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SOME UNIFORM ESTIMATES IN PRODUCTS OF RANDOM MATRICES

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Abstract. For each product of random matrices, there associated an invariant measure on the projective space. The convergence to the invariant measure is exponentially fast. In this paper we give uniform estimates on the exponential convergence when the distribution of the random matrices depends on a compact set of parameters.

1. Introduction

The limit theory of products of random matrices, initiated by Bellman, was fully developed by many mathematicians. The subject matter is to understand the asymptotic behavior of norms and matrix elements of the random products. For each random product there is an associated Lyapunov exponent which gives a measure of the exponential growth rate of the matrix norm. However, unlike the usual situation, the Lyapunov exponent cannot be calculated directly from the distribution of the random matrices in most of the cases. The formula for the Lyapunov exponent involves an invariant measure on the projective space. Le Page [9] shows that the convergence to the invariant measure is exponentially fast. The exponent depends on the distribution of the random matrices. It is important to have uniform estimates when the distribution of the random matrices depends on a set of parameters. The uniform estimates will be useful when one studies the scattering problem for the discrete random Schrödinger wave equations. In this paper we derive uniform estimates for the exponential convergence to the invariant measure and for the large deviation of matrix elements.

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2. Random Matrices

Let $\{A_n, n \geq 1\}$ be a sequence of i.i.d. random $d \times d$ invertible matrices with common distribution μ. We are interested in the asymptotic behavior of products $A_n \cdots A_1$. We assume that μ has support in $SL(d, \mathbb{R})$, set of real $d \times d$ matrices with determinant one. Let $S_n = A_n \cdots A_1$. Suppose that a usual vector norm and a usual matrix norm in $\mathbb{R}^d$ have been chosen. Let $\log^+ x = \max\{\log x, 0\}$.

Definition 1. Suppose that $E[\log^+ \|A_1\|] < \infty$. The Lyapunov exponent γ associated with μ is defined by

$$\gamma = \lim_{n\to\infty} \frac{1}{n} E[\log \|A_n \cdots A_1\|].$$

The existence of the Lyapunov exponent can be easily proved by considering the subadditive sequence $E[\log \|A_n \cdots A_1\|]$. It is proved by Furstenberg and Kesten [4], and is an easy consequence of Kingman's subadditive ergodic theorem [7], that under same hypothesis we have, with probability one,

$$\gamma = \lim_{n\to\infty} \frac{1}{n} \log \|A_n \cdots A_1\|.$$

Unlike the classical law of large numbers the Lyapunov exponent cannot be calculated directly from μ in most of the cases. The formula for γ involves an auxiliary measure on the projective space $P(\mathbb{R}^d$ of $\mathbb{R}^d$ [3].

Let x be a unit vector in $\mathbb{R}^d$. Let $u_1 = x$, and for $n = 2, 3, \cdots$,

$$u_n = \frac{A_{n-1} \cdots A_1 x}{\|A_{n-1} \cdots A_1 x\|}.$$

It is clear that the process $\{(A_n, u_n), n \geq 1\}$ is a Markov chain on the phase space $SL(d, \mathbb{R}) \times S^{d-1}$, where S^{d-1} is the unit sphere in $\mathbb{R}^d$, and that

$$\log \|A_n \cdots A_1 x\| = \sum_{k=1}^{n} \log \|A_k u_k\|.$$

Thus if the process is ergodic then one expects that the limit

$$\beta(x) = \lim_{n\to\infty} \frac{1}{n} \log \|A_n \cdots A_1 x\|$$

exists almost surely, and can be expressed as an average with respect to an invariant measure on the phase space. This leads to the consideration of the invariant measure on the projective space $P(\mathbb{R}^d)$. The limit $\beta(x)$ is closely related to the Lyapunov exponent. A full account of the discussion of the relationship between γ and $\beta(x)$ can be found in [10].

For two non-zero vectors $x, y \in \mathbb{R}^d$, we say $x \sim y$ if $x = cy$ for some $c \in \mathbb{R}$. The projective space $P(\mathbb{R}^d)$ is the quotient space $\mathbb{R}^d \backslash \{0\}/\sim$. For $x \in \mathbb{R}^d \backslash \{0\}$, $\overline{x}$ denotes its equivalent class in $P(\mathbb{R}^d)$. For $M \in SL(d, \mathbb{R}^d)$, we set $M \cdot \overline{x} = \overline{Mx}$. For $\bar{u}, \bar{v} \in P(\mathbb{R}^d)$, define

$$\delta(\bar{u}, \bar{v}) = \left[1 - \left|\left\langle \frac{u}{\|u\|}, \frac{v}{\|v\|} \right\rangle\right|^2\right]^{\frac{1}{2}} = |\sin\theta|,$$

where θ is the angle between $\bar{u}$ and $\bar{v}$. It is not hard to check that $\delta(\bar{u}, \bar{v})$ is a metric on $P(\mathbb{R}^d)$, and is equal to $\|u \wedge v\|/(\|u\| \cdot \|v\|)$, where $\|u \wedge v\|$ is the norm of the exterior product of u, v. Let μ be a probability measure on $SL(d, \mathbb{R}^d)$, and ν be a probability measure on $P(\mathbb{R}^d)$.

Definition 2. $\mu * \nu$ is the probability measure on $P(\mathbb{R}^d)$ which satisfies

$$\int f(\overline{x}) d\mu * \nu(\overline{x}) = \int \int f(M \cdot \overline{x}) d\mu(M) d\nu(\overline{x})$$

for all bounded Borel function f on $P(\mathbb{R}^d)$.

We say that ν is μ-invariant if $\mu * \nu = \nu$. The following theorem [5] shows the relationship between the invariant measures and the Lyapunov exponent.

Theorem 1. *Let $\{A_n, n \geq 1\}$ be a sequence of i.i.d. random matrices with common distribution μ. Suppose that μ has support in $SL(d, \mathbb{R})$, and that $E[\log^+ \|A_1\|] < \infty$. Then, with probability one,*

$$\gamma = \lim_{n \to \infty} \frac{1}{n} \log \|A_n \cdots A_1\|$$

and

$$\gamma = \sup \int \int \log \frac{\|Mx\|}{\|x\|} d\mu(M) d\nu(\overline{x}),$$

where the sup *is taken over all μ-invariant measures.*

In the situation that there is only one μ-invariant measure ν on $P(\mathbb{R}^d)$, we have a simple expression for γ. But this does not mean we can calculate γ easily. Since in most of the cases, the μ-invariant measures are not available.

Let T_μ be the smallest closed semigroup which contains the support of μ. Let $l(M) = \max(\log^+ \|M\|, \log^+ \|M^{-1}\|)$ for $M \in SL(d, \mathbb{R})$. A distribution μ on $SL(d, \mathbb{R})$ is said to have a finite exponential moment if $\int e^{\tau l(M)} d\mu(M)$ is finite for some $\tau > 0$. We assume throughout this paper that all the distributions in our discussion have finite exponential moments.

Definition 3. A subset S of $SL(d,\mathbb{R})$ is said to be irreducible if there is no proper linear subspace V of $\mathbb{R}^d$ such that $M(V)=V$ for all $M\in S$. S is said to be strongly irreducible if there does not exist a finite union of proper linear subspaces of $\mathbb{R}^d$ which is invariant under the action of the elements of S. S is said to be contracting if there is a sequence $\{M_n, n\geq 0\}$ in S for which $\|M_n\|^{-1}M_n$ converges to a rank-one matrix.

The importance of being strongly irreducible and contracting is indicated by the fact that if T_μ is strongly irreducible and contracting, then there exists a unique μ-invariant distribution on $P(\mathbb{R}^d)$ and $\gamma>0$ [6]. The uniqueness of the invariant distribution plays an extremely important role in the theory of products of random matrices.

Let ϕ be a continuous function on $P(\mathbb{R}^d)$ and $\alpha>0$. We define

$$\|\phi\|_\alpha=\sup_{\bar{u}\in P(\mathbb{R}^d)}|\phi(\bar{u})|+\sup_{\bar{u}\neq\bar{v}}\frac{|\phi(\bar{u})-\phi(\bar{v})|}{\delta(\bar{u},\bar{v})^\alpha},$$

$$P\phi(\bar{u})=\int\phi(M\cdot\bar{u})d\mu(M),$$

$$Q\phi(\bar{u})=\int\phi(\bar{v})d\nu(\bar{v}).$$

With these notations, we have

$$E[\phi(S_n\cdot\bar{u})]=\int\phi(M\cdot\bar{u})d\mu^n(M)=P^n\phi(\bar{u}),$$

where μ^n is the nth convolution power of μ. Let $C(\alpha)$ be the set of continuous functions ϕ on $P(\mathbb{R}^d)$ for which $\|\phi\|_\alpha$ is finite. Note that $\|\phi\|_\alpha\leq\|\phi\|_{\alpha'}$ if $\alpha\leq\alpha'$. The following proposition is due to Le Page [9]. It shows that the distribution of $S_n\cdot\bar{u}$ converges to the μ-invariant distribution ν exponentially fast.

Proposition 1. *If T_μ is strongly irreducible and contracting, then there exist $\alpha_0, c_1>0, 0<\rho_1<1$ such that for $0<\alpha\leq\alpha_0$, the operators P and Q defined on $C(\alpha)$ are bounded and satisfy*

$$\|P^n-Q\|_\alpha\leq c_1\rho_1^n.$$

The next proposition is from Guivarc'h and Raugi [6]. It tells about the regularity of the invariant distribution.

Proposition 2. *If T_μ is strongly irreducible and contracting, then there exists $\beta>0$ such that*

$$\sup_{\|v\|=1}\int\left[\frac{\|u\|}{|\langle u,v\rangle|}\right]^\beta d\nu(\bar{u})<\infty.$$

We know that the Lyapunov exponent gives the growth rate of the vectors and matrix elements. A large deviation result for vectors has been proved by Le Page [9].

Proposition 3. *If T_μ is strongly irreducible and contracting, then for each $\epsilon > 0$ there exists $a > 0$ such that for all unit vectors $u \in \mathbb{R}^d$,*

$$Pr\left\{|\frac{1}{n}\log\|S_n u\| - \gamma| \geq \epsilon\right\} \leq e^{-an}$$

for large n.

We now prove a large deviation result for matrix elements.

Theorem 2. *If T_μ is strongly irreducible and contracting, then for all $\epsilon > 0$ there exists $a > 0$ such that for all unit vectors $u, v \in \mathbb{R}^d$,*

$$Pr\left\{|\frac{1}{n}\log|\langle S_n u, v\rangle| - \gamma| \geq \epsilon\right\} \leq e^{-an}$$

for large n.

Proof.

$$\begin{aligned} &Pr\{|\langle S_n u, v\rangle| \leq e^{(\gamma-\epsilon)n}\} \\ &\leq Pr\{\|S_n u\| \leq e^{(\gamma-\frac{\epsilon}{2})n}\} + Pr\{|\langle S_n u, v\rangle| \leq e^{-\frac{\epsilon}{2}n}\|S_n u\|\}. \end{aligned} \tag{1}$$

The first term in (1) is bounded by $e^{-a_1 n}$ for some $a_1 > 0$ by Proposition 3. To find an upper bound for the second term in (1), we shall use Propositions 1 and 2. Define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq e^{-\frac{\epsilon}{2}n}, \\ 2 - te^{\frac{\epsilon}{2}n} & \text{if } e^{-\frac{\epsilon}{2}n}, \leq t \leq 2e^{-\frac{\epsilon}{2}n}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $0 \leq f_n(t) \leq 1$. It is easy to see that for all $t, t' \in [0,1]$

$$|f_n(t) - f_n(t')| \leq |t - t'|e^{\frac{\epsilon}{2}n}.$$

Therefore if we define $\phi_n : P(\mathbb{R}^d) \to \mathbb{R}$ by $\phi_n(\bar{u}) = f_n(\frac{|\langle u,v\rangle|}{\|u\|})$, we have $\|\phi_n\|_\alpha \leq$

$e^{\frac{\epsilon}{2}n} + 1$ for $\alpha \leq 1$. Then

$$\begin{aligned}
&Pr\{|\langle S_n u, v\rangle| \leq e^{-\frac{\epsilon}{2}n}\|S_n u\|\} \\
&= Pr\left\{\frac{|\langle S_n u, v\rangle|}{\|S_n u\|} \leq e^{-\frac{\epsilon}{2}n}\right\} \\
&\leq E\left[f_n\left(\frac{|\langle S_n u, v\rangle|}{\|S_n u\|}\right)\right] \\
&\leq |E\left[f_n\left(\frac{|\langle S_n u, v\rangle|}{\|S_n u\|}\right)\right] - \int f_n\left(\frac{|\langle u, v\rangle|}{\|u\|}\right) d\nu(\bar{u})| \\
&\quad + \int f_n\left(\frac{|\langle u, v\rangle|}{\|u\|}\right) d\nu(\bar{u}) \\
&\leq \|P^n - Q\|_\alpha \|\phi_n\|_\alpha + E_\nu\left[f_n\left(\frac{|\langle u, v\rangle|}{\|u\|}\right)\right].
\end{aligned} \tag{2}$$

From Proposition 1 we know that $\|P^n - Q\|_\alpha \leq e^{-bn}$ for some $b > 0$. The first term in (2) is bounded by $e^{-(b-\frac{\epsilon}{2})n} + e^{-bn}$. For the second term in (2), since $0 \leq f_n(t) \leq 1$, we have

$$\begin{aligned}
E_\nu\left[f_n\left(\frac{|\langle u, v\rangle|}{\|u\|}\right)\right] &\leq Pr\left\{\frac{|\langle u, v\rangle|}{\|u\|} \leq 2e^{-\frac{\epsilon}{2}n}\right\} \\
&= Pr\left\{\left[\frac{\|u\|}{|\langle u, v\rangle|}\right]^\beta \geq 2^{-\beta} e^{\frac{\beta\epsilon}{2}n}\right\} \\
&\leq 2^\beta e^{-\frac{\beta\epsilon}{2}n} E\left[\left[\frac{\|u\|}{|\langle u, v\rangle|}\right]^\beta\right].
\end{aligned}$$

The large deviation result follows by making ϵ small and by using Proposition 2. The other case is a direct consequence of Proposition 3. ∎

Since the metric δ is related to the exterior power of vectors, we will apply the theory of products of random matrices to subspaces of $\wedge^r \mathbb{R}^d$ $(1 \leq r \leq d)$ instead of $\mathbb{R}^d$. Let $\wedge^r \mathbb{R}^d$ (the *exterior power* of $\mathbb{R}^d$) denote the space of alternating r-linear forms on the dual space $(\mathbb{R}^d)^\star$ of $\mathbb{R}^d$. For $v_1, v_2, \ldots, v_r \in \mathbb{R}^d$ and $f_1, f_2, \ldots, f_r \in (\mathbb{R}^d)^\star$, we set

$$(v_1 \wedge v_2 \wedge \cdots \wedge v_r)(f_1, f_2, \ldots, f_r) = \det[\{f_i(v_j)\}_{i,j}].$$

The linear space $\wedge^r \mathbb{R}^d$ is a C_r^d-dimensional vector space (C_m^n is the binomial coefficient). If $\{e_1, e_2, \ldots, e_d\}$ is the standard basis of $\mathbb{R}^d$, then $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r}, 1 \leq i_1 < i_2 < \cdots < i_r \leq d\}$ is an orthonormal basis of $\wedge^r \mathbb{R}^d$ [2, Chap. II].

We define an inner product on $\wedge^r\mathbb{R}^d$ by

$$\langle u_1 \wedge u_2 \wedge \cdots \wedge u_r, v_1 \wedge v_2 \wedge \cdots \wedge v_r\rangle = \det[\{\langle u_i, v_j\rangle\}_{i,j}],$$

where $\langle\cdot,\cdot\rangle$ is the usual inner product in $\mathbb{R}^d$. For $M \in SL(d,\mathbb{R})$, we define a linear mapping $\wedge^r M$ in $\wedge^r\mathbb{R}^d$ by

$$\wedge^r M(v_1 \wedge v_2 \wedge \cdots \wedge v_r) = Mv_1 \wedge Mv_2 \wedge \cdots \wedge Mv_r.$$

Note that $\wedge^r(AB) = (\wedge^r A)(\wedge^r B)$. We now define the rth Lyapunov exponent γ_r for $r = 1, \dots, d$ inductively by

$$\sum_{i=1}^{r} \gamma_i = \lim_{n\to\infty} \frac{1}{n} E[\log \| \wedge^r S_n\|].$$

In other words, $\sum_{i=1}^{r} \gamma_i$ is the Lyapunov exponent associated with the matrices $\wedge^r S_n$ acting on $\wedge^r\mathbb{R}^d$. Obviously, $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_d$. The set $\{\gamma_i\}$ also describes the exponential growth of the moduli of the eigenvalues of S_n as n becomes large.

In our application the random matrices are symplectic. A $2d \times 2d$ matrix M is said to be *symplectic* if it satisfies the equation $M^t JM = J$, where

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

and I is the $d \times d$ identity matrix. The Lyapunov exponents associated with symplectic matrices satisfy the equation $\gamma_{2d+1-i} = -\gamma_i$ for $i = 1, 2, \dots, d$. This can be seen by the fact that the products of symplectic matrices are also symplectic and if λ is an eigenvalue of a symplectic matrix M, then so is λ^{-1}:

$$\det\{M - \lambda^{-1}I\} = \det\{MJ - \lambda^{-1}MJM^t\} = \det\{MJ\}\det\{I - \lambda^{-1}M^t\} = 0.$$

3. Main Results

In this section we consider that the distribution of the random matrices, denoted as μ_E, depends on a parameter E. We will show that the propositions in section 2 can be extended from a fixed real parameter E to a compact set F of real numbers. We assume that T_{μ_E} is strongly irreducible and contracting, and that μ_E has a uniform finite exponential moment for $E \in F$. We can show that there exist $\tau, K > 0$ such that $\int e^{\tau l(M)} d\mu_E(M) \leq K < \infty$ for all $E \in F$. It is shown in [1] that $n^{-1}E[\log \|S_n^E u\|]$ converges to $\gamma_1(E)$ uniformly in $E \in F$ and $u \in \mathbb{R}^d$ with $\|u\| = 1$. This implies that the top Lyapunov exponent is continuous in E. From the assumption T_{μ_E} being strongly irreducible and contracting, we know that the top Lyapunov exponent $\gamma_1(E)$ is strictly positive, and that the top two Lyapunov exponents, $\gamma_1(E)$ and $\gamma_2(E)$, are distinct [6, Thm 6.1, §III.6, Part A]. The uniform convergence and the continuity of the Lyapunov exponent will play the central role in our derivations.

Theorem 3. *There exist $\alpha_0, b_1 > 0, 0 < r_1 < 1$ such that for all $0 < \alpha \leq \alpha_0$,*

$$\|P_E^n - Q_E\|_\alpha < b_1 r_1^n$$

for all $E \in F$.

We will need the following lemmas.

Lemma 1. *There exist $N, c_1 > 0$ such that for all $n > N$,*

$$\sup_{\bar{u} \neq \bar{v}} \frac{1}{n} E\left[\log \frac{\delta(S_n^E \bar{u}, S_n^E \bar{v})}{\delta(\bar{u}, \bar{v})}\right] \leq -c_1 < 0$$

for all $E \in F$.

Proof. From the equation $\delta(\bar{u}, \bar{v}) = \|u \wedge v\| \times \|u\|^{-1}\|v\|^{-1}$, we deduce that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} E\left[\log \frac{\delta(S_n^E \bar{u}, S_n^E \bar{v})}{\delta(\bar{u}, \bar{v})}\right] \\
&= \lim_{n\to\infty} \frac{1}{n} E\left[\log \frac{\|S_n^E u \wedge S_n^E v\|}{\|u \wedge v\|}\right] - 2 \lim_{n\to\infty} \frac{1}{n} E\left[\log \frac{\|S_n^E u\|}{\|u\|}\right] \\
&= \gamma_2(E) - \gamma_1(E).
\end{aligned}$$

Since the top two Lyapunov exponents are distinct, we have $\gamma_1(E) > \gamma_2(E)$. The proof follows by the uniform convergence to the top Lyapunov exponent. ∎

Lemma 2. *There exist $\alpha_0, c_3 > 0, 0 < \rho_3 < 1$ such that for all $0 < \alpha \leq \alpha_0$,*

$$\sup_{\bar{u} \neq \bar{v}} E\left[\frac{\delta(S_n^E \bar{u}, S_n^E \bar{v})^\alpha}{\delta(\bar{u}, \bar{v})^\alpha}\right] \leq c_3 \rho_3^n$$

for all $E \in F$.

Proof. Let

$$a_n = \log\left\{\sup_{\bar{u} \neq \bar{v}} E\left[\frac{\delta(S_n^E \bar{u}, S_n^E \bar{v})^\alpha}{\delta(\bar{u}, \bar{v})^\alpha}\right]\right\}.$$

It is not hard to check that a_n is subadditive: $a_{n+m} \leq a_n + a_m$. This implies that

$$\lim_{n\to\infty} \frac{1}{n} \log \left\{ \sup_{\bar{u}\neq\bar{v}} E\left[\frac{\delta(S_n^E\bar{u}, S_n^E\bar{v})^\alpha}{\delta(\bar{u},\bar{v})^\alpha} \right] \right\} = \lim_{n\to\infty} \frac{a_n}{n} = \inf_n \frac{a_n}{n}.$$

From the inequality $e^x \leq 1 + x + (x^2/2)e^{|x|}$, we deduce that

$$\begin{aligned}
&\sup_{\bar{u}\neq\bar{v}} E\left[\frac{\delta(S_n^E\bar{u}, S_n^E\bar{v})^\alpha}{\delta(\bar{u},\bar{v})^\alpha} \right] \\
(3) \qquad \leq\ & 1 + \alpha \sup_{\bar{u}\neq\bar{v}} E\left[\log \frac{\delta(S_n^E\bar{u}, S_n^E\bar{v})}{\delta(\bar{u},\bar{v})} \right] \\
&+ \frac{\alpha^2}{2} E\left[\left(\log \frac{\delta(S_n^E\bar{u}, S_n^E\bar{v})}{\delta(\bar{u},\bar{v})} \right)^2 \exp\left\{ \alpha |\log \frac{\delta(S_n^E\bar{u}, S_n^E\bar{v})}{\delta(\bar{u},\bar{v})}| \right\} \right].
\end{aligned}$$

Note that $|\log \|\wedge^r Mu\|| \leq rl(M)\|u\|$ [1]. From the assumption on the moment of the random potential, the expectation in the last term of (3) is uniformly bounded. The right-hand side of (3) can be made strictly less than 1 uniformly by choosing α small and using Lemma 1. ∎

We now prove Theorem 3. For any $\phi \in C(\alpha)$,

$$\begin{aligned}
\frac{|(P_E^n - Q_E)\phi(\bar{u}) - (P_E^n - Q_E)\phi(\bar{v})|}{\delta(\bar{u},\bar{v})^\alpha} &= \frac{|P_E^n\phi(\bar{u}) - P_E^n\phi(\bar{v})|}{\delta(\bar{u},\bar{v})^\alpha} \\
&\leq m_\alpha(\phi) \int \frac{\delta(M\bar{u}, M\bar{v})^\alpha}{\delta(\bar{u},\bar{v})^\alpha} d\mu_E^n(M) \\
&\leq c_3\rho_3^n m_\alpha(\phi).
\end{aligned}$$

On the other hand, since ν_E is μ_E-invariant,

$$\begin{aligned}
|(P_E^n - Q_E)\phi(\bar{u})| &= |\int \phi(M\bar{u}) d\mu_E^n(M) - \int \phi(M\bar{v}) d\mu_E^n(M) d\nu_E(\bar{v})| \\
&\leq \int |\phi(M\bar{u}) - \phi(M\bar{v})| d\mu_E^n(M) d\nu_E(\bar{v}) \\
&\leq m_\alpha(\phi) E\left[\frac{\delta(S_n^E\bar{u}, S_n^E\bar{v})^\alpha}{\delta(\bar{u},\bar{v})^\alpha} \right] \\
&\leq m_\alpha(\phi) c_3 \rho_3^n.
\end{aligned}$$

This implies $\|P_E^n - Q_E\|_\alpha \leq 2c_3\rho_3^n$. ∎

A proof of the proposition on the regularity of the invariant measure (Proposition 2) can be found in [1, Thm 2.1, §VI.2, Part A]. Following the proof and using Lemma 2, it is not hard to check that the result of Proposition 2 can be extended from a fixed parameter to a compact set of parameters. The proof is tedious but straightforward, and is omitted.

Theorem 4. *For each $\epsilon > 0$ there exists an $a > 0$ such that for all $\|u\| = 1$,*

$$Pr\left\{|\frac{1}{n}\log\|S_n^E u\| - \gamma_1(E)| > \epsilon\right\} < e^{-an}$$

for all $E \in F$.

Proof. Let t be a small positive number. We have

$$Pr\{\log\|S_n^E u\| - n\gamma_1(E) > n\epsilon\} \le E[\exp\{t(\log\|S_n^E u\| - n(\gamma_1(E) + \epsilon))\}],$$

and

$$(4)\ \log Pr\left\{\frac{1}{n}\log\|S_n^E u\| - \gamma_1(E) > \epsilon\right\} \le -nt\epsilon + \log E[\|S_n^E u\|^t] - nt\gamma_1(E).$$

Again, we use the inequality $e^x \le 1 + x + (x^2/2)e^{|x|}$ to deduce

$$E[\|S_n^E u\|^t] \le 1 + tE[\|S_n^E u\|] + \frac{t^2}{2}E[(\log\|S_n^E u\|)^2\exp\{t|\log\|S_n^E u\||\}].$$

Let $\xi = 1 + (\eta/(\inf_F \gamma_1(E)))$, where η is a small positive number. By applying the Cauchy-Schwarz inequality, the expectation in the last term of the above inequality can be bounded by a positive number c_n depending on F only. From Lemma 1, we can find n_0 such that

$$E[\|S_{n_0}^E u\|^t] \le 1 + t\xi n_0\lambda_1(E) + t^2 c_{n_0}.$$

By iterating the inequality and using the fact that the convergence of the limit $n^{-1}E[\log\|S_n^E u\|]$ to the top Lyapunov exponent is uniform in $\|u\| = 1$, we get

$$E[\|S_n^E u\|^t] \le (1 + t\xi n_0\gamma_1(E) + t^2 c_{n_0})^{\frac{n}{n_0}+1}.$$

This implies that

$$\frac{1}{n}\log E[\|S_n^E u\|^t] \le t\xi\gamma_1(E) + \frac{n_0}{n}t\xi\gamma_1(E) + O(t^2).$$

By choosing $t, \eta > 0$ small enough, the right-hand side of (4) is strictly negative (uniformly in $E \in F$). The other case can be proved in a similar manner. ■

Following the proof of Theorem 2 and using Theorems 3 and 4 and the generalization of Proposition 2, we have the following generalization of Theorem 2.

Theorem 5. *For each $\epsilon > 0$ there exists $a > 0$ such that for all unit vectors $u, v \in \mathbb{R}^d$,*

$$Pr\left\{|\frac{1}{n}\log|\langle S_n^E u, v\rangle| - \gamma| \geq \epsilon\right\} \leq e^{-an}$$

for all $E \in F$.

4. Application

Let $\{V_n\}$ be a sequence of i.i.d. real random variables. We assume that $E[|V_1|^\alpha] < \infty$ for some $\alpha > 0$. Let the Hamiltonian operator H be defined on $l^2(\mathbb{Z})$ as

$$Hu_n = 2u_n - u_{n-1} - u_{n+1} + V_n u_n.$$

The one-dimensional discrete random Schrödinger wave equation is

$$i\frac{d\psi_t}{dt} = H\psi_t.$$

The way to solve the wave equation is to look at the eigenvalue equation

$$Hu = Eu.$$

The real number E is considered as the energy parameter. In scattering theory, one is interested in the superposition of waves with different energy parameters. It would be very helpful to have uniform estimates in the energy parameters.

The eigenvalue equation can be expressed in vector form as

$$\begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix} = \begin{bmatrix} V_n + 2 - E & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_n \\ u_{n-1} \end{bmatrix}.$$

We may rewrite this as

$$y_{n+1} = A_n y_n.$$

Thus, given the initial condition y_1 we can integrate the equation and get the final result

$$y_{n+1} = A_n \cdots A_1 y_1.$$

It is clear that the asymptotic behavior of the solution of the eigenvalue equation is strongly related to the random product $A_n \cdots A_1$. Denote the distribution of the random matrices A_n by μ_E. It has been proved that if the support of the distribution of V_1 contains an open subset of $\mathbb{R}$, then T_{μ_E} is strongly irreducible and contracting [8]. Note that all the matrices A_n have determinant one, and are symplectic. The results in the above section apply to the scattering problem of the discrete random Schrödinger wave equation.

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