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ABOUT SUCCESSIVE GAUSS-SEIDELISATIONS

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Abstract. This note addresses the general problem of the dynamical behavior for successive Gauss-Seidel transformations (shortly called Gauss-Seidelisations) of a given mapping over the n-cube. Complete results are given for $n = 2$ and $n = 3$, and then a natural conjecture is proved to be false for greater n. Thus this interesting problem remains still open for $n \geq 4$.

1. Overview

Let

$$E = \prod_{p=1}^{n} E_p$$

be the Cartesian product of a finite number of sets E_p and let $\mathcal{F}$ be a mapping of E into itself. The relationship $y = \mathcal{F}(x)$ is detailed into

$$y_p = F_p(x_1, \ldots, x_n) \quad \text{with} \quad \text{p=1, 2, \ldots, n},$$

where F_p is the pth component of $\mathcal{F}$, that is to say a mapping from E into E_p, and where x_p (resp. y_p) is the component of x (resp. y) in E_p.

Define the $Gauss-Seidel$ transformation (or $Gauss-Seidelisation$) $\mathcal{G} = (G_p)$ of $\mathcal{F} = (F_p)$ as the following mapping of E into itself (cf. [1]):

$$(1) \qquad \begin{cases} G_1(x) = F_1(x) \\ G_p(x) = F_p(G_1(x), \ldots, G_{p-1}(x), x_p, \ldots, x_n) \end{cases}$$

with $p = 2 \ldots n$. We denote shortly $\mathcal{G} = \mathcal{T}(\mathcal{F})$.

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The *successive Gauss-Seidelisations* of $\mathcal{F}$ are the sequence $(\mathcal{G}_i)$ of mappings from E into itself defined by

$$(2)\qquad \begin{cases} \mathcal{G}_0 &= \mathcal{F} \\ \mathcal{G}_{i+1} &= \text{Gauss-Seidelisation of } \mathcal{G}_i \\ &= \mathcal{T}(\mathcal{G}_i) \textit{ with } i = 0, 1, 2, \dots \ . \end{cases}$$

The general question we address in this note is concerned with the behavior of $\mathcal{G}_i$ when i increases. In a linear algebra context for example, the case where $E = \mathbb{R}^n$, $\mathcal{F}(X) = AX$ and A is an $n \times n$ real matrix has been addressed as early as 1972 in [2]. In this paper we focus over the boolean case, which means that $\mathcal{F}$ is a mapping from the n-cube, $\{0, 1\}^n$, into itself. Let $\mathcal{E}_n$ be the set of the 2^{n2^n} mappings from the n-cube into itself.

It has been quoted in [1] and proved in [3] that if F is a *boolean contraction*, the sequence of $\mathcal{G}_i$ leads to a stable mapping $\mathcal{G}^* = (G^*)$, in at most $n-1$ steps, such that $G_n^*(x)$ is a constant over the n-cube and generally $G_p^*(x)$ depends only on $x_{p+1}, \dots, x_n$ with $p = 1 \dots n-1$.

We now address the general problem of the behavior of the $\mathcal{G}_i$ for any given $\mathcal{G}_0 = \mathcal{F}$ from the n-cube into itself (not necessarily a boolean contraction).

2. Known Results About Both the 2-cube and the 3-cube

2.1. The 2-cube

We are now interested in successive Gauss-Seidelisations of mappings from the 2-cube into itself, that is to say elements of $\mathcal{E}_2$. This set has $\#\mathcal{E}_2 = 256$ elements.

We are to show that this dynamical system reaches either a stable point or a cycle of length two, after at most one step. One uses here the method shown in [4].

Let $\mathcal{F} \in \mathcal{E}_2$ be one of the 256 mappings over the 2-cube. There are four boolean functions h, k, l and m of x_2 such as

$$\mathcal{F}(x_1, x_2) = \begin{pmatrix} F_1(x_1, x_2) \\ F_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} x_1 h(x_2) + \overline{x_1} k(x_2) \\ x_1 l(x_2) + \overline{x_1} m(x_2) \end{pmatrix}$$

which means

$$\mathcal{F} \begin{cases} F_1(x_1, \cdot) = x_1 h + \overline{x_1} k \\ F_2(x_1, \cdot) = x_1 l + \overline{x_1} m \end{cases}$$

Similarly, one denotes

$$(3)\qquad \mathcal{G}_i \begin{cases} G_1^{(i)}(x_1, \cdot) = x_1 h + \overline{x_1} k \\ G_2^{(i)}(x_1, \cdot) = x_1 l_i + \overline{x_1} m_i. \end{cases}$$

Knowing $\mathcal{G}_i$ is equivalent to knowing how to build $G_2^{(i)}$; that means knowing how l_i and m_i depend on h, k, l and m. Because of (1), one can write

$$\begin{aligned} G_2^{(i)}(x_1, x_2) &= G_2^{(i-1)}\left(G_1^{(i)}(x_1, x_2), x_2\right) \\ &= F_1(x_1, x_2) l_{i-1}(x_2) + \overline{F_1(x_1, x_2)} m_{i-1}(x_2) \end{aligned}$$

because $G_1^{(i)} = F_1$ for all $i \in \mathbb{N}$. But

$$\begin{aligned} \overline{F_1(x_1, x_2)} &= \overline{x_1 h(x_2) + \overline{x_1} k(x_2)} \\ &= x_1 \overline{h(x_2)} + \overline{x_1}\,\overline{k(x_2)} + \overline{h(x_2)}\,\overline{k(x_2)}. \end{aligned} \tag{4}$$

So $G_2^{(i)}$ can be expanded, and then identified with (3), which means:

$$\begin{aligned} G_2^{(i)}(x_1, \cdot) &= \left(x_1 h + \overline{x_1}\overline{k_1}\right) l_{i-1} + \left(x_1 \overline{h} + \overline{x_1}\overline{k} + \overline{h}\,\overline{k}\right) m_{i-1} \\ &= x_1 \left(h l_{i-1} + \overline{h} m_{i-1}\right) + \overline{x_1}\left(k l_{i-1} + \overline{k} m_{i-1}\right) + \overline{h}\,\overline{k} m_{i-1} \\ &= x_1 l_i + \overline{x_1} m_i \quad \text{(by definition).} \end{aligned}$$

Then l_i and m_i can be identified[1] and, as a result, we have a recurrent definition of these two functions. Hence the Gauss-Seidelisation $\mathcal{G}_i = \mathcal{T}(\mathcal{G}_{i-1})$ is equivalent to the boolean recurrence:

$$v_i = \begin{bmatrix} l_i \\ m_i \end{bmatrix} = \begin{bmatrix} h & \bar{h} \\ k & \bar{k} \end{bmatrix} \begin{bmatrix} l_{i-1} \\ m_{i-1} \end{bmatrix} = M v_{i-1}. \tag{5}$$

Proposition 1 *Such a matrix M satisfies $M^3 = M$.*

This means $\mathcal{G}_3 = \mathcal{G}_1$. If $M^2 \neq M$, the sequence of $\mathcal{G}_i$ reaches a cycle of length 2 after at most one step. If $M^2 = M$, the sequence reaches a stable point. The two situations can be represented by:

As an example, we can consider the following mapping from the 2-cube into itself:

$$\mathcal{G}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \overline{x_1} + x_2 \\ x_1 \end{pmatrix}.$$

[01] The constant coefficient is dealt with since $1 = x_1 + \overline{x_1}$.

Then the successive Gauss-Seidelisations lead to the mappings:

$$\mathcal{G}_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \overline{x_1} + x_2 \\ \overline{x_1} + x_2 \end{pmatrix}.$$

and

$$\mathcal{G}_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \overline{x_1} + x_2 \\ \overline{(\overline{x_1} + x_2)} + x_2 \end{pmatrix} = \begin{pmatrix} \overline{x_1} + x_2 \\ x_1 + x_2 \end{pmatrix}$$

and as expected $\mathcal{G}_3 = \mathcal{G}_1$.

For this example, the successive Gauss-Seidelisations can be presented as follows:

2.2. The 3-cube

Here $\#\mathcal{E}_3 = 16777216$, and the decomposition is not as easy as (3), because there are three dimensions. (The third component is much tougher to compute!) However, it is still possible to apply the method used for the 2-cube, but it is much heavier and requires computer algebra. Moreover, this kind of method can no longer be applied to dimensions higher than three.

One can consider to use some computer algebra in the ring $\mathbb{Z}/2\mathbb{Z}$ with the operations $\oplus$ and $\otimes$ (method shown and fully explained in [5]). This ring is introduced because it allows faster computation than Boolean algebras[2]. We can switch from Boolean algebra to the ring operations, and conversely, using:

$$\left\{ \begin{array}{rcl} x \oplus y & = & x \cdot \bar{y} + \bar{x} \cdot y \\ x \otimes y & = & x \cdot y \\ x + y & = & x \oplus y \oplus x \otimes y \\ \bar{x} & = & 1 \oplus x \end{array} \right. \tag{6}$$

and $x = y$ if and only if $x \oplus y = 0$.

[02] Roughly, this is due to the fact that the ideal generated by the Gröbner basis containing the $X_i^2 - X_i$ like polynoms simulates calculus in $\mathbb{Z}/2\mathbb{Z}$ and allows to use the *fast* operations of $\mathbb{Z}[X_1 \ldots X_n]$.

We want now to show that for $n = 3$, we reach a cycle whose length divides 4 after at most 3 steps, whatever is the mapping $\mathcal{G}_0 = \mathcal{F}$ from which we have been iterating. Let us use the same kind of notation for the iterated function as when $n = 2$. This means we can describe a mapping $\mathcal{G}_i$ from the 3-cube into itself with:

$$(7) \qquad \mathcal{G}_i \begin{cases} G_1^{(i)}(x_1, x_2, \cdot) = ax_1x_2 \oplus bx_1 \oplus cx_2 \oplus d \\ G_2^{(i)}(x_1, x_2, \cdot) = e_ix_1x_2 \oplus f_ix_1 \oplus g_ix_2 \oplus h_i \\ G_3^{(i)}(x_1, x_2, \cdot) = p_ix_1x_2 \oplus q_ix_1 \oplus r_ix_2 \oplus s_i, \end{cases}$$

where $\mathcal{G}_{i+1} = \mathcal{T}(\mathcal{G}_i)$ are the successive Gauss-Seidelisations obtained from the initial function $\mathcal{G}_0$. The functions $a, b, c, d, e_i, f_i, g_i, h_i, p_i, q_i, r_i, s_i$ from the ring[3] $\mathbb{Z}/2\mathbb{Z}$ into itself depend on x_3.

We compute, expand and identify $G_2^{(i+1)}$ as we did in (2.1). Hence:

$$(8) \qquad \begin{bmatrix} e_{i+1} \\ f_{i+1} \\ g_{i+1} \\ h_{i+1} \end{bmatrix} = \begin{bmatrix} a \oplus b & a & 0 & 0 \\ 0 & b & 0 & 0 \\ c \oplus d & c & 1 & 0 \\ 0 & d & 0 & 1 \end{bmatrix} \begin{bmatrix} e_i \\ f_i \\ g_i \\ h_i \end{bmatrix}$$

whose matrix is denoted by S, and satisfies $S^3 = S$. As a result, G_2^i, the second component of $\mathcal{G}_i$, reaches a cycle of length two after at most one step, like in the 2-cube (the square).

The computation of G_3^i is heavier, hence done through computer algebra[4], and we finally carry out:

$$(9) \qquad \begin{cases} p_{i+1} = A_ip_i \oplus aq_i \oplus e_ir_i \\ q_{i+1} = B_ip_i \oplus bq_i \oplus f_ir_i \\ r_{i+1} = C_ip_i \oplus cq_i \oplus g_ir_i \\ s_{i+1} = D_ip_i \oplus dq_i \oplus h_ir_i \oplus s_i, \end{cases}$$

where A_i, B_i, C_i, D_i are basic functions from the ring $\mathbb{Z}/2\mathbb{Z}$ into itself[5]. This

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[03] We do not need to write the $\otimes$ because it's the same operation as$\cdot$, hence no mistake can be done.

[04] Maple V was used to compute this, with rules like $x^2 = x$, and a few other well-chosen rules in the ring, to speed up the computation.

[05] Actually:

$$\begin{cases} A_i = a(e_i \oplus f_i \oplus g_i \oplus h_i) \oplus b(e_i \oplus g_i) \oplus c(e_i \oplus f_i) \oplus de_i \\ B_i = bf_i \oplus bh_i \oplus df_i \\ C_i = cg_i \oplus ch_i \oplus dg_i \quad \text{and} \quad D_i = dh_i. \end{cases}$$

leads to the recurrence:

$$\begin{bmatrix} p_{i+1} \\ q_{i+1} \\ r_{i+1} \\ s_{i+1} \end{bmatrix} = \begin{bmatrix} A_i & a & e_i & 0 \\ B_i & b & f_i & 0 \\ C_i & c & g_i & 0 \\ D_i & d & h_i & 1 \end{bmatrix} \begin{bmatrix} p_i \\ q_i \\ r_i \\ s_i \end{bmatrix}, \tag{10}$$

which can be written

$$\nu_{i+1} = T_i \nu_i.$$

The coefficients of T_i depend on the first two components of $\mathcal{G}_i$. As a result, because of the periodicity property of the first two components, we have $T_{i+2} = T_i$. That means for all $i \geq 1$, $T_i = T_2$ or T_1, and:

$$\nu_{i+1} = T_i T_{i-1} \ldots T_2 T_1 T_2 T_1 T_0 \nu_0.$$

Proposition 2 *These matrices T_i satisfy*

$$(T_2 T_1)^3 T_0 = T_2 T_1 T_0. \tag{11}$$

In other words: $\mathcal{G}_7 = \mathcal{G}_3$. This shows that we necessarily reach a cycle of length 4 after at most 3 steps:

We used computer algebra for the proof (cf. [5]), that means we checked that:

$$(T_2 T_1)^3 T_0 \oplus T_2 T_1 T_0 \equiv 0.$$

Since for $n = 2$, we reach a cycle of length 2 after at most one step, and for $n = 3$ we reach a cycle whose length divides 4 after at most three steps, we now address the following question: *Find the general behavior of the successive Gauss-Seidelisations for any n.*

3. The General Case: the n-cube

In the 4-cube or the higher-dimensional case, computer algebra tools can no longer be applied, neither to check all the mappings nor to symbolically validate a property, whether the computation is done in the Boolean algebra or in the ring $\mathbb{Z}/2\mathbb{Z}$. The 4-cube is the first real step in abstraction and technical difficulties in computation in order to reach a general result. Since $\#\mathcal{E}_4 \simeq 2 \cdot 10^{19}$, applying the same kind of methods as before would lead to 800 times the estimated age of the universe under computation...

It has been conjectured (cf. [3, 6]) that in the n-cube, whatever the function we start from, a cycle is reached, whose length divides $2^{(n-1)}$ after at most $2^{(n-1)} - 1$ steps. This unsophisticated conjecture fits well with both the results in the 2-cube and the 3-cube, and also with the boolean contractions.

In the sequel we show that this conjecture *no longer holds*, and give a method in order to find counterexamples for $n = 4$.

3.1. Description of a Mapping from the n-cube into Itself

There are 2^{n2^n} mappings of the n-cube into itself. Such a mapping $\mathcal{F}$ is described by its table (or its n components):

$$F_i : \{0,1\}^n \longrightarrow \{0,1\} \quad (i = 1, \cdots, n).$$

A basis is required in order to build a component, so each component can be described by its coordinates in this basis.

In the n-cube, this basis $\prod_j$ can be written[6] in $\mathbb{Z}/2\mathbb{Z}$:

$$\Pi_j : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \longmapsto \bigotimes_{k=1}^{n} (x_k \oplus \xi_{kj}), \tag{12}$$

where $j = 0 \ldots 2^n - 1$ and

$$\xi_{kj} = \begin{cases} 0 & \text{if 2 divides the enclidian quotient } \ j|2^{k-1}, \\ 1 & \text{otherwise.} \end{cases}$$

As a result, giving a mapping from the n-cube into itself is equivalent to giving a set $(a_{ij}) \in \{0,1\}$ with $i = 1 \ldots n$ and $j = 1 \ldots 2^n$, that is to say in $\{0,1\}^{n2^n}$. Given a set (a_{ij}), we can rebuild the mapping with:

$$\mathcal{F} : \{0,1\}^n \longrightarrow \{0,1\}^n$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \longmapsto \begin{bmatrix} F_1(x) \\ \vdots \\ F_n(x) \end{bmatrix},$$

where the F_i are the boolean sums:

$$F_i(x) = \sum_{j=1}^{2^n} a_{ij} \Pi_{j-1}(x). \tag{13}$$

This definition (13) gives a complete description and enumerates all the mappings from the n-cube into itself. Since it is impossible to make a test on all the mappings, this description allows to catch random mappings and test a given property on them.

[06] These $\prod_j$ are only the known $x_1\, x_2\, x_3\, x_4$, $\overline{x_1}\, x_2\, x_3\, x_4$, $\overline{x_1}\,\overline{x_2}\, x_3\, x_4, \ldots \overline{x_1}\,\overline{x_3}\,\overline{x_3}\,\overline{x_4}$ written in $\mathbb{Z}/2\mathbb{Z}$

3.2. Cycles in the 4-cube

As a result of (3.1), a random mapping from the 4-cube into itself is equivalent to a random set of 64 parameters in $\{0,1\} \equiv \mathbb{Z}/2\mathbb{Z}$, without having any mapping aside.

If true, the conjecture quoted above would lead to reach a cycle whose length divides 8 *after at most* 7 *steps*[7], *which means* $\mathcal{G}_{15} = \mathcal{G}_7$.

A test for $\mathcal{G}_{15} = \mathcal{G}_7$ is $\mathcal{G}_{15}(x) = \mathcal{G}_7(x)$ for all $x \in \{0,1\}^4$ or $\mathcal{G}_{15} \oplus \mathcal{G}_7 \equiv 0$.

A code has been written in order to compute the successive Gauss-Seidelisations of a mapping defined by such a set, and to test the property $\mathcal{G}_{15} = \mathcal{G}_7$. If the property is not satisfied, the code returns the transient and cycle lengths. Besides, another code has been made to check these results in both $\mathbb{Z}/2\mathbb{Z}$ and Boolean algebra. After one and half an hour on a Pentium class computer, one of the random mappings appeared to satisfy $\mathcal{G}_{10} = \mathcal{G}_8$.

This example with a transient of length 8 contradicts the conjecture above, which is thus proved to be definitively false.

It took almost four hours of computation to find another counterexample. These examples are shown in [6]. Of course, no hand-made counterexample has ever been found, and the two above have been checked in several ways.

Another kind of mapping satisfying $\mathcal{G}_8 = \mathcal{G}_2$ has been found. This is a mapping whose cycle length, here 6, does not divide 8. Thus the second part of the conjecture about a cycle length dividing 8 is false as well.

[07] That means that after at most 7 steps, we would reach either a stable point, or a cycle of length 2, 4 or 8.

A third kind of counterexample has even a cycle longer than 8. This one satisfies $\mathcal{G}_{15} = \mathcal{G}_3$, which means a cycle of length 12.

This last example is indeed the following:

$$\mathcal{F}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2\, x_3\, \overline{x_4} + \overline{x_1}\, x_2\, x_4 + \overline{x_1}\, x_2\, \overline{x_3}\, \overline{x_4} + x_1\, \overline{x_2} \\ x_2\, \overline{x_3} + \overline{x_2}\, x_3\, \overline{x_4} + \overline{x_1}\, x_2\, x_3 + x_1\, \overline{x_2}\, x_3\, x_4 + x_1\, x_2\, \overline{x_3}\, x_4 \\ x_2\, x_3\, \overline{x_4} + x_1 \overline{x_2}\, x_3 + x_1\, x_2 \overline{x_3} \\ \overline{x_2}\, \overline{x_3}\, x_4 + x_1\, \overline{x_2}\, x_3\, \overline{x_4} + \overline{x_1}\, x_2\, \overline{x_3}\, \overline{x_4} \end{pmatrix}.$$

The whole experiment tested 6800 mappings and took 44 hours on a Pentium class computer.

4. Conclusion

The dynamical behavior of the successive boolean Gauss-Seidelisations is known for $n = 2$ and $n = 3$. Besides, since a natural conjecture has been proved to be false as soon as $n = 4$, the problem remains still open in dimension greater than or equal to 4. Thus, such a generalization is not as simple as it had once been thought.

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