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COMPACT AND WEAKLY COMPACT DERIVATIONS OF CERTAIN CSL ALGEBRAS

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Abstract. In this paper, we investigate weakly compact and compact derivations of certain CSL algebras. These algebras contain finite tensor product algebras of nest algebras and some nest subalgebras of a von Neumann algebra.

1. Introduction

In [10], Peligrad determines the structure of all weakly compact derivations of a nest algebra and also obtains necessary and sufficient conditions in order that a nest algebra admit a nonzero compact derivation. For a nest algebra $\mathcal{A}$, by [3, 7], we have $H^n(\mathcal{A}, \mathcal{B}) = 0$ for any positive integer n and all ultraweakly closed subalgebras $\mathcal{B}$ of $\mathcal{L}(H)$ containing $\mathcal{A}$. In [4], it was shown that $H^1(\mathcal{A}, \mathcal{A})$ need not be trivial even when $\mathcal{A}$ is the intersection of two nest algebras. Thus for a reflexive algebra, even a CSL algebra, it is difficult to determine the structure of all compact and weakly compact derivations of the algebra. In this note, we prove that some of Peligrad's results can be achieved for some reflexive algebras. These algebras include finite tensor product algebras of nest algebras and some nest subalgebras of a von Neumann algebra.

Throughout the paper, $\mathcal{H}$ denotes a complex separable Hilbert space. Let $\mathcal{L}(H)$ denote the set of all operators on $\mathcal{H}$, and let $\mathcal{K}(H)$ denote the set of compact operators on $\mathcal{H}$. For $e,\ f$ in $\mathcal{H}$, we denote by $e \otimes f$, the operator $x \mapsto \langle x, e\rangle f$. A subspace lattice which consists of mutually commuting projections is called a *commutative subspace lattice*; the associated reflexive algebra is called a *CSL algebra*. A totally ordered subspace lattice is called a *nest* and the associated reflexive algebra is called a *nest algebra*. When there is no

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confusion we identify the subspace and the orthogonal projection on it. For any subalgebra $\mathcal{A}$ of $\mathcal{L}(H)$ and any subspace $\mathcal{S} \subseteq L(H)$ which is a 2-sided $\mathcal{A}$-module, let $C(\mathcal{A}, \mathcal{S})$ denote the set

$$\{X \in \mathcal{L}(H): \ AX - XA \in \mathcal{S} \quad \text{for all} \quad A \in \mathcal{A}\},$$

that is, $C(\mathcal{A}, \mathcal{S})$ is the *commutant of* $\mathcal{A}$ *modulo* $\mathcal{S}$.

A derivation of an algebra $\mathcal{A}$ into a (2-sided) $\mathcal{A}$-module $\mathcal{S}$ is a linear map δ such that $\delta(ab) = a\delta(b) + \delta(a)b$. A derivation of the form $\delta_x(a) = xa - ax$ with x in $\mathcal{S}$ is said to be inner. In [1], Christensen proves that all derivations from a CSL algebra $\mathcal{A}$ into itself are norm continuous.

2. Some Results

If $\mathcal{L}$ is a subspace lattice, for $M \in \mathcal{L}$, let

$$M_{-} = \vee\{N \in \mathcal{L}: \ M \not\subseteq N\}.$$

Let $\mathcal{J}_{\mathcal{L}}$ denote the subset of $\mathcal{L}$ defined by $\{L \in \mathcal{L} : L \neq 0 \text{ and } L_{-} \neq I\}$. Define $\mathcal{A}^* = \{A^* : A \in \mathcal{A}\}$ and $\mathcal{L}^{\perp} = \{I - P : P \in \mathcal{L}\}$. For completely distributive lattices, we need not be concerned with the actual definition here. For our purpose it is enough to know the following result.

Lemma 1[8]. *Suppose that $\mathcal{L}$ is a commutative subspace lattice. Then the following statements are equivalent*:

(1) *$\mathcal{L}$ is completely distributive.*

(2) *The collection of finite sums of rank-one operators in* alg $\mathcal{L}$ *is ultraweakly dense in* alg $\mathcal{L}$.

In the following, let $\mathcal{A}$ be a CSL algebra, let $\mathcal{K}(A) = \mathcal{A} \cap K(H)$, and let δ be a derivation from $\mathcal{A}$ into itself.

Using the above Lemma 1, it is easy to prove the following result.

Lemma 2. *If* lat $\mathcal{A}$ *is completely distributive, then* $\mathcal{K}(A)^{**} = \mathcal{A}$.

Theorem 3. *Let δ be a derivation from $\mathcal{A}$ into itself, and* lat $\mathcal{A}$ *be completely distributive such that $H^1(\mathcal{A}, A) = 0$ and $C(\mathcal{A}, K(\mathcal{A})) \subseteq \mathcal{A}' + K(H)$. Then the following statements are equivalent*:

(1) *δ is weakly compact.*

(2) $\delta(\mathcal{A}) \subseteq K(\mathcal{A})$.

(3) *$\delta = \delta_x$ for some $x \in \mathcal{K}(A)$.*

Proof. We only prove that (2) implies (3). The rest is left to the reader. Since $H^1(\mathcal{A}, A) = 0$, we have that $\delta = \delta_T$ with T in $\mathcal{A}$. By $\delta(\mathcal{A}) \subseteq K(\mathcal{A})$, it follows that for A in $\mathcal{A}$, $AT - TA \in \mathcal{K}(\mathcal{A})$. For $C(\mathcal{A}, K(\mathcal{A})) \subseteq \mathcal{A}' + K(H)$, we have $T = m + x, m \in \mathcal{A}' \subseteq \mathcal{L}' \subseteq \mathcal{A}$ and $x \in \mathcal{K}(A)$. Hence $\delta_T = \delta_x$. ∎

Remark 1. Suppose that $\mathcal{L} = N_1 \otimes \cdots \otimes \mathcal{N}_n$, where $\mathcal{N}_i$ is a nest acting on $\mathcal{H}_i$. Using Proposition 2.7, Theorem 3.1 and Remark 3.3 [4], respectively, we have that $\mathcal{L}$ is completely distributive, $H^1(alg\ \mathcal{L},\ alg\ \mathcal{L}) = 0$, and $C(alg\ \mathcal{L},\ K(alg\ \mathcal{L})) \subseteq \mathcal{A}' + K(H)$.

Let $\mathcal{B}$ be a von Neumann algebra contained in $\mathcal{L}(H)$ and let $\mathcal{N}$ be a nest contained in $\mathcal{B}$. Let $\mathcal{A} = \mathcal{B} \cap alg\ \mathcal{N}$. The algebra $\mathcal{A}$ is called (*nsva*) the *nest subalgebra of the von Neumann algebra* $\mathcal{B}$ relative to the nest $\mathcal{N}$.

Remark 2. If $\mathcal{A}$ is a nsva of a von Neumann algebra $\mathcal{B}$, by Theorem 6 [3], then $H^1(\mathcal{A}, L(H)) = 0$. As in [1, 4], this implies that $H^1(\mathcal{A}, B) = 0$ for any ultraweakly closed subalgebra of $\mathcal{L}(H)$ containing $\mathcal{A}$. Hence $H^1(\mathcal{A}, A) = 0$. By Theorem 6.4 [5], we have that

$$C(\mathcal{A}, K(H)) = C(\mathcal{B}, K(H)).$$

By Theorem I [11] and Theorem 2.5 [5], it follows that $C(\mathcal{B}, K(H)) = \mathcal{B}' + K(H)$ and $\mathcal{A}' = \mathcal{B}'$. Hence $C(\mathcal{A}, K(\mathcal{A})) \subseteq \mathcal{A}' + K(H)$. Clearly, by Remarks 1, 2 and Theorem 3, we have the following result.

Corollary 4. *Suppose that* $\mathcal{A}$ *is a nsva such that* lat $\mathcal{A}$ *is completely distributive or that* $\mathcal{A} = alg\ \mathcal{N}_1 \otimes \cdots \otimes alg\ \mathcal{N}_n$*, where* $\mathcal{N}_i$ *is a nest acting on* $\mathcal{H}_i$*. Then the following statements are equivalent*:

(1) δ *is weakly compact.*

(2) $\delta(\mathcal{A}) \subseteq \mathcal{K}(A)$.

(3) $\delta = \delta_x$ *for some* $x \in \mathcal{K}(A)$.

Lemma 5. *Let* $\mathcal{A}$ *be as in Theorem* 3*. If dim* $P\mathcal{H} = \infty$ *for any* P *in* $\mathcal{J}_{\mathcal{L}}$*, then* $\mathcal{A}$ *has no nonzero compact derivation.*

Proof. Suppose that $\mathcal{A}$ has a nonzero compact derivation δ. By Theorem 3, we have that $\delta = \delta_T$ with $T \in \mathcal{K}(A)$. We consider the following two cases.

(i) If there is an element P in $\mathcal{J}_{\mathcal{L}}$ and $e_0 \in (I - P_{-})\mathcal{H}, \|e_0\| = 1$, such that $\langle Te_0, e_0\rangle \neq 0$. Let $Te_0 = \lambda e_0 + y_0$, where $y_0 \perp e_0$. Since $Te_0 = \lambda e_0 + y_0$, it follows $\lambda \neq 0$. Let $\{f_i\}_{i=1}^{\infty}$ be an orthogonal family in $P\mathcal{H}$. By Lemma 3 [9], we have $e_0 \otimes f_i \in \mathcal{A}$. Since δ is compact and $\|e_0 \otimes f_i\| = 1$, it follows that $\{\delta(e_0 \otimes f_i)\}$ contains a convergent subsequence. We may assume that $\{\delta(e_0 \otimes f_i)\}$ converges. Since

$$\delta(e_0 \otimes f_i)e_0 = T(e_0 \otimes f_i)e_0 - (e_0 \otimes f_i)Te_0 = Tf_i - \langle Te_0, e_0\rangle f_i = Tf_i - \lambda f_i$$

and T is compact, it follows that $\{f_i\}$ has a convergent subsequence. Since $\{f_i\}$ is an orthogonal family, clearly this is impossible.

(ii) If for any P in $\mathcal{J}_{\mathcal{L}}$ and for all $e \in (I - P_-)\mathcal{H}$ we have $< Te, e >= 0$, then

$$(I - P_-)T(I - P_-) = (I - P_-)T = 0.$$

Since $\mathcal{L}$ is completely distributive, we have that $\wedge\{P_- : P \in \mathcal{J}_L\} = 0$. Hence $T = 0$. Since $\delta = \delta_T \neq 0$, we have a contradiction. Hence $\mathcal{A}$ has no nonzero compact derivation. ∎

Remark 3. Let $\mathcal{A}$ be as in Theorem 3. Since lat $\mathcal{A}^* = \{I - P : P \in lat\ \mathcal{A}\}$ and alg $\mathcal{L}^\perp = \mathcal{A}^*$, it follows that $\mathcal{A}^*$ is a CSL algebra and lat $\mathcal{A}^*$ is completely distributive. It is easy to prove that $H^1(\mathcal{A}^*, \mathcal{A}^*) = 0$, and $C(\mathcal{A}^*, K(\mathcal{A}^*)) \subseteq \mathcal{A}^{*\prime} + K(\mathcal{A}^*)$ if and only if $H^1(\mathcal{A}, A) = 0$ and $C(\mathcal{A}, K(\mathcal{A})) \subseteq \mathcal{A}' + K(\mathcal{A})$.

By the above Remark and Lemma 5, we can prove the following result.

Lemma 6. *Let $\mathcal{A}$ be as in Theorem 3. If dim $P\mathcal{H} = \infty$, for any $P \in \mathcal{J}_{\mathcal{L}^\perp}$, then $\mathcal{A}$ has no nonzero compact derivation.*

Theorem 7. *Let $\mathcal{A} = alg\ \mathcal{N}_1 \otimes \cdots \otimes alg\ \mathcal{N}_n$, where $\mathcal{N}_i$ is a nest on $\mathcal{H}_i$. Then the following statements are equivalent:*

(1) *$\mathcal{A}$ has a nonzero compact derivation.*
(2) *There exist an L in $\mathcal{N}_1 \otimes \cdots \otimes \mathcal{N}_n,\ L \neq 0$, with dim $L\mathcal{H} < \infty$ and an M in $\mathcal{N}_1 \otimes \cdots \otimes \mathcal{N}_n,\ M \neq I$ with dim $(I - M)\mathcal{H} < \infty$.*

Proof. By Lemmas 5 and 6, it is obvious that (1) implies (2).

Conversely, suppose that $\mathcal{A}$ has no nonzero compact derivation. Let $P \in \mathcal{N}_1 \otimes \cdots \otimes \mathcal{N}_n, P \neq 0$, and dim $P\mathcal{H} < \infty$. By Proposition 2.4 [4], since

$$P = \vee\{P_1 \otimes \cdots \otimes P_n \mid P_i \in \mathcal{N}_i,\ P_1 \otimes \cdots \otimes P_n \subseteq P\},$$

there exists $\tilde{P} = \tilde{P}_1 \otimes \cdots \otimes \tilde{P}_n \neq 0$ in $\mathcal{N}_1 \otimes \cdots \otimes \mathcal{N}_n$ such that dim $\tilde{P}\mathcal{H} < \infty$. Thus dim $\tilde{P}_i\mathcal{H} < \infty, i = 1, ..., n$.

Suppose that $Q \neq I$ and $Q \in \mathcal{N}_1 \otimes \cdots \otimes \mathcal{N}_n$ such that dim $(I - Q)\mathcal{H} < \infty$. Let $\mathcal{L} = N_1 \otimes \cdots \otimes \mathcal{N}_n$. By Theorem 2.6 [4],

$$\begin{aligned} alg\ \mathcal{L}^\perp &= (alg\ \mathcal{L})^* = (alg\ \mathcal{N}_1 \otimes \cdots \otimes alg\ \mathcal{N}_n)^* \\ &= (alg\ \mathcal{N}_1)^* \otimes \cdots \otimes (alg\ \mathcal{N}_n)^* = alg(\mathcal{N}_1^\perp \otimes \cdots \otimes \mathcal{N}_n^\perp). \end{aligned}$$

So $\mathcal{L}^\perp = \mathcal{N}_1^\perp \otimes \cdots \otimes \mathcal{N}_n^\perp$. Since $I - Q \in \mathcal{L}^\perp = \mathcal{N}_1^\perp \otimes \cdots \otimes \mathcal{N}_n^\perp$ and dim $(I-Q)\mathcal{H} < \infty$, similarly, there exists $\tilde{Q}_1 \otimes \cdots \otimes \tilde{Q}_n \neq 0$ in $\mathcal{N}_1^\perp \otimes \cdots \otimes \mathcal{N}_n^\perp$ with dim $\tilde{Q}_i\mathcal{H}_i < \infty$. Since $\tilde{P}_i\mathcal{L}(H)\tilde{Q}_i \subseteq alg\ \mathcal{N}_i$, it follows that $\tilde{P}\mathcal{L}(H_1 \otimes \cdots \otimes \mathcal{H}_n)\tilde{Q} \subseteq alg\ \mathcal{L}$. For any $K \in \tilde{P}\mathcal{L}(H_1 \otimes \cdots \otimes \mathcal{H}_n)\tilde{Q}$ and $K \neq 0$, let $\delta = \delta_K$. Since dim $\tilde{P}\mathcal{L}(H_1 \otimes \cdots \otimes \mathcal{H}_n)\tilde{Q} < \infty$ and for any $a \in alg\ \mathcal{L}, \delta(a) = \delta_K(a) \in \tilde{P}\mathcal{L}(H_1 \otimes \cdots \otimes \mathcal{H}_n)\tilde{Q}$, we have δ_K is compact. By the proof of Theorem 3.2 [4], we have that $\mathcal{A}' = \mathbb{C}I$. Hence $\delta_K \neq 0$. ∎

Lemma 8. *The following statements are equivalent*:

(1) $\mathcal{A}$ *is a nsva of a von Neumann algebra* $\mathcal{B}$ *such that* lat $\mathcal{A}$ *is completely distributive.*

(2) $\mathcal{A} = \sum_{i\in\Lambda} \oplus$ alg $\mathcal{N}_i$, *where* $\mathcal{N}_i$ *is a nest acting on* $\mathcal{H}_i$, *and* $\Lambda = \{1, ..., n\}$ *or all positive integers.*

Proof. Suppose that (1) is true. Since lat $\mathcal{B} \subseteq lat\ \mathcal{A}$, by the hypothesis, it follows that lat $\mathcal{B}$ is both commutative and completely distributive. So lat $\mathcal{B}$ is a complete Boolean algebra. By Tarski's Theorem [6, p.287], we have that lat $\mathcal{B}$ is totally atomic. Hence $\mathcal{B}'$ is an atomic abelian von Neumann algebra and $\mathcal{B}' = \sum_{i\in\Lambda} \oplus \mathbb{C}I_{H_i}$, where $\Lambda = \{1, ..., n\}$ or all positive integers. Thus $\mathcal{B} = \sum_{i\in\Lambda} \oplus \mathcal{L}(H_i)$. Let E_i be the projection onto $\mathcal{H}_i$ for each $i \in \Lambda$. Since $\mathcal{A} = B \cap (alg\ \mathcal{N})$, letting $\mathcal{N}_i = (E_i - E_{i-1})\mathcal{N}(E_i - E_{i-1})$, we have that $\mathcal{A} = \sum_{i\in\Lambda} \oplus alg\ \mathcal{N}_i$.

Conversely, let $\mathcal{B} = \sum_{i\in\Lambda} \oplus \mathcal{L}(H_i)$ and let $\mathcal{N}$ be the ordinal sum of the $\mathcal{N}_i$. This is the nest on $\mathcal{H} = \sum_{i\in\Lambda} \oplus \mathcal{H}_i$ consisting of 0 and I together with all projections of the form $I_{H_1} \oplus \cdots \oplus I_{H_{i-1}} \oplus P \oplus 0 \oplus 0 \oplus \cdots$, $P \in \mathcal{N}_i$ for $i \in \Lambda$. It follows that $\mathcal{A} = B \cap alg\ \mathcal{N}$ and lat $\mathcal{A}$ is both commutative and completely distributive. ∎

Theorem 9. *Let* $\mathcal{A} = \sum_{i\in\Lambda} \oplus alg\ \mathcal{N}_i$, *where* $\mathcal{N}_i$ *is a nest acting on* $\mathcal{H}_i, \Lambda = \{1, ..., n\}$ *or all positive integers. The following statements are equivalent*:

(1) $\mathcal{A}$ *has a nonzero compact derivation.*

(2) *There is a* $j \in \Lambda$ *such that alg* $\mathcal{N}_j$ *has a nonzero compact derivation.*

Proof. Suppose that (2) is true. Let δ_j be a nonzero compact derivation of $\mathcal{A}_j$. For any $a = \sum_{i\in\Lambda} \oplus a_i$, define $\delta(a) = \delta_j(a_j)$. Then δ is a nonzero compact derivation. Hence (2) implies (1).

Conversely, let δ be a nonzero compact derivation of $\mathcal{A}$. By Corollary 4 and Lemma 8, we have that $\delta = \delta_x$ with $x \in \mathcal{K}(\mathcal{A})$. Let $x = \sum\limits_{i\in\Lambda} \oplus x_i, x_i \in \mathcal{K}(alg\, \mathcal{N}_i)$. For any $a = \sum\limits_{i\in\Lambda} \oplus a_i \in \mathcal{A}$, it follows that

$$\delta(a) = xa - ax = \sum_{i\in\Lambda} \oplus(x_i a_i - a_i x_i). \tag{2.1}$$

Since δ is a nonzero compact derivation, by (2.1) we have that there exists a $j \in \Lambda$ such that alg $\mathcal{N}_j$ has a nonzero compact derivation. ∎

Theorem 10. *Let $\mathcal{A}$ be as in Theorem 9. Every compact derivation of $\mathcal{A}$ is the norm limit of finite-rank derivations of $\mathcal{A}$.*

Proof. Let δ be a nonzero compact derivation of $\mathcal{A}$. By Corollary 4 and Lemma 8, $\delta = \delta_x$ where $x = \sum\limits_{i\in\Lambda} \oplus x_i, x_i \in \mathcal{K}(alg\ \mathcal{N}_i)$. For any $a \in alg\ \mathcal{N}_i$, define $\delta^{(i)}(a) = x_i a - a x_i$. Then $\delta^{(i)} = 0$ or $\delta^{(i)}$ is a nonzero compact derivation of alg $\mathcal{N}_i$. From Theorem 10 [10], it follows that there exists a sequence $\{\delta_n^{(i)}\}$ such that every $\delta_n^{(i)}$ is a finite-rank derivation of alg $\mathcal{N}_i$ and $\delta_n^{(i)} \longrightarrow \delta^{(i)}$. When $\Lambda = \{1, ..., n\}$, let

$$\delta_m = \delta_m^{(1)} \oplus \cdots \oplus \delta_m^{(n)}.$$

When Λ is all positive integers, let

$$\delta_m = \delta_m^{(1)} \oplus \cdots \oplus \delta_m^{(m)} \oplus 0 \oplus 0 \oplus \cdots .$$

Then we have that δ_m is a finite-rank compact derivation of $\mathcal{A}$ and $\lim\limits_{m\to\infty} \delta_m = \delta$. ∎

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