of (2) or (3) is a sequence $\{x_n\}$ defined for $n \geq \sigma_*$, where $\sigma_* = \inf_{n \geq 0} \sigma(n) > -\infty$. A solution of (2) or (3) or, in general, a sequence is said to be *oscillatory* if it is neither eventually positive nor eventually negative.

Equations of the form (2) or (3) have been studied by a number of authors (see e.g. [2, 3, 8, 10, 11]). Under appropriate conditions on the coefficient sequences and the functions σ and f, it is known that all solutions are oscillatory. Here we only require oscillatory solutions exist. Such existence criteria are relatively scarce but not nonexistent (see e.g. [12]).

The ideas in this paper are not completely new since they are variations of arguments leading to Lyapunov inequalities (see e.g. Cheng [1]), and Kusano and Onose [5] have employed similar ideas in deriving stability criteria for oscillatory solutions of second order functional differential equations.

For the sake of convenience, the notations $a^+ = \max\{a, 0\}$ and $a^- = -\min\{a, 0\}$ will be adopted. Also, as is customary, an empty sum will be taken to be zero.

We will need the following properties of an oscillatory sequence. First we need several terminologies. Let us first say that a sequence $x = \{x_n\}_{n=a}^{b}$ possesses a *positive arch* $x(\alpha, \beta)$ if $x(\alpha, \beta)$ is a finite subsequence $\{x_\alpha, x_{\alpha+1}, ..., x_\beta\}$ of x such that $x_{\alpha-1} \leq 0$, $x_{\beta+1} \leq 0$ and $x_i > 0$ for $\alpha \leq i \leq \beta$. A negative arch of x is similarly defined. Given two positive arches $x(\alpha, \beta)$ and $x(s, t)$ of x, the arch $x(\alpha, \beta)$ is said to be the *immediate predecessor* of the positive arch $x(s, t)$ if the sequence $\{x_\beta, x_{\beta+1}, ..., x_s\}$ does not have any positive arches. Similarly, given any two positive or negative arches $x(\alpha, \beta)$ and $x(s, t)$ of x, the arch

$x(\alpha, \beta)$ is said to be the immediate predecessor of the arch $x(s,t)$ of x if the sequence $\{x_\beta, x_{\beta+1}, ..., x_s\}$ does not have any arches. An immediate predecessor and its successor form neighbors.

It is not difficult to see that if a sequence $x = \{x_n\}_{n=a}^\infty$ is oscillatory and if it has a positive subsequence, then x has a positive arch $x(\alpha, \beta)$. Indeed, the set Ω of integers where $x_n > 0$ is infinite. Since $\{x_n\}$ is not eventually positive, $\Omega \neq \{a, a+1, ...\}$. Thus there is a least integer α in Ω such that $x_{\alpha-1} \leq 0$ and $x_\alpha > 0$. If $x_{\alpha+1} \leq 0$, we may take $\beta = \alpha$. Otherwise, there is some integer m such that $x_{\alpha+1} > 0, ..., x_m > 0$ and $x_{m+1} \leq 0$. In such a case, we may let $\beta = m$. By induction, we may now easily show that there is a unique sequence $\{x(\alpha_i, \beta_i)\}_{i=1}^\infty$ of positive arches of x such that each $x(\alpha_i, \beta_i)$ is the immediate predecessor of the positive arch $x(\alpha_{i+1}, \beta_{i+1})$ and that every positive arch of x is contained in this sequence. Suppose in addition that $\limsup_{n\to\infty} x_n = \delta > 0$. Then by examining the positive arches in the sequence $\{x(\alpha_i, \beta_i)\}_{i=1}^\infty$ one by one, we may conclude that x has a sequence $\{x(s_i, t_i)\}_{i=1}^\infty$ of positive arches such that $\max_{s_i \leq j \leq t_i} x_j > \delta/2$; and if in addition $\limsup_{n\to\infty} x_n = \infty$, the same reasoning will lead to a sequence $\{x(u_i, v_i)\}_{i=1}^\infty$ such that $\max_{a \leq j \leq v_i} x_j$ is equal to $\max_{u_i \leq j \leq v_i} x_j$ and tends monotonically to ∞ as $i \to \infty$. A similar statement can be made for oscillatory sequences with negative subsequences. Finally, if $\{x_n\}_{n=a}^\infty$ oscillates and if $\limsup_{n\to\infty} |x_n| = \delta > 0$, then we may conclude that x has a sequence $\{x(s_i, t_i)\}_{i=1}^\infty$ of positive arches such that $\max_{s_i \leq j \leq t_i} |x_j| > \delta/2$; and if $\{x_n\}_{n=a}^\infty$ oscillates and $\limsup_{n\to\infty} |x_n| = \infty$, then there is a sequence $\{x(u_i, v_i)\}_{i=1}^\infty$ of arches of x such that $\max_{a \leq j \leq v_i} |x_j|$ is equal to $\max_{u_i \leq j \leq v_i} |x_j|$ and tends monotonically to ∞ as $i \to \infty$. When the unique sequence $\{x(\alpha_i, \beta_i)\}_{i=1}^\infty$ of neighboring arches of an oscillatory sequence satisfies $\beta_i - \alpha_i \leq c$ for $i \geq 1$, we will say that such a sequence has oscillation distances bounded by c. An example of such a sequence is a periodic and oscillatory sequence.

2. Equation (2)

Let $\{x_n\}$ be a bounded and oscillatory solution of (2) and assume that $\sup_{n \geq \sigma_*} |x_n| = M < \infty$ and

$$\limsup_{n\to\infty} |x_n| > 2\delta > 0. \tag{4}$$

As asserted in Section 1, there is a sequence $\{x(\alpha_i, \beta_i)\}_{i=1}^\infty$ of neighboring arches of x. By choosing a subsequence if necessary, we further assume that

$$M_i = \max_{\alpha_i \leq j \leq \beta_i} |x_j| = |x_{\gamma_i}| > \delta, \quad i = 1, 2, ...,$$

where $\alpha_i \le \gamma_i \le \beta_i$. By summing (2) from $\alpha_i - 1$ to $\gamma_i - 1$, we obtain

$$x_{\gamma_i} - x_{\alpha_i - 1} = -\sum_{j=\alpha_i-1}^{\gamma_i-1} p(j) f\left(x_{\sigma(j)}\right) + \sum_{j=\alpha_i-1}^{\gamma_i-1} g(j).$$

Since $|x_{\gamma_i}| \le |x_{\gamma_i} - x_{\alpha_i-1}|$, we see that

$$M_i = |x_{\gamma_i}| \le \sum_{j=\alpha_i-1}^{\gamma_i-1} \left|p(j) f\left(x_{\sigma(j)}\right)\right| + \sum_{j=\alpha_i-1}^{\gamma_i-1} |g(j)|. \tag{5}$$

If, in addition to the properties of the function f, assume further that $|f(x)| \le f(|x|)$, then

$$\left|f\left(x_{\sigma(j)}\right)\right| \le f\left(\left|x_{\sigma(j)}\right|\right) \le f(M)$$

so that

$$\delta < M_i \le f(M) \sum_{j=\alpha_i-1}^{\infty} |p(j)| + \sum_{j=\alpha_i-1}^{\infty} |g(j)|.$$

By imposing conditions on $\{p(n)\}$ and $\{g(n)\}$ such that the two infinite sums in the last inequality tend to zero as α_i tends to infinity, we see that a contradiction will be reached. The following is now clear.

Lemma 2.1. *Suppose $|f(x)| \le f(|x|)$ for all x and*

$$\sum_{j=0}^{\infty} |p(j)| < \infty \text{ and } \sum_{j=0}^{\infty} |g(j)| < \infty. \tag{6}$$

Then every bounded oscillatory solution $\{x_n\}$ of (2) tends to zero as $n \to \infty$.

Theorem 2.1. *Suppose that $|f(x)| \le f(|x|)$ for all x, that $\sigma(n) \le n+1$ for $n \ge 0$, that (6) holds and that*

$$\limsup_{|x|\to\infty} \frac{f(x)}{x} = \Gamma < \infty. \tag{7}$$

Then every oscillatory solution $\{x_n\}$ of (2) is bounded (and hence tends to zero as $n \to \infty$).

Proof. Let $\{x_n\}$ be an oscillatory solution of (2). We assert that it must be bounded. Assume to the contrary that $\{x_n\}$ is unbounded. As asserted before, there exists a sequence $\{x(\alpha_i, \beta_i)\}_{i=1}^{\infty}$ of neighboring arches of x. Let

$$M_i = \max_{\alpha_i \le j \le \beta_i} |x_j|, \quad i \ge 1.$$

Since $\{x_n\}$ is unbounded, by choosing a subsequence if necessary, we may assume that $\{M_i\}$ is nondecreasing and tends to positive infinity. Thus, we see that

$$M_i = \max_{\alpha_i \le j \le \beta_i} |x_j| = \max_{\sigma_* \le j \le \beta_i} |x_j|,$$

where $M_i = |x_{\gamma_i}|$ tends to ∞ monotonically as $i \to \infty$. Clearly, (5) is again valid. Furthermore, since $\sigma(n) \le n+1$, we have

$$\max_{\alpha_i \le j \le \gamma_i - 1} \left| f\left(x_{\sigma(j)}\right)\right| \le \max_{\alpha_i \le j \le \gamma_i - 1} f\left(\left|x_{\sigma(j)}\right|\right) \le f\left(M_i\right).$$

It follows that

$$1 \le \frac{f(M_i)}{M_i} \sum_{j=\alpha_i - 1}^{\infty} |p(j)| + \frac{1}{M_i} \sum_{j=\alpha_i - 1}^{\infty} |g(j)|. \tag{8}$$

By taking limits on both sides of (8), we see that

$$1 \le \Gamma \lim_{i \to \infty} \sum_{j=\alpha_i - 1}^{\infty} |p(j)| + \frac{1}{M_1} \lim_{i \to \infty} \sum_{j=\alpha_i - 1}^{\infty} |g(j)| = 0,$$

which is a contradiction. ∎

As an example, consider the equation

$$x_{n+1} - x_n + \frac{1}{n!} x_{n+1} = (-1)^{n+1} \frac{n!(n+2)+1}{n!(n+1)!}.$$

It is easy to see that the assumptions in Theorem 2.1 hold. Thus any of its oscillatory solution will tend to zero. Indeed, the sequence $\{(-1)^n/n!\}$ is such a solution.

We remark that in case $\sigma(n) = n+1$, then without any additional assumptions on f, (5) is replaced by

$$\begin{aligned} M_i &\le f(M) \sum_{j=\alpha_i - 1}^{\gamma_i - 1} p^-(j) + \sum_{j=\alpha_i - 1}^{\gamma_i - 1} |g(j)| \\ &\le f(M) \sum_{j=\alpha_i - 1}^{\infty} p^-(j) + \sum_{j=\alpha_i - 1}^{\infty} |g(j)|, \end{aligned}$$

and (8) changes to

$$1 \le \frac{f(M_i)}{M_i} \sum_{j=\alpha_i - 1}^{\infty} p^-(j) + \frac{1}{M_i} \sum_{j=\alpha_i - 1}^{\infty} |g(j)|.$$

Therefore, the following variant of Lemma 2.1 holds.

Lemma 2.2. *Suppose $\sigma(n) = n + 1$ for $n \geq 0$ and further that*

$$\sum_{j=0}^{\infty} p^-(j) < \infty \text{ and } \sum_{j=0}^{\infty} |g(j)| < \infty. \tag{9}$$

Then every bounded oscillatory solution of (2) tends to zero as $n \to \infty$.

Theorem 2.2. *Suppose that $\sigma(n) = n + 1$ for $n \geq 0$, and that (7) and (9) hold. Then every oscillatory solution of (2) is bounded (and hence tends to zero as $n \to \infty$).*

We remark further that if the solution $\{x_n\}$ of (2) has oscillation distances bounded by c, then (5) is replaced by

$$\delta < M_i \leq f(M) \sum_{j=\alpha_i - 1}^{\alpha_i + c} |p(j)| + \sum_{j=\alpha_i - 1}^{\alpha_i + c} |g(j)|,$$

and (8) is replaced by

$$1 \leq \frac{f(M_i)}{M_i} \sum_{j=\alpha_i - 1}^{\alpha_i + c} p^-(j) + \frac{1}{M_i} \sum_{j=\alpha_i - 1}^{\alpha_i + c} |g(j)|.$$

The following result is now clear.

Theorem 2.3. *Suppose $|f(x)| \leq f(|x|)$ for all x, and*

$$\lim_{n \to \infty} \sum_{j=n}^{n+c+1} |p(j)| = \lim_{n \to \infty} \sum_{j=n}^{n+c+1} \| g(j)| = 0. \tag{10}$$

Then every bounded (and oscillatory) solution of (2) with oscillatory distances bounded by c will converge to zero. Suppose in addition that $\sigma(n) \leq n + 1$ for $n \geq 0$, and that (7) holds. Then every (oscillatory) solution $\{x_n\}$ of (2) with oscillatory distances bounded by c is bounded (and hence converges to zero).

A similar statement can be made in case $\sigma(n) = n + 1$ for $n \geq 0$.

Theorem 2.4. *Suppose $|f(x)| \leq f(|x|)$ for all x, and $\sigma(n) \leq n + 1$ for $n \geq 0$, and*

$$\lim_{n \to \infty} \sum_{j=n}^{n+c+1} p^-(j) = \lim_{n \to \infty} \sum_{j=n}^{n+c+1} |g(j)| = 0.$$

Then every bounded (*and oscillatory*) *solution with oscillation distances bounded by* c *will converge to zero. If in addition that* (7) *holds, then every* (*oscillatory*) *solution of* (2) *with oscillation distances bounded by* c *will converge to zero.*

Under the condition that the forcing term in (2) is identically zero, (8) becomes

$$1 \leq \frac{f(M_i)}{M_i} \sum_{j=\alpha_i-1}^{\alpha_i+c} |p(j)|.$$

Therefore, if

$$\limsup_{n\to\infty} \sum_{j=n}^{n+c+1} |p(j)| < \frac{1}{\Gamma}, \tag{11}$$

then a contradiction will be obtained.

Theorem 2.5. *Suppose that* $g(n) \equiv 0$, *and that* (7) *holds. If in addition, either* (*i*) $|f(x)| \leq f(|x|)$ *for all* x, *and* $\sigma(n) \leq n+1$ *for* $n \geq 0$ *and* (11) *holds, or,* (*ii*) $\sigma(n) = n+1$ *for* $n \geq 0$ *and*

$$\limsup_{n\to\infty} \sum_{j=n}^{n+c+1} p^-(j) < \frac{1}{\Gamma}$$

holds, then every (*oscillatory*) *solution* $\{x_n\}$ *with oscillatory distances bounded by* c *is bounded.*

As an example, consider the case where $p(n) \geq 0$ for $n \geq 0$ and $g(n) = \Delta G(n)$ for $n \geq 0$, where $\{G(n)\}$ is a real sequence which has a nonpositive subsequence $\{G(n_k)\}$. If (2) has an eventually positive solution, then

$$\Delta(x_n - G(n)) = -p(n)f(x_{\sigma(n)}) \leq 0$$

for all large n. The nonincreasing sequence $\{x_n - G(n)\}$ cannot be eventually nonpositive, for otherwise

$$0 < x_{n_k} \leq G(n_k) \leq 0$$

for all large k, which is a contradiction. Thus $\{x_n - G(n)\}$ is eventually positive. This implies

$$x_n > G^+(n)$$

for all large n. Furthermore,

$$\Delta\left(x_n - G(n)\right) \leq -p(n)f\left(x_{\sigma(n)}\right) \leq -p(n)f\left(G^+(\sigma(n))\right)$$

for all large n. Summing the above inequality from N to k, we obtain

$$0 > -\left(x_{k+1} - G(k+1)\right) \geq -x(N) + G(N) + \sum_{n=N}^{k} p(n) f\left(G^+(\sigma(n))\right).$$

If the condition

$$\sum_{j=N}^{\infty} p(n) f\left(G^+(\sigma(n))\right) = \infty$$

is imposed, a contradiction will be reached. The following is now not difficult to see: Suppose $p(n) \geq 0$ for $n \geq 0$, suppose there is an oscillatory sequence $\{G(n)\}$ which satisfies $\Delta G(n) = g(n)$ for $n \geq 0$, and suppose further that

$$\sum_{n=0}^{\infty} p(n) f\left(G^+(\sigma(n))\right) = \sum_{n=0}^{\infty} p(n) f\left(G^-(\sigma(n))\right) = \infty. \tag{12}$$

Then every solution of (2) oscillates.

In particular, if $p(n) \geq n$ for $n \geq 0$, then all solutions of the following equation

$$\Delta x_n + p(n) x_{n+1} = (-1)^{n+1} \left\{ \frac{1}{(n+1)^2} + \frac{1}{n^2} \right\}$$

converge to zero. This is easily seen by taking $G(n) = (-1)^n / n^2$ and then verifying the validity of the conditions (7), (9) and (12).

3. Equation (3)

Let $\{x_n\}$ be a bounded and oscillatory solution of (3) and assume that $\sup_{n \geq \sigma_*} |x_n| = M < \infty$, and

$$\limsup_{n \to \infty} |x_n| > 2\delta > 0.$$

As asserted in Section 1, the sequence x has a sequence $\{x(\alpha_i, \beta_i)\}$ of arches. By choosing a subsequence if necessary, we may further assume that

$$M_i = \max_{\alpha_i \leq j \leq \beta_i} |x_j| = |x_{\gamma_i}| > \delta, \ j = 1, 2, ...,$$

where $\alpha_i \leq \gamma_i \leq \beta_i$. Upon summing (3) from $j \in \{\alpha_i - 1, \alpha_i, ..., \gamma_i - 1\}$ to $\gamma_i - 1$, we obtain

$$h(\gamma_i) \Delta x_{\gamma_i} - h(j) \Delta x_j = -\sum_{k=j}^{\gamma_i - 1} p(k) f\left(x_{\sigma(k)}\right) + \sum_{k=j}^{\gamma_i - 1} g(k).$$

Assume that $x(\alpha_i, \beta_i)$ is a positive arch. Then $\Delta x_{\gamma_i} \le 0$, so that

$$\Delta x_j \le \frac{1}{h(j)} \sum_{k=j}^{\gamma_i - 1} p(k) f\left(x_{\sigma(k)}\right) - \frac{1}{h(j)} \sum_{k=j}^{\gamma_i - 1} g(k),$$

and

$$\sum_{j=\alpha_i-1}^{\gamma_i-1} \Delta x_j = x_{\gamma_i} - x_{\alpha_i - 1} \le \sum_{j=\alpha_i-1}^{\gamma_i-1} \frac{1}{h(j)} \sum_{k=j}^{\gamma_i-1} p(k) f\left(x_{\sigma(k)}\right) - \sum_{j=\alpha_i-1}^{\gamma_i-1} \frac{1}{h(j)} \sum_{k=j}^{\gamma_i-1} g(k).$$

Since $x_{\alpha_i - 1} \le 0$, we see that

$$M_i = x_{\gamma_i} \le \sum_{j=\alpha_i-1}^{\gamma_i-1} \frac{1}{h(j)} \sum_{k=j}^{\gamma_i-1} \left|p(k) f\left(x_{\sigma(k)}\right)\right| + \sum_{j=\alpha_i-1}^{\gamma_i-1} \frac{1}{h(j)} \sum_{k=j}^{\gamma_i-1} |g(k)|. \tag{13}$$

Under the additional assumption that $|f(x)| \le f(|x|)$, we see further that

$$\begin{aligned} M_i &\le f(M) \sum_{j=\alpha_i-1}^{\gamma_i-1} \frac{1}{h(j)} \sum_{k=j}^{\gamma_i-1} |p(k)| + \sum_{j=\alpha_i-1}^{\gamma_i-1} \frac{1}{h(j)} \sum_{k=j}^{\gamma_i-1} |g(k)| \\ &\le f(M) \sum_{j=\alpha_i-1}^{\infty} \sum_{k=j}^{\infty} \frac{|p(k)|}{h(j)} + \sum_{j=\alpha_i-1}^{\infty} \sum_{k=j}^{\infty} \frac{|g(k)|}{h(j)}. \end{aligned}$$

In case the arch $x(\alpha_i, \beta_i)$ is negative, a similar argument leads to the above inequality again. By imposing conditions on $\{h(n)\}$ and $\{p(n)\}$ such that the two double sums tend to zero as j tends to infinity, we see that a contradiction will be reached.

Lemma 3.1. *Suppose $|f(x)| \le f(|x|)$ for all x, and*

$$\sum_{i=0}^{\infty} \frac{1}{h(i)} = \infty, \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{|p(k)|}{h(j)} < \infty \text{ and } \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{|g(k)|}{h(j)} < \infty, \tag{14}$$

or

$$\sum_{i=0}^{\infty} \frac{1}{h(i)} < \infty, \sum_{k=0}^{\infty} |p(k)| < \infty \text{ and } \sum_{k=0}^{\infty} |q(k)| < \infty. \tag{15}$$

Then every bounded oscillatory solution of (3) *tends to zero as $n \to \infty$.*

Indeed, it is not difficult to see that (15) implies

$$\lim_{i\to\infty}\sum_{j=i}^{\infty}\sum_{k=j}^{\infty}\frac{|p(k)|}{h(j)}=\lim_{i\to\infty}\sum_{j=i}^{\infty}\sum_{k=j}^{\infty}\frac{|g(k)|}{h(j)}=0. \tag{16}$$

Furthermore, note that (14) implies

$$\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\frac{|p(k)|}{h(j)}<\infty$$

and

$$\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\frac{|g(k)|}{h(j)}<\infty,$$

and thus the same conclusion (16) holds.

Theorem 3.1. *Suppose that* $|f(x)|\le f(|x|)$ *for all* x, *and, either* (14) *or* (15) *holds. Suppose further that* $\sigma(n)\le n+1$ *for* $n\ge 0$ *and*

$$\limsup_{|x|\to\infty}\frac{f(x)}{x}=\Gamma<\infty. \tag{17}$$

Then every oscillatory solution $\{x_n\}$ *of* (3) *is bounded* (*and hence tends to zero as* $n\to\infty$).

Proof. Let $\{x_n\}$ be an oscillatory solution of (3). We assert that it must be bounded. Otherwise, as seen in the proof of Theorem 2.1, there is a sequence $\{x(\alpha_i,\beta_i)\}_{i=1}^{\infty}$ of arches such that

$$M_i=\max_{\alpha_i\le j\le\beta_i}|x_j|=\max_{\sigma_*\le j\le\beta_i}|x_j|,$$

where $M_i=|x_{\gamma_i}|$ increases to infinity as $i\to\infty$. Clearly, (13) is again valid. Furthermore, since $\sigma(n)\le n+1$, we have

$$\max_{j\le k\le\gamma_i-1}\left|f\left(x_{\sigma(k)}\right)\right|\le f\left(M_i\right).$$

It follows that

$$\begin{aligned}1&\le\frac{f\left(M_i\right)}{M_i}\sum_{j=\alpha_i}^{\infty}\sum_{k=j}^{\infty}\frac{|p(k)|}{h(j)}+\frac{1}{M_i}\sum_{j=\alpha_i}^{\infty}\sum_{k=j}^{\infty}\frac{|g(k)|}{h(j)}\\&\le\Gamma\sum_{j=\alpha_i}^{\infty}\sum_{k=j}^{\infty}\frac{|p(k)|}{h(j)}+\frac{1}{M_1}\sum_{j=\alpha_i}^{\infty}\sum_{k=j}^{\infty}\frac{|g(k)|}{h(j)},\end{aligned}$$

which is contrary to either (14) or (15). The proof is complete. ∎

We remark that in case $\sigma(n) = n + 1$, then (13) is replaced by

$$M_i \leq \sum_{j=\alpha_i-1}^{\gamma_i-1} \frac{1}{h(j)} \sum_{k=j}^{\gamma_i-1} p^+(k) f(x_{k+1}) + \sum_{j=\alpha_i-1}^{\gamma_i-1} \frac{1}{h(j)} \sum_{k=j}^{\gamma_i-1} |g(k)|$$
$$\leq f(M_i) \sum_{j=\alpha_i-1}^{\gamma_i-1} \frac{1}{h(j)} \sum_{k=j}^{\gamma_i-1} p^+(k) + \sum_{j=\alpha_i-1}^{\gamma_i-1} \frac{1}{h(j)} \sum_{k=j}^{\gamma_i-1} |g(k)| .$$

Therefore, the following variant of Lemma 3.1 is valid.

Lemma 3.2. *Suppose $\sigma(n) = n + 1$ for $n \geq 0$. Suppose*

$$(18) \qquad \sum_{i=0}^{\infty} \frac{1}{h(i)} = \infty, \quad \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{p^+(k)}{h(j)} < \infty \text{ and } \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{|g(k)|}{h(j)} < \infty,$$

or

$$(19) \qquad \sum_{i=0}^{\infty} \frac{1}{h(i)} < \infty, \quad \sum_{k=0}^{\infty} p^+(k) < \infty \text{ and } \sum_{k=0}^{\infty} |g(k)| < \infty.$$

Then every bounded oscillatory solution of (3) *tends to zero as $n \to \infty$.*

Furthermore, the following variant of Theorem 3.1 is valid.

Theorem 3.2. *Suppose $\sigma(n) = n + 1$ for $n \geq 0$, that either* (18) *or* (19) *holds, and that* (17) *holds. Then every oscillatory solution $\{x_n\}$ of* (3) *is bounded* (*and hence tends to zero as $n \to \infty$*).

We remark further that if the solution $\{x_n\}$ has oscillation distances bounded by c, then (13) is replaced by

$$(20) \qquad M_i \leq f(M_i) \sum_{j=\alpha_i-1}^{\alpha_i+c} \frac{1}{h(j)} \sum_{k=j}^{\infty} |p(k)| + \sum_{j=\alpha_i-1}^{\alpha_i+c} \frac{1}{h(j)} \sum_{k=j}^{\infty} |g(k)| .$$

By imposing conditions on $\{h(n)\}$, $\{p(n)\}$ and $\{g(n)\}$ such that the two double sums in the last two inequalities tend to zero as j tends to infinity, we see that a contradiction will be reached. The following variant of Lemma 3.1 is now clear.

Lemma 3.3. *Suppose $|f(x)| \leq f(|x|)$ for all x, and*

$$(21) \qquad \sum_{i=0}^{\infty} |p(n)| < \infty, \sum_{i=0}^{\infty} |g(n)| < \infty \text{ and } \limsup_{n\to\infty} \sum_{i=n}^{n+c+1} \frac{1}{h(i)} < \infty.$$

Then every bounded (*and oscillatory*) *solution of* (3) *with oscillation distances bounded by* c *converges to zero.*

Dividing (20) through by M_i and then invoking (17) as in the proof of Theorem 3.1, we see that

$$1 \leq \frac{f(M_i)}{M_i} \sum_{j=\alpha_i-1}^{\alpha_i+c} \frac{1}{h(j)} \sum_{k=j}^{\infty} |p(k)| + \frac{1}{M_1} \sum_{j=\alpha_i-1}^{\alpha_i+c} \frac{1}{h(j)} \sum_{k=j}^{\infty} |g(k)| .$$

The following variant of Theorem 3.1 is also clear.

Theorem 3.3. *Suppose that* $|f(x)| \leq f(|x|)$ *for all* x, *that* $\sigma(n) \leq n+1$ *for* $n \geq 0$, *that* (21) *and* (17) *hold. Then every* (*oscillatory*) *solution of* (3) *with oscillation distances bounded by* c *converges to zero.*

Theorem 3.4. *Suppose* $\sigma(n) = n+1$ *for* $n \geq 0$ *and suppose*

$$\sum_{i=0}^{\infty} p^+(n) < \infty, \sum_{i=0}^{\infty} |g(n)| < \infty \text{ and } \limsup_{n\to\infty} \sum_{i=n}^{n+c+1} \frac{1}{h(i)} < \infty. \tag{22}$$

Then every bounded (*and oscillatory*) *solution of* (3) *with oscillation distances bounded by* c *converges to zero. If in addition* (17) *holds, then every* (*oscillatory*) *solution of* (3) *with oscillation distances bounded by* c *converges to zero.*

As our final example, consider the equation

$$\Delta^2 x_n + \frac{1}{(n+1)^3} x_{n+1}^{1/3} = \frac{1}{(n+2)^3} \sin(n+2) - \frac{2}{(n+1)^3} \sin(n+1) + \frac{1}{n^3} \sin n + \frac{1}{(n+1)^4} \sin^{1/3}(n+1).$$

It is easy to see that the assumptions in Theorem 3.2 are satisfied, so that every oscillatory solution of this equation tends to zero. Such a solution is given by $\{\sin n/n^3\}$.

References

1. S. S. Cheng, Lyapunov inequalities for differential and difference equations, *Fasc. Math.* **23** (1991), 25-41.

2. S. S. Cheng, B. G. Zhang and S. L. Xie, Qualitative theory of partial difference equations (IV): Forced oscillations of hyperbolic type nonlinear partial difference equations, *Tamkang J. Math.* **26** (1995), 337-360.

3. S. R. Grace and B. S. Lalli, Oscillation theorems for second order delay and neutral difference equations, *Utilitas Math.* **45** (1994), 197-211.

4. I. Györi, Interaction between oscillations and global asymptotic stability in delay differential equations, *Differential Integral Equations* **3** (1990), 181-200.

5. T. Kusano and H. Onose, Asymptotic decay of oscillatory solutions of second order differential equations with forcing term, *Proc. Amer. Math. Soc.* **66** (1977), 251-257.

6. G. Ladas, C. Qian, P. N. Vlahos and J. Yan, Stability of solutions of linear nonautonomous difference equations, *Appl. Anal.* **41** (1991), 183-191.

7. G. Ladas, Y. G. Sficas and I. P. Stavroulakis, Asymptotic behavior of solutions of retarded differential equations, *Proc. Amer. Math. Soc.* **88** (1983), 247-250.

8. H. J. Li and S. S. Cheng, Asymptotically monotone solutions of a nonlinear difference equation, *Tamkang J. Math.* **24** (1993), 269-282.

9. B. Singh, Asymptotically vanishing oscillatory trajectories in second order retarded equations, *SIAM J. Math. Anal.* **7** (1976), 37-44.

10. Y. Yang and W. Zhang, Oscillation for second order superlinear difference equations, *J. Math. Anal. Appl.* **189** (1995), 631-639.

11. B. G. Zhang and S. S. Cheng, Oscillation criteria and comparison theorems for delay difference equations, *Fasc. Math.* **25** (1995), 13-32.

12. B. G. Zhang and H. Wang, The existence of oscillatory and nonoscillatory solutions of neutral difference equations, *Chinese J. Math.* **24** (1996), 377-393.

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