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ON SLANT SURFACES

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Abstract. A slant immersion was introduced in [1] as an isometric immersion of a Riemannian manifold into an almost Hermitian manifold with constant Wirtinger angle. It is known that there exist ample examples of slant submanifolds; in particular, slant surfaces in complex-space-forms. In this paper, we establish a sharp inequality for slant surfaces and determine the Riemannian structures of special slant surfaces in complex-space-forms. By applying the special forms of the Riemannian structures on special slant surfaces we prove that proper slant surfaces in $\mathbf{C}^2$ are minimal if and only if they are special slant. We also determine proper slant surfaces in complex-space-forms which satisfy the equality case of the inequality identically.

1. Introduction

Let M be a Riemannian manifold and $\widetilde{M}$ an almost Hermitian manifold with almost complex structure J. An isometric immersion $f : M \to \widetilde{M}$ of M in $\widetilde{M}$ is called *holomorphic* if at each point $p \in M$ we have $J(T_pM) = T_pM$, where T_pM denotes the tangent space of M at p. The immersion is called *totally real* if $J(T_pM) \subset T_p^{\perp}M$ for each $p \in M$, where $T_p^{\perp}M$ is the normal space of M at p. A totally real immersion $f : M \to \widetilde{M}$ is called *Lagrangian* if $\dim_{\mathbf{R}} M = \dim_{\mathbf{C}} \widetilde{M}$.

Let $\widetilde{M}^m(4\epsilon)$ denote a Kählerian m-manifold with constant holomorphic sectional curvature 4ϵ and $f : M \to \widetilde{M}^m(4\epsilon)$ an isometric immersion. We denote by $\langle\ ,\ \rangle$ the inner product for M as well as for $\widetilde{M}^m(4\epsilon)$.

For any vector X tangent to M, we put

$$JX = PX + FX, \tag{1.1}$$

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where PX and FX denote the tangential and normal components of JX, respectively. For each nonzero vector X tangent to M at p, the angle $\theta(X)$ between JX and T_pM is called the *Wirtinger angle* of X. An immersion $f : M \to \widetilde{M}^m(4\epsilon)$ is called *slant* if the Wirtinger angle θ is a constant [1]. The Wirtinger angle θ of a slant immersion is called the *slant angle*. A slant submanifold with slant angle θ is said to be θ-*slant*. Holomorphic and totally real immersions are slant immersions with slant angle 0 and $\frac{\pi}{2}$, respectively. A slant immersion is called *proper slant* if it is neither holomorphic nor totally real. It is well-known that there exist ample examples of proper slant submanifolds in complex-space-forms (see $[1, 5-8]$).

In this paper we prove that the squared mean curvature H^2 and the Gauss curvature K of a proper slant surface M in $\widetilde{M}^2(4\epsilon)$ satisfy the inequality:

$$H^2(p) \geq 2K(p) - 2(1 + 3\cos^2\theta)\epsilon, \quad p \in M, \tag{1.2}$$

where θ is the slant angle of the slant surface. For each $\theta \in (0, \frac{\pi}{2})$, we show that there exist non-minimal slant surfaces in $\mathbf{C}^2$ satisfying the equality case of (1.2) at some points in M. In contrast, we prove that, except the totally geodesic ones, there do not exist proper slant surfaces in $\mathbf{C}^2$ which satisfy the equality case on some nonempty open subset of M. In this paper, we also determine the Riemannian structure of special slant surfaces in complex-space-forms. By applying the obtained special forms of the Riemannian structure we prove that proper slant surfaces in $\mathbf{C}^2$ are minimal if and only if they are special slant. Finally, we prove that there exist non-minimal special slant surfaces in complex hyperbolic plane $CH^2(-4\epsilon)$ which satisfy the equality case of (1.2) identically.

Several applications of the results of this paper are given in [3].

2. Basic Formulas

Let $f : M \to \widetilde{M}^m(4\epsilon)$ be an isometric immersion of a Riemannian n-manifold into $\widetilde{M}^m(4\epsilon)$. We denote by h and A the second fundamental form and the shape operator of f and by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M and $\widetilde{M}^m(4\epsilon)$, respectively. The Gauss and Weingarten formulas of M in $\widetilde{M}$ are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2.2}$$

where X, Y are vector fields tangent to M and ξ is normal to M. The second fundamental form h and the shape operator A are related by

$$\langle A_\xi X, Y\rangle = \langle h(X, Y), \xi\rangle\,. \tag{2.3}$$

The mean curvature vector $\vec{H}$ of the immersion is defined by $\vec{H} = (1/n)$ trace h, where $\{e_1, \dots, e_n\}$ is a local orthonormal frame field of the tangent bundle TM.

Denote by R the Riemann curvature tensor of M and by R^D the curvature tensor of the normal connection D. Then the *equation of Gauss* and the *equation of Ricci* are given respectively by

$$\tilde{R}(X,Y;Z,W) = R(X,Y;Z,W) + \langle h(X,Z), h(Y,W)\rangle - \langle h(X,W), h(Y,Z)\rangle, \tag{2.4}$$

$$R^D(X,Y;\xi,\eta) = \tilde{R}(X,Y;\xi,\eta) + \langle [A_\xi, A_\eta](X), Y\rangle \tag{2.5}$$

for vectors X, Y, Z, W tangent to M and ξ, η normal to M.

For the second fundamental form h, we define the covariant derivative $\bar{\nabla} h$ of h with respect to the connection on $TM \oplus T^\perp M$ by

$$(\bar{\nabla}_X h)(Y,Z) = D_X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.6}$$

The *equation of Codazzi* is given by

$$(\tilde{R}(X,Y)Z)^\perp = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z), \tag{2.7}$$

where $(\tilde{R}(X,Y)Z)^\perp$ denotes the normal component of $\tilde{R}(X,Y)Z$.

For an endomorphism Q on the tangent bundle of the submanifold, we define ∇Q by

$$(\nabla_X Q)Y = \nabla_X(QY) - Q(\nabla_X Y). \tag{2.8}$$

For any vector field ξ normal to the submanifold M in $\widetilde{M}^n(4\epsilon)$, we put

$$J\xi = t\xi + f\xi, \tag{2.9}$$

where $t\xi$ and $f\xi$ are the tangential and the normal components of $J\xi$, respectively.

Suppose M is θ-slant in $\widetilde{M}^n(4\epsilon)$, then we have [1]

$$P^2 = -(\cos^2\theta)I, \quad \langle PX, Y\rangle + \langle X, PY\rangle = 0, \tag{2.10}$$

$$(\nabla_X P)Y = th(X,Y) + A_{FY}X, \tag{2.11}$$

$$D_X(FY) - F(\nabla_X Y) = fh(X,Y) - h(X,PY), \tag{2.12}$$

where I is the identity map. For simplicity, for each $X \in TM$, we put

$$X^* = (\csc\theta)FX. \tag{2.13}$$

We define a symmetric bilinear TM-valued form α on M by

$$\alpha(X,Y)=th(X,Y). \tag{2.14}$$

(1.1) and (2.13) imply

$$J\alpha(X,Y)=P\alpha(X,Y)+(\sin\theta)\alpha^*(X,Y). \tag{2.15}$$

Also (2.14) implies

$$Jh(X,Y)=\alpha(X,Y)+\beta^*(X,Y), \tag{2.16}$$

where β is also a symmetric bilinear TM-valued form on M. From (2.13), (2.15) and (2.16), we have

$$-h(X,Y)=P\alpha(X,Y)+(\sin\theta)\alpha^*(X,Y)-(\sin\theta)\beta(X,Y)-P\beta(X,Y)^*.$$

Thus $\beta(X,Y)=(\csc\theta)P\alpha(X,Y)$ and $h(X,Y)=-(\csc\theta)\alpha^*(X,Y)$. Consequently, the second fundamental form satisfies

$$h(X,Y)=(\csc^2\theta)\,(P\alpha(X,Y)-J\alpha(X,Y)). \tag{2.17}$$

For an n-dimensional θ-slant submanifold in $\widetilde{M}^n(4\epsilon)$ with $\theta\neq 0$, the equations of Gauss and Codazzi in $\widetilde{M}^n(4\epsilon)$ become

$$\begin{aligned}R(X,Y;Z,W)=\ &(\csc^2\theta)\{\langle\alpha(X,W),\alpha(Y,Z)\rangle-\langle\alpha(X,Z),\alpha(Y,W)\rangle\}\\&+\epsilon\{\langle X,W\rangle\langle Y,Z\rangle-\langle X,Z\rangle\langle Y,W\rangle+\langle PX,W\rangle\langle PY,Z\rangle\\&-\langle PX,Z\rangle\langle PY,W\rangle+2\langle X,PY\rangle\langle PZ,W\rangle\},\end{aligned} \tag{2.18}$$

$$\begin{aligned}&(\nabla_X\alpha)(Y,Z)+(\csc^2\theta)\,\{P\alpha(X,\alpha(Y,Z))+\alpha(X,P\alpha(Y,Z))\}\\&\qquad+(\sin^2\theta)c\{\langle X,PY\rangle Z+\langle X,PZ\rangle Y\}\\&=(\nabla_Y\alpha)(X,Z)+(\csc^2\theta)\,\{P\alpha(Y,\alpha(X,Z))+\alpha(Y,P\alpha(X,Z))\}\\&\qquad+(\sin^2\theta)c\{\langle Y,PX\rangle Z+\langle Y,PZ\rangle X\}.\end{aligned} \tag{2.19}$$

We need the following Existence Theorem from [6].

Existence Theorem. *Let c and θ be two constants with $0<\theta\leq\frac{\pi}{2}$ and M a simply-connected Riemannian n-manifold with inner product $\langle\ ,\ \rangle$. Suppose there exist an endomorphism P of the tangent bundle TM and a symmetric bilinear TM-valued form α on M such that for $X,Y,Z,W\in TM$, we have*

$$P^2=-(\cos^2\theta)I, \tag{2.20}$$

$$\langle PX,Y\rangle+\langle X,PY\rangle=0, \tag{2.21}$$

(2.22) $$\langle(\nabla_X P)Y, Z\rangle = \langle\alpha(X,Y), Z\rangle - \langle\alpha(X,Z), Y\rangle\,,$$

$$\begin{aligned}R(X,Y;Z,W) = &(\csc^2\theta)\,\{\langle\alpha(X,W),\alpha(Y,Z)\rangle - \langle\alpha(X,Z),\alpha(Y,W)\rangle\}\\ &+\epsilon\{\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle + \langle PX,W\rangle\langle PY,Z\rangle\\ &-\langle PX,Z\rangle\langle PY,W\rangle + 2\langle X,PY\rangle\langle PZ,W\rangle\},\end{aligned}$$

(2.23)
and

(2.24) $$\begin{aligned}(\nabla_X\alpha)(Y,Z) + &(\csc^2\theta)\,\{P\alpha(X,\alpha(Y,Z)) + \alpha(X,P\alpha(Y,Z))\}\\ &+(\sin^2\theta)\epsilon\{\langle X,PZ\rangle Y + \langle X,PY\rangle Z\}\end{aligned}$$

is totally symmetric. Then there exists a θ-slant isometric immersion from M into a complete simply-connected complex-space-form $\widetilde{M}^n(4\epsilon)$ whose second fundamental form h is given by

(2.25) $$h(X,Y) = \csc^2\theta(P\alpha(X,Y) - J\alpha(X,Y)).$$

Let M be a proper θ-slant surface in a Kählerian surface $\widetilde{M}^2$. Let e_1 be a unit vector tangent to M. We choose a canonical orthonormal basis $\{e_1, e_2, e_3, e_4\}$ defined by

(2.26) $$e_2 = (\sec\theta)Pe_1, \quad e_3 = (\csc\theta)Fe_1, \quad e_4 = (\csc\theta)Fe_2.$$

We call such an orthonormal basis an *adapted orthonormal basis*.

3. A Basic Inequality for Slant

First we give the following.

Theorem 1. *Let M be a proper slant surface in a complex-space-form $\widetilde{M}^2(4\epsilon)$. Then the squared mean curvature and the Gauss curvature of M satisfy*

(3.1) $$H^2(p) \geq 2K(p) - 2(1 + 3\cos^2\theta)\epsilon$$

at each point $p \in M$, where θ is the slant angle of the slant surface.

The equality sign of (3.1) holds at a point $p \in M$ if and only if, with respect to a suitable adapted orthonormal basis $\{e_1, e_2, e_3, e_4\}$ at p, the shape operators of M at p take the following form:

(3.2) $$A_{e_3} = \begin{pmatrix} 3\lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}.$$

Proof. Let M be a proper slant surface in a complex-space-form $\widetilde{M}^2(4\epsilon)$ with slant angle θ. Then, according to Proposition 3.3 of [1], M is Kählerian slant, i.e., M satisfies $\nabla P = 0$ identically. Hence, by (2.11), we have

$$\langle A_{FX}Y, Z\rangle = \langle A_{FY}X, Z\rangle \tag{3.3}$$

for any vectors X, Y, Z tangent to M.

Let e_1 be a unit tangent vector of M. We put

$$e_2 = (\sec\alpha)Pe_1, \quad e_3 = (\csc\alpha)Fe_1, \quad e_4 = (\csc\alpha)Fe_2.$$

Then, with respect to the adapted orthonormal frame $\{e_1, e_2, e_3, e_4\}$, we obtain from (3.3) that

$$A_{e_3} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} b & c \\ c & d \end{pmatrix}. \tag{3.4}$$

From (2.18) and (3.4) we obtain

$$4H^2 = (a+c)^2 + (b+d)^2, \quad K = ac - b^2 + bd - c^2 + (1 + 3\cos^2\theta)\epsilon.$$

Thus, we get

$$4H^2(p) - 8K(p) + 8(1 + 3\cos^2\theta)\epsilon = (a - 3c)^2 + (3b - d)^2 \geq 0, \tag{3.5}$$

which implies (3.1). From (3.5) we know that the equality case of (3.1) holds at a point $p \in M$ if and only if $a = 3c$ and $d = 3b$ at p. Therefore, if we choose e_1 in such a way that Fe_1 is parallel to the mean curvature vector $\vec{H}$, then the shape operators at p take the form (3.2).

Conversely, by applying (2.18), it is easy to verify that (3.2) implies the equality case of (3.1). ■

The following result shows that inequality (3.1) is sharp for each $\theta \in (0, \frac{\pi}{2})$.

Proposition 2. *For each $\theta \in (0, \frac{\pi}{2})$, there exists a non-totally geodesic θ-slant surface M in $\mathbf{C}^2$ which satisfies the equality sign of* (3.1) *at some points in M.*

Proof. Let $\phi = \phi(x)$ be a function defined on an open interval containing 0 such that $\phi(0) = 3b \neq 0$.

Consider the following system of first order ordinary differential equations:

$$\begin{aligned} y_1'(x) &= -3y_1y_3 + (\csc\theta\cot\theta)(y_2 + \phi)y_2, \\ y_2'(x) &= (\phi - 2y_2)y_3 - (\csc\theta\cot\theta)(y_2 + \phi)y_1, \\ y_3'(x) &= -y_3^2 + (\csc^2\theta)(2y_1^2 + y_2^2 - \phi y_2), \end{aligned} \tag{3.6}$$

with the initial conditions

$$y_1(0) = 0, \quad y_2(0) = b, \quad y_3(0) = c, \tag{3.7}$$

where c is a real number. It is well-known that the system (3.6) with the initial condition (3.7) has a unique solution: $y_1 = \phi_1(x), y_2 = \phi_2(x), y_3 = \phi_3(x)$ on some open interval containing 0.

Put

$$f(x) = \exp\left(\int^x \phi_3(x)dx\right). \tag{3.8}$$

Let M be a simply-connected open neighborhood of the origin $(0,0) \in E^2$ endowed with the warped metric tensor:

$$g = dx \otimes dx + f^2(x)dy \otimes dy. \tag{3.9}$$

Put $e_1 = \frac{\partial}{\partial x}, e_2 = \frac{1}{f}\frac{\partial}{\partial y}$. Then $\{e_1, e_2\}$ is an orthonormal frame field of TM such that

$$\nabla_{e_1}e_1 = \nabla_{e_1}e_2 = 0, \quad \nabla_{e_2}e_1 = \phi_3 e_2, \quad \nabla_{e_2}e_2 = -\phi_3 e_1.$$

We define a symmetric bilinear TM-valued form α on M by

$$\begin{aligned} &\alpha(e_1, e_1) = \phi e_1 + \phi_1 e_2,\ \alpha(e_1, e_2) = \phi_1 e_1 + \phi_2 e_2, \\ &\alpha(e_2, e_2) = \phi_2 e_1 - \phi_1 e_2. \end{aligned} \tag{3.10}$$

The oriented Riemannian 2-manifold (M, g) admits a canonical Kählerian structure $J = (\sec\theta)P$. By a direct long computation, we can prove that (M, g, P, α) satisfies the conditions of the Existence Theorem with $\epsilon = 0$. Thus, by applying the Existence Theorem, we know that there exists a θ-slant isometric immersion of M into $\mathbf{C}^2$ whose second fundamental form is given by $h = P\alpha - J\alpha$, where α is defined by (3.10) and $P = (\cos\theta)J$.

From the initial condition (3.7), it follows that the shape operators of M take the form of (3.2) at the point $p = (0,0)$. Thus, the slant surface satisfies the equality case of (3.1) at p. Clearly, the slant surface so obtained is a non-totally geodesic one. ■

4. Minimal and Special Slant Surfaces

A slant surface M in a Kählerian surface $\widetilde{M}^2$ is called *special slant* if, with respect to some suitable adapted orthonormal frame $\{e_1, e_2, e_3, e_4\}$, the shape operators of M take the following special form:

$$A_{e_3} = \begin{pmatrix} c\lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} \tag{4.1}$$

for some constant c.

Proposition 3. *Every proper slant minimal surface in a Kählerian surface is special slant which satisfies* (4.1) *with* $c = -1$.

Proof. Let M be a proper slant minimal surface in a Kählerian surface. Let p be a non-totally geodesic point in M. We define a function γ_p by

$$\gamma_p : UM_p \to \mathbf{R} : v \mapsto \gamma_p(v) = \langle h(v,v), Jv\rangle\,, \tag{4.2}$$

where $UM_p = \{v \in T_pM : \langle v, v\rangle = 1\}$. Since UM_p is a compact set, there exists a vector v in UM_p such that γ_p attains its absolute minimum at v. Since p is a non-totally geodesic point, it follows from (3.3) that $\gamma_p \neq 0$. By linearity, we have $\gamma_p(v) < 0$. Because γ_p attains an absolute minimum at v, it follows from (3.3) that $\langle h(v,v), Jw\rangle = 0$ for all w orthogonal to v. So, using (3.3), v is an eigenvector of the symmetric operator A_{Jv}. By choosing an orthonormal basis $\{e_1, e_2\}$ of T_pM with $e_1 = v$, we obtain

$$h(e_1, e_1) = -\lambda J e_1, \;\; h(e_1, e_2) = \lambda J e_2, \;\; h(e_2, e_2) = \lambda J e_1$$

for some real number λ. This gives (4.1) with $c = -1$.

If p is a totally geodesic point, (4.1) holds trivially. ■

Lemma 4. *Let* M *be a proper slant surface in a complex-space-form* $\widetilde{M}^2(4\epsilon)$ *with slant angle* θ. *If* M *is special slant such that, with respect to some suitable adapted orthonormal frame* $\{e_1, e_2, e_3, e_4\}$, *the shape operators satisfy* (4.1) *for some constant* $c \neq -1$, *then we have*

$$e_1\lambda = (2-c)\lambda\omega_2^1(e_2), \tag{4.3}$$

$$e_2\lambda = -\lambda\omega_2^1(e_1) + \left(\frac{3}{1+c}\right)\epsilon\sin 2\theta, \tag{4.4}$$

$$\lambda\omega_2^1(e_1) = -\left(\frac{1+c}{2}\right)\lambda^2\cot\theta + \frac{3(c-1)}{4(1+c)}\,\epsilon\sin 2\theta, \tag{4.5}$$

where $\omega_2^1 = -\omega_1^2$ *are the connection forms defined by*

$$\nabla_X e_1 = \omega_1^2(X)e_2, \quad \nabla_X e_2 = \omega_2^1(X)e_1. \tag{4.6}$$

In particular, if $c \neq -1, 2$, *then the metric tensor on* M *is given by*

$$g = \left(\frac{k(x)}{\lambda}e^W\right)^2 dx^2 + \left(\phi(y)\lambda^{1/(c-2)}\right)^2 dy^2 \tag{4.7}$$

for some nonzero functions $k = k(x)$ and $\phi = \phi(y)$, where

$$W = W(x,y) = \left(\frac{3\epsilon}{c+1}\right)\sin 2\theta \int^{y} \phi(y)\lambda^{(3-c)/(c-2)}dy. \tag{4.8}$$

Proof. Let M be a proper θ-slant surface in a complex-space-form $\widetilde{M}^2(4\epsilon)$. If, with respect to some suitable adapted orthonormal frame $\{e_1, e_2, e_3, e_4\}$, the shape operators are given by (4.1), then we have

$$h(e_1,e_1) = c\lambda e_3, \quad h(e_1,e_2) = \lambda e_4, \quad h(e_2,e_2) = \lambda e_3. \tag{4.9}$$

Put

$$D_X e_3 = \omega_3^4(X)e_4, \quad D_X e_4 = \omega_3^4(X)e_3. \tag{4.10}$$

From Lemma 4.1 of [1, p.29] we have

$$\omega_3^4 = \omega_1^2 - \cot\theta\{(\operatorname{trace} h^3)\omega^1 + (\operatorname{trace} h^4)\omega^2\}, \tag{4.11}$$

where $\{\omega^1, \omega^2\}$ is the dual basis of $\{e_1, e_2\}$ and $h = h^3 e_3 + h^4 e_4$.

From (2.6), (4.9) and (4.11) we obtain

$$\begin{aligned}
&(\bar\nabla_{e_2} h)(e_1,e_1) = c(e_2\lambda)e_3 + (c-2)\lambda\omega_1^2(e_2)e_4,\\
&(\bar\nabla_{e_1} h)(e_2,e_1) = (e_1\lambda)e_4 + (c+1)\lambda^2\cot\theta e_3 + (2-c)\lambda\omega_2^1(e_1)e_3,\\
&(\bar\nabla_{e_1} h)(e_2,e_2) = (e_1\lambda)e_3 - (c+1)\lambda^2\cot\theta e_4 + 3\lambda\omega_1^2(e_1)e_4,\\
&(\bar\nabla_{e_2} h)(e_1,e_2) = (e_2\lambda)e_4 + \lambda\omega_2^1(e_2)e_3 + (1-c)\lambda\omega_2^1(e_2)e_3.
\end{aligned} \tag{4.12}$$

Because $\widetilde{M}^2(4\epsilon)$ is a complex-space-form with constant holomorphic sectional curvature 4ϵ, the Riemann curvature tensor $\tilde R$ of $\widetilde{M}^2(4\epsilon)$ satisfies

$$\begin{aligned}
\widetilde R(\widetilde X,\widetilde Y)\widetilde Z = c(&\langle \widetilde Y,\widetilde Z\rangle \widetilde X - \langle \widetilde X,\widetilde Z\rangle \widetilde Y + \langle J\widetilde Y,\widetilde Z\rangle J\widetilde X\\
&- \langle J\widetilde X,\widetilde Z\rangle J\widetilde Y + 2\langle \widetilde X, J\widetilde Y\rangle J\widetilde Z).
\end{aligned} \tag{4.13}$$

From (4.13), we find

$$\begin{aligned}
(\tilde R(e_2,e_1)e_1)^{\perp} &= 3\epsilon\sin\theta\cos\theta e_3,\\
(\tilde R(e_1,e_2)e_2)^{\perp} &= -3\epsilon\sin\theta\cos\theta e_4.
\end{aligned} \tag{4.14}$$

Substituting (4.12) and (4.14) into equation (2.7) of Codazzi gives rise to (4.3)-(4.5) whenever $c \neq -1$.

Since Span $\{e_1\}$ and Span $\{e_2\}$ are one-dimensional distributions, there exists a local coordinate system $\{x, y\}$ on M such that $\partial/\partial x$ and $\partial/\partial y$ are parallel to e_1, e_2, respectively. Thus, the metric tensor g on M takes the following form:

$$g = E^2 dx^2 + G^2 dy^2, \tag{4.15}$$

where E and G are positive functions of x, y. Without loss of generality, we may assume

$$e_1 = \frac{1}{E}\frac{\partial}{\partial x}, \quad e_2 = \frac{1}{G}\frac{\partial}{\partial y}. \tag{4.16}$$

From (4.16) we find

$$\omega_2^1(e_1)e_1 = \nabla_{e_1} e_2 = \frac{E_y}{E^2 G}\frac{\partial}{\partial x}, \quad E_y = \frac{\partial E}{\partial y}. \tag{4.17}$$

Using (4.4), (4.16) and (4.17), we get

$$\frac{E_y}{E} + \frac{\lambda_y}{\lambda} = \left(\frac{3\epsilon \sin 2\theta}{1+c}\right)\frac{G}{\lambda}, \tag{4.18}$$

from which we obtain

$$E = \frac{k(x)}{\lambda} e^W, \tag{4.19}$$

where $W = W(x, y)$ is given by

$$W = \left(\frac{3\epsilon}{1+c}\right)\sin 2\theta \int^y \frac{G}{\lambda} dy. \tag{4.20}$$

Similarly, we have

$$\omega_1^2(e_2)e_2 = \frac{G_x}{EG^2}\frac{\partial}{\partial y}, \quad G_x = \frac{\partial G}{\partial x}. \tag{4.21}$$

If $c \neq 2$, (4.3) and (4.21) imply

$$\frac{G_x}{G} = \left(\frac{1}{2-c}\right)\frac{\lambda_x}{\lambda}. \tag{4.22}$$

Therefore

$$G = \phi(y)\lambda^{1/(c-2)} \tag{4.23}$$

for some function $\phi = \phi(y)$. Combining (4.19), (4.20) and (4.23), we obtain (4.7)-(4.8). This completes the proof of Lemma 4. ∎

Lemma 5. *If M is a proper slant surface in a complex-space-form $\widetilde{M}^2(4\epsilon)$ such that, with respect to some suitable adapted orthonormal frame $\{e_1, e_2, e_3, e_4\}$, the shape operators take the form (4.1) with $c = 2$, then, with respect to the coordinate system $\{x, y\}$ with $\partial/\partial x = Ee_1, \partial/\partial y = Ge_2$, we have*

$$\lambda = \lambda(y), \quad e_2\lambda = \frac{3}{2}\lambda^2 \cot\theta + \frac{3}{4}\epsilon \sin 2\theta, \tag{4.24}$$

$$\lambda\omega_2^1(e_1) = -\frac{3}{2}\lambda^2 \cot\theta + \frac{1}{4}\,\epsilon \sin 2\theta, \tag{4.25}$$

where θ is the slant angle. Moreover, the metric tensor on M is given by

$$g = \left(\frac{f(x)}{\lambda(y)}\, e^{Z(y)}\right)^2 dx^2 + \left(\frac{4\lambda'}{6\lambda^2 \cot\theta + 3\epsilon \sin 2\theta}\right)^2 dy^2 \tag{4.26}$$

for some function $f = f(x)$, where

$$Z(y) = \epsilon \sin 2\theta \int^y \frac{4\lambda'}{6\lambda^3 \cot\theta + 3\epsilon\lambda \sin 2\theta}\, dy. \tag{4.27}$$

Proof. Under the hypothesis, we have

$$h(e_1, e_1) = 2\lambda e_3, \quad h(e_1, e_2) = \lambda e_4, \quad h(e_2, e_2) = \lambda e_3. \tag{4.28}$$

Applying (4.28) and the equation of Codazzi, we obtain (4.24) and (4.25).
Using (4.24) we find

$$G = G(y) = \frac{4\lambda'}{6\lambda^2 \cot\theta + 3\epsilon \sin 2\theta}. \tag{4.29}$$

Applying (4.17), (4.25) and (4.29) we obtain

$$E = \frac{f(x)}{\lambda(y)} e^{Z(y)}, \tag{4.30}$$

where $Z = Z(y)$ is given by (4.27). ∎

The following theorem determines completely special slant surfaces in $\mathbf{C}^2$.

Theorem 6. *A proper slant surface M in the complex Euclidean plane $\mathbf{C}^2$ is special slant if and only if it is a slant minimal surface.*

Proof. Let M be a special slant surface with slant angle θ whose shape operators satisfy (4.1) for some constant $c \neq -1$. We divide the proof into two cases.

Case (i): $c \neq 2$. In this case, Lemma 4 implies that the metric tensor of M takes the following form:

$$g = \left(\frac{k(x)}{\lambda}\right)^2 dx^2 + \left(\phi(y)\lambda^{1/(c-2)}\right)^2 dy^2 \tag{4.31}$$

for some functions $k = k(x)$ and $\phi = \phi(y)$.

It is well-known that the Gauss curvature K of a surface with metric tensor $g = E^2dx^2 + G^2dy^2$ is given by

$$K = -\frac{1}{EG}\left\{\frac{\partial}{\partial y}\left(\frac{E_y}{G}\right) + \frac{\partial}{\partial x}\left(\frac{G_x}{E}\right)\right\}. \tag{4.32}$$

Applying (4.31), (4.32) and a direct computation, we find

$$(2-c)k\lambda^{(3-c)/(c-2)}K = \frac{\partial}{\partial x}\left(\frac{\lambda^{1/(c-2)}\lambda_x}{k}\right). \tag{4.33}$$

On the other hand, from (4.4), (4.5) and (4.31), we obtain

$$\lambda_y = \left(\frac{1+c}{2}\right)(\cot\theta)\phi(y)\lambda^{(2c-3)/(c-2)}. \tag{4.34}$$

Integrating (4.34) with respect to y yields

$$\lambda^{(1-c)/(c-2)} = \left(\frac{1-c^2}{2c-4}\right)\cot\theta\int^y \phi(y)dy + F(x) \tag{4.35}$$

for some function $F = F(x)$. Applying (4.33) and (4.35) we conclude that λ satisfies the following equation:

$$2\left(\frac{F'(x)}{k(x)}\right)^2\lambda^{2(c-1)/(c-2)} - \left(\frac{F'(x)}{k(x)}\right)'\lambda^{(c-1)/(c-2)} + (1-c)^2k(x) = 0.$$

Therefore, $\lambda^{(c-1)/(c-2)}$ is a function of x only. Hence, $\lambda_y = 0$. Consequently, (4.34) yields $\cot\theta = 0$ which is a contradiction.

Case (ii): $c = 2$**.** In this case, since $\epsilon = 0$, Lemma 5 implies that the metric tensor on the slant surface is given by

$$g = \left(\frac{f(x)}{\lambda(y)}\right)^2 dx^2 + \left(\frac{2\lambda'(y)}{3\lambda^2(y)\cot\theta}\right)^2 dy^2. \tag{4.36}$$

(4.32), (4.36) and a direct computation yield $K = 0$. On the other hand, since $\epsilon = 0$ and $c = 2$, the assumption on special slantness yields $\lambda^2 = K = 0$. Hence, M must be totally geodesic in this case.

The converse follows from Proposition 3. ∎

The following proposition shows that Theorem 6 is false if $\mathbf{C}^2$ were replaced by a non-flat complex-space-form $\widetilde{M}^2(4\epsilon)$.

Proposition 7. *For any $\theta \in (0, \frac{\pi}{2})$, there exists a non-minimal special slant surface with slant angle θ and with constant Gauss curvature $-4\cos^2\theta < 0$ in the complex hyperbolic plane $CH^2(-4)$.*

Proof. Let M be a simply-connected open subset of the half-plane of E^2 endowed with metric tensor

$$g = y^2 dx^2 + \frac{\sec\theta^2}{4y^2} dy^2. \tag{4.37}$$

Then the Gauss curvature of M is constant given by $-4\cos^2\theta < 0$ by applying (4.32) and (4.37).

We put

$$\lambda = \sin\theta, \quad e_1 = \frac{1}{y}\frac{\partial}{\partial x}, \quad e_1 = 2y\cos\theta\frac{\partial}{\partial y} \tag{4.38}$$

and let P denote the endomorphism of the tangent bundle TM defined by

$$Pe_1 = (\cos\theta)e_2, \quad Pe_2 = -(\cos\theta)e_1.$$

Define a symmetric bilinear form α on M by

$$\begin{aligned} &\alpha(e_1, e_1) = -2\sin^2\theta e_1, \quad \alpha(e_1, e_2) = -\sin^2\theta e_2, \\ &\alpha(e_2, e-2) = -\sin^2\theta e_1. \end{aligned} \tag{4.39}$$

Then, by a direct long computation, we can verify that (M, g, P, α) satisfies the conditions (2.20)–(2.24) of the Existence Theorem for $\epsilon = -1$. Therefore, by applying the Existence Theorem, we know that there exists a θ-slant isometric immersion from (M, g) into the complex hyperbolic plane $CH^2(-4)$. Using (4.39), we conclude that the slant immersion is special slant with $c = 2$. ■

5. A Further Result

Although there exist proper slant surfaces in $\mathbf{C}^2$ which satisfy the equality sign of (3.1) at some points, the following result shows that the equality sign of (3.1) cannot hold identically on any nonempty open subset of a proper slant surface in $\mathbf{C}^2$ except the totally geodesic one.

Theorem 8. *Let M be a proper slant surface in complex-space-form $\widetilde{M}^2(4\epsilon)$ which satisfies the equality sign of* (3.1) *identically. Then either*

(1) *M is a totally geodesic slant surface in a flat Kählerian surface ($\epsilon = 0$) or*

(2) *$\epsilon < 0$, M has constant Gauss curvature $K = (2/3)\epsilon$, and M is a slant surface with slant angle $\theta = \cos^{-1}(1/3)$.*

Proof. We divide the proof into two cases.

Case (1). If M is a non-totally geodesic proper slant surface in a flat Kählerian surface which satisfies the equality sign of (3.1) identically on a nonempty subset U of M, then U is a special slant surface satisfying (4.1) with $c = 3$ according to Theorem 1. Thus, by applying Theorem 6, U is minimal which is impossible unless $\lambda = 0$ identically on U. This implies that U is totally geodesic which is a contradiction.

Case (2). Assume M is a proper slant surface in a non-flat complex-space-form satisfying the equality sign of (3.1) identically on a nonempty subset U of M. Then, U is non-totally geodesic according to the well-known classification theorem of totally geodesic submanifolds of non-flat complex-space-forms. Thus, according to Theorem 1, U is a special slant surface satisfying (4.1) with $c = 3$ and $\lambda \neq 0$. Hence, by Lemma 4, the metric tensor g on U is given by

$$g = \left(\frac{k(x)}{\lambda} e^{W}\right)^2 dx^2 + (\phi(y)\lambda)^2\, dy^2 \tag{5.1}$$

for some nonzero functions $k = k(x)$ and $\phi = \phi(y)$, where

$$W = W(y) = \left(\frac{3\epsilon}{4}\right) \sin 2\theta \int^y \phi(y) dy. \tag{5.2}$$

From Lemma 4, we also have

$$e_1\lambda = -\lambda\omega_2^1(e_2), \tag{5.3}$$

$$e_2\lambda = -\lambda\omega_2^1(e_1) + \frac{3}{4}\,\epsilon \sin 2\theta, \tag{5.4}$$

$$\lambda\omega_2^1(e_1) = -2\lambda^2 \cot\theta + \frac{3}{8}\,\epsilon \sin 2\theta, \tag{5.5}$$

where

$$e_1 = \frac{\lambda}{k} e^{-W} \frac{\partial}{\partial x}, \quad e_2 = \frac{1}{\lambda\phi}\frac{\partial}{\partial y}. \tag{5.6}$$

Using (5.4), (5.5) and (5.6), we obtain

$$\lambda_y - \frac{3}{8}(\phi\epsilon \sin 2\theta)\lambda = 2(\phi \cot\theta)\lambda^3. \tag{5.7}$$

Solving differential equation (5.7) yields

$$\lambda^{-2} = e^{-W}\,(Z(y) + F(x)) \tag{5.8}$$

for some function $F = F(x)$, where

$$Z(y) = -4\cot\theta \int^y \phi(y) e^{W(y)} dy. \tag{5.9}$$

From (4.32) and (5.1), we know that the Gauss curvature of U is given by

$$K = 6\epsilon\cos^2\theta + e^{-2W} k^{-3}\left(\lambda\lambda_x k'(x) - k(x)\lambda_x^2 - k(x)\lambda\lambda_{xx}\right). \tag{5.10}$$

Combining (5.10) with equation (2.23) of Gauss yields

$$\lambda\lambda_x k' - k\lambda_x^2 - k\lambda\lambda_{xx} = k^3(x)e^{2W(y)}\left(2\lambda^2 + \epsilon - 3\epsilon\cos^2\theta\right). \tag{5.11}$$

Differentiating (5.8) with respect to x yields

$$\begin{aligned} \lambda_x &= -\frac{1}{2}e^{-W}\lambda^3 F'(x), \\ \lambda_{xx} &= \frac{3}{4}e^{-2W}\lambda^5 F'^2(x) - \frac{1}{2}e^{-W}\lambda^3 F''(x). \end{aligned} \tag{5.12}$$

Combining (5.8), (5.11) and (5.12) gives

$$\begin{aligned} &(Z(y) + F(x))\left(\frac{k(x)F''(x) - k'(x)F'(x)}{k^3(x)}\right) - 2\left(\frac{F'(x)}{k(x)}\right)^2 \\ &= 4e^{2W}(Z+F)^2 + 2e^W\epsilon(Z+F)^3(1 - 3\cos^2\theta). \end{aligned} \tag{5.13}$$

Taking the partial derivative of (5.13) with respect to y yields

$$\begin{aligned} \frac{kF'' - k'F'}{k^3} = &-3\epsilon\sin^2\theta e^W(Z+F)^2 + 8e^{2W}(Z+F) \\ &-\frac{3}{4}\epsilon^2\sin^2\theta(1 - 3\cos^2\theta)(Z+F)^3 \\ &+6\epsilon e^W(1 - 3\cos^2\theta)(Z+F)^2. \end{aligned} \tag{5.14}$$

Differentiating (5.14) with respect to y yields

$$\begin{aligned} 0 = &\frac{9}{4}\epsilon^2\sin 2\theta(3 - 11\cos^2\theta)(Z+F)^2 + 24\epsilon e^W(\sin 2\theta)(Z+F) \\ &-32e^{2W}\cot\theta - 48\epsilon e^W(1 - 3\cos^2\theta)\cot\theta. \end{aligned} \tag{5.15}$$

By taking partial derivative of (5.15) with respect to y we find

$$\begin{aligned} &\epsilon\{\sin 2\theta - \cot\theta(3 - 11\cos^2\theta)\}(Z+F) \\ &= \{8e^W + 2\epsilon(1 - 3\cos^2\theta)\}\cot\theta, \end{aligned} \tag{5.16}$$

which implies that either $F = F(x)$ is a constant or

$$\sin 2\theta = \cot\theta(3 - 11\cos^2\theta). \tag{5.17}$$

If (5.17) holds, then (5.16) implies that $W = W(y)$ is constant. Hence, $\phi(y) = 0$ by virtue of (5.2). This is impossible. Thus, $F = F(x)$ is constant. Hence, by using (5.12), we get $\lambda_x = \lambda_{xx} = 0$. Therefore, by (5.11), λ is a constant satisfying

$$2\lambda^2 = 3\epsilon\cos^2\theta - \epsilon. \tag{5.18}$$

On the other hand, since λ is constant, (5.7) yields

$$\lambda^2 = -\frac{3}{8}\epsilon\sin^2\theta. \tag{5.19}$$

Combining (5.18) and (5.19), we obtain

$$\cos^2\theta = \frac{1}{9}, \quad \sin^2\theta = \frac{8}{9}, \quad \lambda^2 = -\frac{\epsilon}{3}. \tag{5.20}$$

From (5.20), we get $\epsilon < 0$ and $K = \frac{2}{3}\epsilon$. ■

Remark 5.1. See [2,4] for Lagrangian surfaces in $\mathbf{C}^2$ whose shape operators take the form (4.1).

Remark 5.2. For an n-dimensional Kählerian slant submanifold in a complex-space-form $\widetilde{M}^n(4\epsilon)$, one may prove that the scalar curvature τ and the squared mean curvature H^2 of M satisfy

$$H^2 \geq \frac{2(n+2)}{n^2(n-1)}\tau - \frac{n+2}{n}\left(1 + \frac{3}{n-1}\cos^2\theta\right)\epsilon, \tag{5.21}$$

where θ is the slant angle and τ is the scalar curvature defined by

$$\tau = \sum_{i<j} K(e_i \wedge e_j)$$

for an orthonormal frame $\{e_1, \ldots, e_n\}$ of TM.

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