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POINCARÉ RECURRENCES OF COUPLED SUBSYSTEMS IN SYNCHRONIZED REGIMES

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Abstract. We introduce a notion of topological synchronization by using Poincaré recurrences of coupled subsystems. We show that the dimension of Poincaré recurrences may indicate synchronized behavior.

1. Introduction

It is well-known now that dynamical systems with chaotic behavior can be synchronized provided that they are coupled by a dissipative coupling (see, for instance, [11] and references therein). In other words, a system

$$\begin{cases} \dot{x} = X(x) + cF(x, y, c), \\ \dot{y} = Y(y) + cG(x, y, c), \end{cases} \tag{1.1}$$

where $x \in R^m,\ y \in R^n,\ c$ is a coupling parameter, can behave in such a way that the x-component and the y-component of a solution $x(t, x_0, y_0),\ y(t, x_0, y_0)$, manifest "similar properties" for $t \geq t_0 \gg 1$, independent of initial conditions (x_0, y_0) in a large region in R^{m+n}. In the case of "exact" synchronization, the following identity holds:

$$\lim_{t\to\infty} |x(t, x_0, y_0) - y(t, x_0, y_0)| = 0. \tag{1.2}$$

Of course, in this case $m = n$ and the right-hand side of the system (1.1) should satisfy the identity

$$X(x) + cF(x, x, c) \equiv Y(x) + cG(x, x, c). \tag{1.3}$$

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For example, it is so if $X(x) \equiv Y(x)$ and $F(x, x, c) = G(x, x, c) \equiv 0$.

However, if coupled subsystems in (1.1) are nonidentical then we cannot expect the validity of (1.2) and the notion of "synchronization" may be treated differently. Specialists introduced notions of phase synchronization [14], asymptotic synchronization [9], stochastic synchronization [3], generalized synchronization [1] and others to point out significant features of the synchronization phenomena. The present work is of the same spirit. A definition of synchronization will be introduced which is based on the notion of Poincaré recurrences.

We start with periodic oscillations. Assume that the system $\dot{x} = X(x)$ has a linearly stable limit cycle L_1 with the period τ_1 and the system $\dot{y} = Y(x)$ has a linearly stable limit cycle L_2 with the period τ_2. Then the system (1.1) for $c = 0$ has the attracting torus $T_0 = L_1 \times L_2$. If the rotation number $\rho_0 = \tau_1/\tau_2$ is rational, then T_0 consists of periodic orbits of the system (1.1) for $c = 0$; if ρ_0 is irrational then every orbit on T_0 is dense (on it). For $c \neq 0$ and small enough, still there exists an invariant attracting torus T_c in a neighborhood of T_0 [7]. Generally, for an open region in the parameter space, the system (1.1) has stable limit cycles. The synchronization regime corresponds to the existence of the stable limit cycle, say L_c, on the torus T_c. The Poincaré rotation number for these values of parameters is rational, say, $m_0/n_0 \in Q$, and it means that the closed curve L_c makes m_0 rotations along the generator L_1 of the torus T_0 and n_0 rotations along another one. In terms of individual subsystems, we can describe the regime as follows. The orbit L_c corresponds to the solution $x = x_c(t),\ y = y_c(t)$ of the system (1.1), where x_c, y_c are τ_c-periodic vector functions. One can introduce "polar coordinates" (a_i, θ_i) in a neighborhood of $L_i,\ i = 1, 2$, in R^m for $i = 1$, and R^n for $i = 2$, such that θ_i is an angular coordinate along L_i and a_1(resp. a_2) is an "amplitude" coordinate on a transversal to L_1 in R^m (resp. to L_2 in R^n). Then (for small values of c) the solution $(x_c(t), y_c(t))$ can be expressed in the new coordinates in the form

$$\begin{aligned} a_1 = a_1(t),\ \theta_1 = w_1 t + \alpha_1(t),\ \bmod \tau_c, \\ a_2 = a_2(t),\ \theta_2 = w_2 t + \alpha_2(t),\ \bmod \tau_c, \end{aligned} \tag{1.4}$$

where $a_1, a_2, \alpha_1, \alpha_2$ are τ_c-periodic functions and $w_1/w_2 = n_0/m_0$. For the sake of simplicity, assume that a_1 and a_2 are constants, $\alpha_1 \equiv 0,\ \alpha_2 \equiv 0,\ m_0 = 1$. Then at the moment $t = t_x = \tau_c/w_1$ we have $\theta_1(t_x) = \theta_1(0) \bmod \tau_c$ and $x_c(t_x) = x_c(0)$. However, only at the moment $t = t_y = \tau_c/w_2 = n_0 t_x$, the second coordinate $y_c(t_y) = y_c(0)$. In other words, the "period" t_c of oscillations in the x-subspace can be different from the period of those in the y-subspace, and this difference can be written as the following equation

$$\frac{t_x}{t_y} = \frac{1}{n_0}. \tag{1.5}$$

The same is true if $m_0 \neq 1$, and, then,

$$\frac{t_x}{t_y} = \frac{m_0}{n_0}. \tag{1.6}$$

Assume now that for some parameter values, the system (1.1) has an attractor A_c containing infinitely many orbits, such that for $(x_0, y_0) \in A_c$ the projections $x(t, x_0, y_0)$ and $y(t, x_0, y_0)$ of the solution $(x, y)(t, x_0, y_0)$ onto the x-subspace and the y-subspace behave similarly. In order to define rigorously this similarity, we have to be sure that if (x_0, y_0) belongs to a periodic orbit then something like the inequality (1.6) holds and the number m_0/n_0 is independent of the choice of the periodic orbit in the attractor. Furthermore, if (x_0, y_0) belongs to a nonperiodic orbit we should define some quantities which are similar to the periods of periodic orbits, and we again have to have something like the equality (1.6) for these quantities. We use *Poincaré recurrences* in the capacity of desired quantities, and we use Carathéodory-Pesin [13] approach to compare the Poincaré recurrences for different subsystems.

2. Poincaré Recurrences

Orbits in Hamiltonian systems and limiting orbits in dissipative systems possess the property of a repetition of their behavior in time. This repetition can be expressed in terms of Poincaré recurrences.

Consider, first, a dynamical system with discrete time, generated by a map $f : M \to M$, where M is the phase space of the system which is assumed to be a complete metric space. Let A be a compact f-invariant subset (for example, A is an attractor). Given a set $U \in A$ and a point $z \in U$, let us denote by $t(z, U)$ the smallest positive integer for which $f^{t(z,U)} z \in U$ again.

Definition 2.1. 1. *We call $t(z, U)$ the Poincaré recurrence for the set U specified by the point z (it can be ∞, of course).*
2. *The quantity*

$$\tau(U) = \inf_{z \in U} t(z, U) \tag{2.1}$$

is called the Poincaré recurrence for the set U.

It is clear that $\tau(U) < \infty$ if U is open, and nonwandering points (in A) are dense in A. For volume-preserving maps of a Riemannian manifold, $\tau(U) < \infty$ if U is open, thanks to the Poincaré recurrence theorem (see for instance [10]).

Consider now the flow (f^t, M), where $t \in R^+$ and M is the phase space. Given an open U and a point $z \in U$, let us denote by $t_1(z, U)$ the following number: if $f^t z \in U$ for all $t \in R^+$, then $t_1(z,U) = \infty$; if there is $t_0 \in R^+$ such that $f^{t_0} z \notin U$, then

$$t_1(z,U) = \inf\{t_0 | f^{t_0} z \notin U\}.$$

Since U is open, therefore if there is $\bar{t} > t_1(z,U)$ such that $f^{\bar{t}} z \in U$, then there exists a maximal interval $(\alpha, \beta) \ni \bar{t}$ such that $f^t z \in U$ for any $t \in (\alpha, \beta)$. Set

$$t(z,U) = 0, \quad \text{if } \; t_1(z,U) = \infty;$$
$$t(z,U) = \inf \frac{\alpha + \beta}{2}, \quad \text{if } \; t_1(z,U) < \infty,$$

where the infimum is taken over all maximal intervals (α, β) such that $\alpha \geq t_1(z,U)$ and $f^t z \in U$ if $t \in (\alpha, \beta)$. In particular, $f^{t(z,U)} z \in U$.

Definition 2.2. *The number $t(z,U)$ is said to be the Poincaré recurrence for the set U specified by the point z. The number*

$$\tau(U) = \inf_{z \in U} t(z,U) \tag{2.2}$$

is called the Poincaré recurrence for the set U.

These definitions are related to the repetition of the motion along orbits of dynamical systems. However, we are going to deal with the properties of the repetition along the x-(or the y-) components of the solution of a system of (1.1) type. We have to extend the definition of Poincaré recurrences to the case of coupled subsystems.

Let X, Y be complete metric spaces and $f^t : X \times Y \to X \times Y$ be a dynamical system, $t \in R^+$ or $t \in R$, in the case of continuous time, and $t \in N$ or $t \in Z$ in the case of discrete time. Let A be a compact invariant subset of $X \times Y$ and $\pi_1 A = A_1 \subset X$, $\pi_2 A = A_2 \subset Y$ be the images under natural projections of A to the first and the second subspaces respectively. Let

$$x(t, x_*, y_*) = \pi_1 f^t(x_*, y_*) \subset A_1,$$
$$y(t, x_*, y_*) = \pi_2 f^t(x_*, y_*) \subset A_2$$

be images of the orbit $\{f^t(x_*, y_*)\}$ going through an initial point $(x_*, y_*) \in A$.

Consider, first, the case of the discrete time. Let $U_1 \subset X \cap A_1$ (resp. $U_2 \subset Y \cap A_2$) be an open set in A_1 (resp. A_2), and $x_0 \in U_1$ (resp. $y_0 \in U_2$). Let $Y_{x_0} = \pi_2(\pi_1^{-1}(x_0) \cap A)$, the set of the y-coordinates of all preimages of the point x_0 in A (resp. $X_{y_0} = \pi_1(\pi_2^{-1}(y_0) \cap A)$). Denote by $t_x(z, U_1)$ the smallest positive integer for which $\pi_1(f^{t_x(z,U_1)} z) \in U_1$, where $z = (x_0, y)$, $y \in Y_{x_0}$, and by $t_y(\tilde{z}, U_2)$ the smallest positive integer for which $\pi_2(f^{t_y(\tilde{z},U_2)} \tilde{z}) \in U_2$, where $\tilde{z} = (x, y_0)$, $x \in X_{y_0}$.

Definition 2.3. 1. *The number*

$$t(x_0, U_1) := \inf_{y \in Y_{x_0}} t_x(z, U_1), \tag{2.3}$$

$z = (x_0, y)$, $y \in Y_{x_0}$, *is said to be the x-Poincaré recurrence for the set* U_1 *specified by the point* $x_0 \in U_1$. *The number*

$$\tau_x(U_1) := \inf_{x_0 \in U_1} t(x_0, U_1) \tag{2.4}$$

is said to be the x-Poincaré recurrence for the set U_1.

2. *The number*

$$t(y_0, U_2) := \inf_{x \in X_{y_0}} t_y(\tilde{z}, U_2), \tag{2.5}$$

$\tilde{z} = (x, y_0)$, $x \in X_{y_0}$, *is said to be the y-Poincaré recurrence for the set* U_2 *specified by the point* $y_0 \in U_2$. *The number*

$$\tau_y(U_2) := \inf_{y_0 \in U_2} t(y_0, U_2) \tag{2.6}$$

is said to be the y-Poincaré recurrence.

For the case of continuous time, we proceed in the way of Definition 2.2. Let $U_1 \subset X \cap A_1$ (resp. $U_2 \subset Y \cap A_2$) be an open set in A_1 (resp. A_2), and $x_0 \in U_1$ (resp. $y_0 \in U_2$). As above, denote by Y_{x_0} the set $\pi_2(\pi_1^{-1}(x_0) \cap A)$ of all π_1-preimages of the point $x_0 \in A_1$ (resp. $X_{y_0} = \pi_1(\pi_2^{-1}(y_0) \cap A$, the set of all π_2-preimages of the point $y_0 \in A_2$). Introduce the number $t_1(x_0, U_1)$ (resp. $t_2(y_0, U_2)$) as follows. If $\pi_1(f^t z) \in U_1$, $t \in R^+$, (resp. $\pi_2(f^t \tilde{z}) \in U_2$, $t \in R^+$), where $z = (x_0, y)$, $y \in Y_{x_0}$ (resp. $\tilde{z} = (x, y_0)$, $x \in X_{y_0}$), then $t_1(x_0, U_1) := \infty$ (resp. $t_2(y_0, U_2) =: \infty$). If there exists $z = (x_0, y)$, $y \in Y_{x_0}$ (resp. $\tilde{z} = (x, y_0), x \in X_{y_0}$) such that $\pi_1(f^{t_0} z) \notin U_1$ for some $t_0 = t_0(y)$ (resp. $\pi_2(f^{t_0} \tilde{z}) \notin U_2$ for some $t_0 = t_0(x)$), then

$$t_1(x_0, U_1) =: \inf_{y \in Y_{x_0}} \inf \left\{ t_0(y) | f^{t_0(y)} z \notin U_1 \right\} \tag{2.7}$$

$$\left(\text{resp. } t_2(y_0, U_2) =: \inf_{x \in X_{y_0}} \inf \left\{ t_0(x) | f^{t_0(x)} \tilde{z} \notin U_2 \right\} \right). \tag{2.8}$$

Since the set U_1 (resp. U_2) is open, then if there exists $\bar{t} > t_1(x_0, U_1)$ (resp. $\bar{t} > t_2(y_0, U_2)$) such that $\pi_1(f^{\bar{t}}(x_0, y)) \in U_1$ for some $y \in Y_{x_0}$ (resp. $\pi_2(f^{\bar{t}}(x, y_0)) \in$

U_2 for some $x \in X_{y_0}$), then there is a maximal interval $(\alpha, \beta) \ni \bar{t}$ such that $\pi_1(f^t(x_0, y)) \in U_1$ (resp. $\pi_2(f^t(x, y_0)) \in U_2$) for any $t \in (\alpha, \beta)$. Set

$$\begin{aligned} &t(x_0, U_1) := 0 \ \text{ if } \ t_1(x_0, U_1) = \infty; \\ &t(x_0, U_1) := \inf_{y \in Y_{x_0}} \inf \frac{\alpha + \beta}{2} \ \text{if } t_1(x_0, U_1) < \infty, \end{aligned} \tag{2.9}$$

where the first infimum is taken over all maximal intervals (α, β) such that $\alpha \geq t_1(x_0, U_1)$ and $\pi_1(f^t(x_0, y)) \in U_1$ if $t \in (\alpha, \beta)$, $y \in Y_{x_0}$. In particular, $\pi_1(f^{t(x_0, U_1)}(x_0, y)) \in U_1$ for some $y \in Y_{x_0}$. Similarly, introduce

$$\begin{aligned} &t(y_0, U_2) := 0 \ \text{ if } \ t_2(y_0, U_2) = \infty; \\ &t(y_0, U_2) := \inf_{x \in X_{y_0}} \inf \frac{\alpha + \beta}{2} \ \text{if } t_2(y_0, U_2) < \infty, \end{aligned} \tag{2.10}$$

where the first infimum is taken over all maximal intervals (α, β) such that $\alpha \geq t_2(y_0, U_2)$ and

$$\pi_2(f^t(x, y_0)) \in U_2 \ \text{ if } \ t \in (\alpha, \beta), \ x \in X_{y_0}.$$

Definition 2.4. *The number $t(x_0, U_1)$ is said to be the x-Poincaré recurrence for the set U_1 specified by the point x_0. The number*

$$\tau_x(U_1) := \inf_{x_0 \in U_1} t(x_0, U_1) \tag{2.11}$$

is said to be the x-Poincaré recurrence for the set U_1. The number $t(y_0, U_2)$ is said to be the y-Poincaré recurrence for the set U_2 specified by the point $y_0 \in U_2$. The number

$$\tau_y(U_2) := \inf_{y_0 \in U_2} t(y_0, U_2) \tag{2.12}$$

is said to be the y-Poincaré recurrence for the set U_2.

The main difference between Definitions 2.1, 2.2 and 2.3, 2.4 is the additional infimum. We take the infimum not only over all points in the open set but also over all possible "branches" going through the point in it. Roughly speaking, the curves $x(t)$ for different initial condititions can intersect each other (they are not the orbits but only their projections) and at a point of the intersection we should take into account all possible itineraries. Of course, it is possible to introduce not the infimum but some other function of different "branches", but at the moment, the infimum seems to be a satisfactory one.

3. Topological Synchronization

We defined in the previous section the quantities which play the role of the periods. By using them, we may define some kind of "synchronization equalities" of the type (1.5), (1.6). We should take into account that $\tau_x(U_1)$ and $\tau_y(U_2)$ depend on the sets U_1, U_2. For example, they may go to infinity as diam $(U_1) \to 0$ (resp. diam $(U_2) \to 0$). Moreover, we have to be prepared for the fact that not all points on a fixed "curve" $x(t), y(t)$ determine the same value p/q.

Consider the following example of a periodically perturbed oscillator:

$$\ddot{x} + k\dot{x} + f(x) = a\sin\theta, \quad \dot{\theta} = 1, \tag{3.1}$$

where the nonlinearity $f(x)$ is of the Duffing-type. It is well-known (see, for instance, [9]) that for same values of the parameters the system (3.1) undergoes the period-doubling bifurcation, and has a stable 4π-periodic limit cycle, say, L. For the system (3.1), the phase space is the direct product $R^2 \times S^1$, where $X = \{(x, \dot{x})\} \subset R^2$, $Y = \{\theta, \mathrm{mod}\ 2\pi\} = S^1$. Let $\{x = x_0(t),\ \dot{x} = \dot{x}_0(t)\} \subset X$, $\{\theta = t, \mathrm{mod}\ 2\pi\} \subset S^1$ be a solution corresponding to L. It is simple to understand that the curve $x = x_0(t),\ \dot{x} = \dot{x}_0(t),\ t \in [0, 4\pi]$, which is the projection of L onto X, might possess points of self-intersection. At each of these points, say, $(x_*, \dot{x}_*)$, we have $x_* = x_0(t_1) = x_0(t_2),\ \dot{x}_* = \dot{x}_0(t_1) = \dot{x}_0(t_2),\ t_1 \neq t_2,\ t_1,\ t_2 \in (0, 4\pi)$. If L is close to limit cycle at the bifurcation moment, then such points have to exist by simple geometrical reasons. Evidently, if U_1 is a small neighborhood of $(x_*, \dot{x}_*)$, then $\tau_x(U_1) < 4\pi - \delta$, where δ is independent of diam (U_1), while $\tau_x(\tilde{U}) \approx 4\pi$, where $\tilde{U}$ does not contain points of self-intersection and is small enough. The example shows that not all points on $x(t)$ (or $y(t)$) are responsible for the "right" value of Poincaré recurrences. However, generally, most of them are the desired ones.

Let us emphasize that each point $\theta(t_0) = \theta(t_0 + 2\pi), \mathrm{mod}\ 2\pi$, corresponds to at most two different points on the curve $(x, \dot{x})(t) : (x, \dot{x})(t_0)$ and $(x, \dot{x})(t_0 + 2\pi)$, if $t_0 \in [0, 2\pi)$.

All features mentioned in the example may serve as the basic ones for a definition of a synchronization. But before that, we recall that a system $f^t : X \times Y \to X \times Y$ has an attractor A if A is compact and there is an open set $U \supset A$ such that $f^t\bar{U} \subset U,\ t > 0$, and $A = \bigcap_{t>0} f^t U$. (In principle, we might use any definition of the attractor. However, for the sake of definiteness, we restrict ourselves to this sufficiently appropriate definition).

Definition 3.1. *A dynamical system $f^t : X \times Y \to X \times Y$ is said to be (m_0/n_0)-topologically synchronized if the following hold.*

(i). *It has an attractor A such that nonwandering (in A) orbits are dense in A.*

(ii). *There is a number $N \in Z_+$ such that for any point $x_0 \in \pi_1(A)$, the set Y_{x_0} contains at most N points, and for any point $y_0 \in \pi_2(A)$ the set X_{y_0} contains at most N points.*

(iii). *There is a compact set $B \subset A$ (B might be empty) such that if $A_1 = \pi_1(A),\ A_2 = \pi_2(A),\ B_1 = \pi_1(B),\ B_2 = \pi_2(B)$, then*

$$\dim_B(B_i) < \dim_H(A_i),\ i = 1, 2, \tag{3.2}$$

where $\dim_B$ (resp. $\dim_H$) *is the upper box* (resp. *Hausdorff*) *dimension* (*see below*).

(iv). *For any point $(x_0, y_0) \in A \setminus B$, there are numbers $\varepsilon_0 > 0$ and $a_1 \geq a_2 \geq 1$ such that: for any open set $U_1 \subset X,\ U_1 \ni x_0$, $\mathrm{diam}\ U_1 \leq \varepsilon \leq \varepsilon_0$, there is an open set $U_2 \subset Y,\ \mathrm{diam}\ U_2 \leq a_1 \cdot \mathrm{diam}\ U_1,\ U_2 \ni y_0$, and for any open set $\tilde{U}_2 \subset Y,\ \tilde{U}_2 \ni y_0$, $\mathrm{diam}\ \tilde{U}_2 \leq \varepsilon \leq \varepsilon_0$, there is an open set $\tilde{U}_1$, $\mathrm{diam}\ \tilde{U}_1 \leq a_2 \cdot \mathrm{diam}\ U_2,\ \tilde{U}_1 \ni x_0$ such that*

$$\tau_y(U_2) = \frac{m_0}{n_0}\tau_x(U_1) + \beta_2, \quad \tau_x(\tilde{U}_1) = \frac{n_0}{m_0}\tau_y(\tilde{U}_2) + \beta_1, \tag{3.3}$$

where $m_0, n_0 \in Z_+$, and $\beta_1 = \beta_1(\tilde{U}_1, \tilde{U}_2),\ \beta_2 = \beta_2(U_1, U_2)$ are bounded as $\varepsilon \to 0$.

(v). *If $\delta(B)$ is an open δ-neighborhood of the set B in A where δ is small enough, then the constants ε_0, a_1, a_2 can be chosen to be the same for any point $(x_0, y_0) \in A \setminus \delta(B)$. They depend only on δ. Furthermore, the functions $\beta_{1,2} = \beta_{1,2}(U_1, U_2)$ can be estimated from above by a constant $\bar{\beta} > 0$ depending only on δ and $\varepsilon : |\beta_{1,2}| \leq \bar{\beta}$.*

(vi). *Moreover, for any set $U_1,\ U_1 \ni x_*,\ x_* \notin \pi_1(\delta(B))$, $\mathrm{diam}\ U_1 = \varepsilon < \varepsilon_0$, consider the union*

$$\bigcup_{x_0 \in U_1} \bigcup_{y \in Y_{x_0}} U_2(x_0, y),$$

where $U_2(x_0, y)$ is a set with $\mathrm{diam}\ U_2(x_0, y) \leq a_1\varepsilon$, *the existence of which was assumed in the assumption* (iv). *We claim that there are finitely many points $x_{0s} \in U_1,\ s = 1, \cdots, S$, such that*

$$\bigcup_{x_0 \in U_1} \bigcup_{y \in Y_{x_0}} U_2(x_0, y) = \bigcup_{s=1}^{S} \bigcup_{y \in Y_{x_{0s}}} U_2(x_{0s}, y). \tag{3.4}$$

In other words, we claim that if $x_0 \in U_1$ and $U_1 \cap B_1 = \phi$ then the set $U_2(x_0, y)$ depends, in fact, on the set U_1 but not on x_0 in U_1. Furthermore, we assume that there is a constant $S_0 = S_0(\delta)$ such that $S \leq S_0$ for any given set $U_1,\ U_1 \ni x_0,\ x_0 \notin \pi_1(\delta(B))$, $\mathrm{diam}\ U_1 \leq \varepsilon$ and ε is small enough. Similarly, for any set $\tilde{U}_2 \ni y_*,\ y_* \notin \pi_2(\delta(B))$, diam

$\tilde{U}_2 = \varepsilon < \varepsilon_0$, *we consider a set* $\tilde{U}_1(y_0, x)$, *where* $y_0 \in \tilde{U}_2,\ x \in X_{y_0}$, diam $\tilde{U}_1(y_0, x) \leq a_2\varepsilon$, *which was mentioned in* (iv). *We claim that there are finitely many points* $y_{0s} \in \tilde{U}_2,\ s = 1, \cdots, S$, *such that*

$$\bigcup_{y_0 \in \tilde{U}_2} \bigcup_{x \in X_{y_0}} \tilde{U}_1(x, y_0) = \bigcup_{s=1}^{S} \bigcup_{x \in X_{y_{0s}}} \tilde{U}_1(x, y_{0s}). \tag{3.5}$$

Remarks 3.1.

1. The condition (i) tells us that the Poincaré recurrences are finite for any open set, and, moreover, one should observe synchronization for $t \gg 1$ for open set of initial conditions.
2. The condition (ii) claims that the projections π_1 and π_2 are finite-to-one maps. It is a natural assumption which is known to be satisfied, for example, if f^t/A is a minimal flow and coupling is unidirectional [6].
3. The inequalities (3.2) mean that the "bad" points occupy a small part of the attractor.
4. The assumption (iv) tells us that in a neighborhood of a good point we have some kind of synchronization equalities (1.5), (1.6).
5. If A_1, A_2 contain infinitely many points, then condition (ii) implies the impossibility for an uncoupled system ($c = 0$ in (1.1)) to be synchronized.
6. If for any $\varepsilon > 0$ it is possible to cover A_1 (resp. A_2) by finitely many open sets U_i, diam $U_i \leq \varepsilon$, (resp. V_j, diam $V_j \leq \varepsilon$), such that there is a Lipshitz homeomorphism from U_i to $V_{j(i)}$ provided that $U_i \cap B_i = \phi$, $V_{j(i)} \cap B_2 = \phi$, and if the inverse map $V_{j(i)} \to U_i$ is also Lipshitz-continuous, then A_1 and A_2 are locally homeomorphic everywhere except for a neighborhood of the "bad" sets B_1 and B_2. In this case, the assumptions (3.4), (3.5) are satisfied. It is possible to show then that $\dim_H A_1 = \dim_H A_2$ (see below).

4. Carathéodory-Pesin Construction

We describe here a general approach developed by Pesin [13] on the basis of classical Carathéodory results.

Assume that X is a metric space with a distance ρ, and $\mathcal{F}$ is a collection of open subsets of X, or $\mathcal{F}$ is the collection of all balls of all diameters. Consider functions $\xi(u), \eta(u)$ of subsets $u \in \mathcal{F}$ satisfying the following properties:

(1) $\eta(u) > 0$ if $u \neq \emptyset$, $\xi(u) \geq 0$.

(2) For any $\delta > 0$, one can find $\varepsilon > 0$ such that $\eta(u) \leq \delta$ for any $u \in \mathcal{F}$ with diam $u \leq \varepsilon$ (diam $u = \sup_{x,y\in u} \rho(x, y)$).

The collection $\mathcal{F}$ and the functions $\xi(u), \eta(u)$ of sets in $\mathcal{F}$ form a Carathéodory structure [13]. Fix some Carathéodory structure and consider a finite or countable cover $G = \{u_i\}$ of X by sets u_i with diam $u_i \leq \varepsilon$. Then introduce the sum

$$M(\alpha,\varepsilon,G)=\sum_i \xi(u_i)\eta(u_i)^\alpha \tag{4.1}$$

and consider its infimum

$$M(\alpha,\varepsilon)=\inf_G \sum_i \xi(u_i)\eta(u_i)^\alpha, \tag{4.2}$$

where the infimum is taken over all finite or countable covers G with diam $u_j \le \varepsilon,\ u_j \in G$. The quantity $M(\alpha,\varepsilon)$ is a monotone function with respect to ε; therefore, there exists a limit

$$m(\alpha)=\lim_{\varepsilon\to 0} M(\alpha,\varepsilon).$$

It was shown in [13] that there exists a critical value $\alpha_c \in [-\infty,\infty]$ such that

$$m(\alpha)=0,\quad \alpha>\alpha_c,\quad \alpha_c\neq+\infty,$$
$$m(\alpha)=\infty,\quad \alpha<\alpha_c,\quad \alpha_c\neq-\infty.$$

The number α_c is said to be the Carathéodory dimension relative to the structure $(\mathcal{F},\xi,\eta)$.

For example, if $\mathcal{F}$ is a collection of balls $\{B(x,\varepsilon)\}$ of all diameters $\varepsilon>0$, centered at all points $x\in X,\ \xi(B(x,\varepsilon))\equiv 1,\ \eta(B(x,\varepsilon))=\varepsilon$, then

$$M(\alpha,\varepsilon,G)=\sum_{i=1}^{N(\varepsilon)}[\mathrm{diam} B(x_i,\varepsilon_i)]^\alpha,\quad \varepsilon_i\le\varepsilon,$$

and $\alpha_c = \dim_H X$ is the Hausdorff dimension. If the Hausdorff measure $m(\alpha_c)=m_c$ is positive and finite, then the average

$$\langle[\mathrm{diam} B(x_i,\varepsilon_i)]^{\alpha_c}\rangle \sim \frac{m_c}{N(\varepsilon)},\quad \varepsilon\gg 1.$$

Consider another structure $(\mathcal{F},\xi,\eta)$ on the same set X, and let α_c be the corresponding dimension. If $0<m(\alpha_c)=m_c<\infty$, then the average

$$\langle\xi(u_i)\eta(u_i)^{\alpha_c}\rangle=\frac{1}{N(\varepsilon)}\sum_i \xi(u_i)\eta(u_i)^{\alpha_c},$$

where $N(\varepsilon)$ is the minimal number of balls of diameter ε needed to cover X. Hence,

$$\bar\alpha_c=\overline{\dim}_B X,\ \text{the upper box dimension of } X,\ \overline{\dim}_B X=\overline{\lim}_{\varepsilon\to 0}\frac{\ln N(\varepsilon)}{\ln 1/\varepsilon},$$

and

$$\underline{\alpha}_c = \underline{\dim}_B X, \text{ the lower box dimension of } X, \ \underline{\dim}_B X = \underline{\lim}_{\varepsilon\to 0} \frac{\ln N(\varepsilon)}{\ln 1/\varepsilon}.$$

If the box dimension exists, i.e., if

$$\overline{\dim}_B X = \underline{\dim}_B X = b,$$

then

$$N(\varepsilon) \sim \varepsilon^{-b}, \quad \varepsilon \ll 1, \tag{4.3}$$

and for an arbitrary structure $(\mathcal{F}, \xi, \eta)$ we have

$$\langle \xi(u_i)\eta(u_i)^{\alpha_c} \rangle \sim \varepsilon^b \tag{4.4}$$

also behaves as $m_c/N(\varepsilon)$ if $\varepsilon \ll 1$; here $N(\varepsilon)$ is the number of sets u_i in a cover with diam $u_i \leq \varepsilon$.

Now introduce the sum

$$R(\alpha, \varepsilon, G) = \sum_i \xi(u_i)\eta(u_i)^{\alpha}, \tag{4.5}$$

where $\{u_i\} = G$ is a cover of X by sets u_i with diam $u_i = \varepsilon$ (not $\leq \varepsilon$ as above!). Consider the infimum

$$R(\alpha, \varepsilon) = \inf_G R(\alpha, \varepsilon, G),$$

where the infimum is taken over all covers $G = \{u_i\}$ with diam $u_i = \varepsilon$. We may expect that the limit of $R(\alpha, \varepsilon)$ as $\varepsilon \to 0$ does not exist. Consider the upper and lower limits

$$\bar{r}(\alpha) = \overline{\lim}_{\varepsilon\to 0} R(\alpha, \varepsilon)$$

and

$$\underline{r}(\alpha) = \underline{\lim}_{\varepsilon\to 0} R(\alpha, \varepsilon).$$

It was shown in [13] that there are critical values $\bar{\alpha}_c \geq \underline{\alpha}_c$ such that

$$\bar{r}(\alpha) = \begin{cases} 0, & \alpha > \bar{\alpha}_c, \quad \bar{\alpha}_c \neq +\infty, \\ \infty, & \alpha < \bar{\alpha}_c, \quad \bar{\alpha}_c \neq -\infty, \end{cases}$$

and

$$\underline{r}(\alpha) = \begin{cases} 0, & \alpha > \underline{\alpha}_c, \quad \underline{\alpha}_c \neq +\infty, \\ \infty, & \alpha < \underline{\alpha}_c, \quad \underline{\alpha}_c \neq -\infty. \end{cases}$$

The number $\bar{\alpha}_c$ is said to be the upper and $\underline{\alpha}_c$ the lower capacity relative to the structure $(\mathcal{F}, \xi, \eta)$.

If we assume, in addition, that

$$0 < \underline{r}(\underline{\dim}_B X) < \infty, \quad 0 < \bar{r}(\overline{\dim}_B X) < \infty,$$

then

$$N(\bar{\varepsilon}_k) \sim \bar{\varepsilon}_k^{-\bar{\alpha}_c}, \quad N(\underline{\varepsilon}_j) \sim \underline{\varepsilon}_j^{-\underline{\alpha}_c},$$

where $\{\bar{\varepsilon}_k\}$ (resp. $\{\underline{\varepsilon}_j\}$) is a sequence of values of ε such that

$$\overline{\lim}_{\varepsilon \to 0} R(\bar{\alpha}_c, \varepsilon) = \lim_{k \to \infty} R(\bar{\alpha}_c, \bar{\varepsilon}_k)$$

$$\left(\text{resp. } \underline{\lim}_{\varepsilon \to 0} R(\underline{\alpha}_c, \varepsilon) = \lim_{j \to \infty} R(\underline{\alpha}_c, \underline{\varepsilon}_j) \right).$$

For example, if $\mathcal{F}$ is the collection of the balls $\{B(x, \varepsilon)\}$ of all diameters $\varepsilon \geq 0$ centered at all points $x \in X$ and

$$\eta(B(x, \varepsilon)) = \text{diam } B(x, \varepsilon) = \varepsilon,$$

then

$$R(\alpha, \varepsilon) = \inf_G \sum_i [\text{diam } B(x_i, \varepsilon)]^\alpha = N(\varepsilon)\varepsilon^\alpha,$$

provided that $\underline{\alpha}_c = \bar{\alpha}_c = b$. Thus, the described construction allows one to estimate the asymptotic behavior of some average values of functions of sets. We use it to study Poincaré recurrences [2], [4].

5. Dimension and Capacities for Poincaré Recurrences

We give a definition of "fractal" dimension for Poincaré recurrences [2] [4] [5]. Consider a dynamical system (f^t, M). For any open set u, we introduced the Poincaré recurrence $\tau(u)$.

A desired characteristic should be an average value of $\tau(u)$. We use the Carathéodory-Pesin construction to introduce it. Consider the following Carathéodory structure: $\mathcal{F}$ is the collection of all open sets in the phase space M, $\eta(u) = \text{diam } u$, $\xi(u) = \varphi^q[\tau(u)]$, where $\varphi(t)$ is a monotonically decreasing function and $\tau(u)$ is the Poincaré recurrence for the set $u \in \mathcal{F}$. Then, consider the quantities (4.1) in the form

$$M(\alpha, \varepsilon, \varphi, q) = \inf_G \sum_i \varphi(\tau(u_i))^q (\text{diam } u_i)^\alpha, \tag{5.1}$$

where the infimum is taken over all covers $G = \{u_i\}$, diam $u_i \le \varepsilon$, and (4.2) is considered in the form

$$R(\alpha, \varepsilon, \varphi, q) = \inf_H \sum_i \varphi(\tau(u_i))^q (\text{diam } u_i)^\alpha, \tag{5.2}$$

where the infimum is taken over all covers $H = \{u_i\}$, diam $u_i = \varepsilon$. Now apply the general construction and obtain the dimension $\alpha(q)$ and capacities $\underline{\alpha}(q)$ and $\bar{\alpha}(q)$ for values of the parameter q in an interval. *These characteristics are said to be the spectrum of dimensions and the spectra of capacities for Poincaré recurrences.* It follows from [13] that $\underline{\alpha}(q) \le \alpha(q) \le \bar{\alpha}(q)$. Assume that there is a number q_0 (resp. $\bar{q}_0$ or $\underline{q}_0$) such that $q_0 = \sup\{q|\alpha(q) > 0\}$, $\bar{q}_0 = \sup\{q|\bar{\alpha}(q) > 0\}$, $\underline{q}_0 = \sup\{q|\underline{\alpha}(q) > 0\}$. *Then the value q_0 (resp. $\bar{q}_0$ or $\underline{q}_0$) is called the dimension (resp. the upper or lower capacity) for Poincaré recurrences.*

In order to understand the significance of the definition, let us suppose that $q_0 = \bar{q}_0 = \underline{q}_0$ and $\dim_B M = b$ is the box dimension. Then

$$\langle \varphi(\tau(u_i))^{q_0} \rangle \sim \varepsilon^b, \quad \text{diam } u_i = \varepsilon. \tag{5.3}$$

It was shown in [2] that in a nonchaotic situation (minimal sets), the function $\varphi(t) = 1/t$ can serve well. Thus for the case of minimal sets we have

$$\left\langle \frac{1}{\tau(u_i)^{q_0}} \right\rangle \sim \varepsilon^b \tag{5.4}$$

and we can expect that

$$\langle \tau(u_i) \rangle \sim \varepsilon^{-b/q_0}. \tag{5.5}$$

The example in the next section shows that in chaotic cases we should use $\varphi(t) = e^{-t}$, i.e., (5.3) becomes

$$\langle e^{-q_0 \tau(u_i)} \rangle \sim \varepsilon^b. \tag{5.6}$$

6. Spectrum of Capacities for Transitive Topological Markov Chains

Let A be the matrix of transitions for a topological Markov chain satisfying the condition of mixing [10]: there exists $n_0 > 0$ such that A^{n_0} has only positive entries. Therefore, we may introduce such a metric on the space of admissible sequences Ω_A for which

$$\text{diam}([i_0 \cdots i_{n-1}]) = a^{-n}, \quad n > n_0, \quad a > 1, \tag{6.1}$$

where $[i_0 \cdots i_{n-1}]$ is a cylinder, i.e.

$$[i_0 \cdots i_{n-1}] = \{\underline{w} = (j_0\ j_1 \cdots) \in \Omega_A | j_0 = i_0, \cdots, j_{n-1} = i_{n-1}\},$$

the set of all admissible infinite sequences for which the first n coordinates are determined by $[i_0 \cdots i_{n-1}]$.

Remark 6.1. Such topological Markov chains appear, for example, when we describe some repellers of maps with the constant derivative a. The simplest of them is the map $x \to ax$, mod 1, restricted to a set of orbits belonging to [0, 1] and forming a topological Markov chain.

We shall calculate the capacity, so we may consider only values of $\varepsilon = a^{-n}$, $n \in Z_+$, (see [13]) and we consider covers of Ω_A by cylinders $\{[i_0 \cdots i_{n-1}]\}$ for a fixed n. Then the equation (5.2) becomes

$$R_n(\alpha, q) = \sum_{(i_0 \cdots i_{n-1})} e^{-q\tau(i_0 \cdots i_{n-1})} a^{-\alpha n}, \tag{6.2}$$

where the sum is taken over all admissible words $(i_0 \cdots i_{n-1})$. The main idea is to rewrite (6.2) in the form

$$\left(P_1 e^{-q} + P_2 e^{-2q} + \cdots + P_{m_n} e^{-m_n q}\right) a^{-\alpha n}, \tag{6.3}$$

where P_k is the number of cylinders $[i_0 \cdots i_{n-1}]$ for which $\tau([i_0 \cdots i_{n-1}]) = k$. Of course, for that we need to show first that $m_n < \infty$. The following result holds.

Proposition 6.1. *$R_n(\alpha, q)$ can be represented in the form* (6.3) *where*

$$m_n \leq n + n_0$$

(n_0 is a constant in the condition (6.1) *of mixing).*

It means that Poincaré recurrence for a cylinder of length n cannot be greater than $n + n_0$.

Proof. We need to show that each cylinder $[i_0 \cdots i_{n-1}]$ contains an $(n+n_0)$ periodic point. Indeed, since $a^{n_0}_{i_{n-1} i_0} > 0$ by assumption, there exists a collection $(j_1, j_2, \cdots, j_{n_0 - 1})$ such that the periodic point $(\cdots [i_0 \cdots i_{n-1}] j_1 \cdots j_{n_0-1} [i_0 \cdots i_{n-1}] j_1 \cdots j_{n_0-1} \cdots)$ belongs to Ω_A. This point has period $n+n_0$ and therefore $\tau(u_{i_0 \cdots i_{n-1}}) \leq n + n_0$ for any admissible cylinder $[i_0 \cdots i_{n-1}]$. ■

The next proposition tells us that the Poincaré recurrence is realized due to the existence of periodic points.

Proposition 6.2. $\tau([i_0 \cdots i_{n-1}])$ = *the minimal period of all periodic points belonging to* $[i_0 \cdots i_{n-1}]$.

Proof. Let $k = \min\{s : \sigma^s \underline{w} = \underline{w},\ \underline{w} \in [i_0 \cdots i_{n-1}]\}$, and, on the contrary, assume that

$$\sigma^{-k_1}[i_0 \ldots i_{n-1}] \cap [i_0 \cdots i_{n-1}] \neq \emptyset, \quad k_1 < k,$$

i.e., there exists a point

$$(j_1 j_2 \cdots j_{k_1},\ i_0 i_1 \cdots i_{n-1} \cdots) \in [i_0 \cdots i_{n-1}].$$

But this means that

$$j_1 = i_0,\ j_2 = i_1, \cdots, j_{k_1 - 1} = i_{k_1} \ \text{ and } \ a_{i_{k_1-1} i_0} = 1.$$

Therefore, the sequence $\underline{w}' = (i_0 i_1 \cdots i_{k_1-1}, i_0 i_1 \cdots i_{k_1-1}, \cdots)$ is admissible and $\sigma^{k_1} \underline{w}' = \underline{w}'$. It contradicts the assumption that k is the minimal period. ■

Let $\lambda_{\max} = \max\{\lambda | \lambda \in \text{spec } A\}$. The number of k-periodic points behaves asymptotically as $\lambda_{\max}^k$, $k \gg 1$. It allows one to estimate the upper and lower capacities by using Propositions 6.1 and 6.2.

Proposition 6.3. $\underline{\bar{\alpha}}(q) < 0$ *if* $\ln \lambda_{\max} - 1 < 0$, *and*

$$\underline{\bar{\alpha}}(q) \leq \frac{\ln \lambda_{\max} - q}{\ln a} \tag{6.4}$$

if $\ln \lambda_{\max} - q \geq 0$.

Proof. The sum (6.3) satisfies

$$\begin{aligned} &\left(P_1 e^{-q} + P_2 e^{-2q} + P_{m_n} e^{-m_n q}\right) a^{-\alpha n} \\ &\leq \left(N_1 e^{-q} + N_2 e^{-2q} + \cdots + N_{n+n_0} e^{-(n+n_0)q}\right) a^{-\alpha n}, \end{aligned} \tag{6.5}$$

where N_k is the number of periodic points of period k. Indeed, for each cylinder $[i_0 \cdots i_{n-1}]$ with $\tau([i_0 \cdots i_{n-1}]) = k$, there exists a periodic point of period k belonging to it (Proposition 6.2), i.e., $P_k \leq N_k$. Therefore,

$$\begin{aligned} M_n(\alpha) &\leq \sum_{j=1}^{n+n_0} \left(N_j e^{-jq}\right) a^{-\alpha n} \\ &= a^{-\alpha n} \cdot \left(\sum_{j=1}^{n+n_0} \left(e^{-q}\right)^j \cdot trA^j \right) = a^{-\alpha n} \sum_{j=1}^{n+n_0} \left(e^{-q}\right)^j \left(\lambda_{\max}^j + \lambda_2^j + \cdots + \lambda_s^j\right), \end{aligned}$$

where $\lambda_2, \lambda_3, \cdots \lambda_s, \lambda_{\max} \in \text{spec } A$ (A is assumed to be an $s \times s$ matrix). Let us note that the number $N_j = \text{tr} A^j$, and (due to Frobenius-Perron theorem [10]), $\lambda_{\max} > |\lambda_i|,\ i = 2, \cdots, s$. Therefore

$$\begin{aligned} M_n(\alpha) &\leq a^{-\alpha n} \cdot \sum_{j=1}^{n+n_0} e^{-qj} \cdot \lambda_{\max}^j \cdot s \\ &\leq s \cdot (e^{-q}\lambda_{\max}) \cdot a^{-\alpha n} \cdot \frac{(e^{-q}\lambda_{\max})^{n+n_0+1} - 1}{e^{-q}\lambda_{\max} - 1} \end{aligned}$$

and

$$\begin{gathered} M_n(\alpha) \to 0 \quad \text{as} \quad n \to \infty \quad \text{if} \\ a^{-\alpha} e^{-q} \lambda_{\max} < 1 \quad \text{or} \\ \alpha > \frac{\ln \lambda_{\max} - q}{\ln a}. \end{gathered}$$

This is the case when $e^{-q}\lambda_{\max} > 1$.
If $e^{-q}\lambda_{\max} = 1$, then

$$M_n(\alpha) \leq s \cdot a^{-\alpha n} \cdot (n + n_0) \to 0 \quad \text{as} \quad \alpha > 0$$

or $\alpha > (\ln \lambda_{\max} - q)/\ln a$ again.
If $e^{-q}\lambda_{\max} < 1$, then

$$M_n(\alpha) \leq jsa^{-\alpha n} \frac{1}{1 - e^{-q}\lambda_{\max}}$$

and if $\alpha \geq 0$ the series converges. By definition of the spectra of capacities we have that

$$\begin{aligned} &\underline{\alpha}(q) \leq \frac{\ln \lambda_{\max} - q}{\ln a} \quad \text{if} \quad \lambda_{\max} e^{-q} \geq 1 \\ \text{and} \qquad &\underline{\alpha}(q) \leq 0 \quad \text{if} \quad \lambda_{\max} e^{-q} < 1. \end{aligned}$$

■

Proposition 6.4. *The upper capacity $\bar{\alpha}_c(q)$ satisfies*

$$\bar{\alpha}_c(q) \leq \frac{\ln \lambda_{\max} - q}{\ln a}. \tag{6.7}$$

Proof. Trivially, the sum

$$M_n(\alpha) \geq P_n e^{-qn} a^{-\alpha n}.$$

The number $P_n = P_n^{(1)} + P_n^{(2)}$, where $P_n^{(1)}$ is the number of cylinders $[i_0 \cdots i_{n-1}]$ containing the periodic point $((i_0 \cdots i_{n-1})(i_0 \cdots i_{n-1}) \cdots)$ for different collections $(i_0 \cdots i_{n-1})$. Assume that n is a prime number. Then

$$\begin{aligned} &P_n^{(1)} \geq N_n, \quad \text{and} \\ &M_n(\alpha) \geq N_n e^{-qn} a^{-\alpha n} \geq \lambda_{\max}^n e^{-qn} a^{-\alpha n}. \end{aligned} \tag{6.8}$$

The inequality (6.8) implies that

$$\overline{\lim}_{n\to\infty} M_n(\alpha) \geq \lim_{\substack{n\to\infty \\ n \text{ is prime}}} M_n(\alpha) = \infty$$

if

$$\lambda_{\max} e^{-q} a^{-\alpha} > 1, \quad \text{or}$$

$$\alpha < \frac{\ln \lambda_{\max} - q}{\ln a}. \tag{6.9}$$

Thus,

$$\bar{\alpha}_c(q) \geq \frac{\ln \lambda_{\max} - q}{\ln a}. \qquad \blacksquare$$

Comparing (6.4) and (6.7) we have

Theorem 6.1. (1) *If* $q \leq h_{\text{top}}$, *then* $\bar{\alpha}(q) = (h_{\text{top}} - q)/\ln a$.
(2) $\bar{q}_0 = h_{\text{top}} = \ln \lambda_{\max}$.

Remark 6.2. Taking into account that the number of admissible words $\{[i_0 \cdots i_{n-1}]\}$ is asymptotically equal to $\exp(nh_{\text{top}}) = e^{q_0 n}$, we can write that

$$\langle e^{-q_0 \tau([i_0 \cdots i_{n-1}])} \rangle \sim e^{-q_0 n} \quad \text{or}$$

$$\langle \tau([i_0 \cdots i_{n-1}]) \rangle \sim= -\ln \varepsilon / \ln a \tag{6.10}$$

$(\varepsilon = a^{-n})$. Thus, $\varphi(t) = e^{-t}$ works well in an ideal chaotic situation.

In our considerations, the constant a served as a rate of expansion of the shift map σ at every point $\underline{w} = (i_0, i_1, \cdots)$. We saw that the capacity $\bar{q}_0$ is just the topological entropy and does not carry new information. In a more realistic hyperbolic situation when the rates of expansion (and contraction) depend on a point, we may expect that the capacities $\underline{\bar{q}_0}$ and the topological entropy are independent characteristics.

Remark 6.3. It was shown in [12] that a dimension for Poincaré recurrences (which is defined in a slightly different way) equals the topological entropy for subshifts of finite type and β-subshifts. It was conjectured there that the topological entropy is a lower bound for dimension for Poincaré recurrences.

7. Hausdorff Dimensions of Projections of the Attractor

We show here that under the assumptions of Definition 3.1 the "individual attractors" A_1 and A_2 have the same Hausdorff dimensions.

Theorem 7.1. *Assume that a dynamical system $f^t : X \times Y \to X \times Y$ is topologically synchronized* (*with respect to an attractor A*). *Then*

$$\dim_H(A_1) = \dim_H(A_2) \tag{7.1}$$

($\dim_H$ *means the Hausdorff dimension*).

Proof.

1. Let $\delta_n(B)$ be the open δ_n-neighborhood of the set B in A, $A_1\backslash\pi_1(\delta_n(B)) =: A_{1n}$, and $A_2\backslash\pi_2(\delta_n(B)) =: A_{2n}$, where $\delta_n \to 0$ as $n \to \infty$. The set A_{in} does not contain the set B_i together with some of its open neighborhood, $i = 1, 2$. Therefore, thanks to the assumption (vi) in Definition 3.1, the constant $\varepsilon_0 = \varepsilon_{0n}, a_i = a_{in}$, can be chosen to be independent of $(x_0, y_0) \in A\backslash\delta_n(B),\ i = 1, 2$. Furthermore, $A_i\backslash B_i = \bigcup A_{in},\ i = 1, 2$, and thus (see [7])

$$\begin{aligned} &\dim_H A_i = \sup\{\dim_H(A_i\backslash B_i),\ \dim_H B_i\} \\ = \ &\dim_H(A_i\backslash B_i) = \sup_n\{\dim_H A_{in}\}, \quad i = 1, 2. \end{aligned} \tag{7.2}$$

We take into account here that, by (3.2),

$$\dim_H A_i > \dim_B B_i.$$

2. Let $\alpha_{in} = \dim_H A_{in},\ i = 1, 2,\ n \in Z_+$. We show that

$$\alpha_{1n} = \alpha_{2n}, \quad n \ge n_0 \in Z_+. \tag{7.3}$$

Given $\alpha > \alpha_{2n},\ K > 0$, consider a finite cover $\{\tilde{U}_{2j}\}$ of the set A_{2n} by open sets with diam $\tilde{U}_{2i} \le \varepsilon \le \varepsilon_0$, such that

$$\sum_j (\text{diam } \tilde{U}_{2j})^\alpha \le K. \tag{7.4}$$

Such a cover exists by the definition of the Hausdorff dimension. Given $\tilde{U}_{2j}$ and $y_0 \in \tilde{U}_{2j}$, consider X_{y_0}, and for any $x \in X_{y_0}$ choose a set $\tilde{U}_{1j}(x, y_0) \ni x$ with diam $\tilde{U}_{1j}(x, y_0) \le a_2\varepsilon$, which exists thanks to the assumption (iv) in Definition 3.1. We have

$$\bigcup_j \bigcup_{y_0 \in \tilde{U}_{2j}} \bigcup_{x \in X_{y_0}} \tilde{U}_{1j}(x, y_0) \supset A_{1n}.$$

Thanks to (3.5), we have

$$\bigcup_j \bigcup_{y_{os}} \bigcup_{x\in X_{y_{os}}} \tilde{U}_{1j}(x, y_{os}) \supset A_{1n},$$

i.e., the sets $\{\tilde{U}_{1j}(x, y_{os})\}$ form a finite cover of the set A_{1n}. Now, thanks to assumptions (ii), (iv), (v) in Definition 3.1 and (7.4), we obtain

$$\sum \Big(\text{diam } \tilde{U}_{1j}(x, y_{os})\Big)^\alpha = \sum_j \sum_{y_{os}} \sum_{x\in X_{y_{os}}} \Big(\text{diam } \tilde{U}_{1j}(x, y_{os})\Big)^\alpha$$

$$\leq N \cdot S_0(\delta) \cdot a_2^\alpha \cdot \sum_j (\text{diam } \tilde{U}_{2j})^\alpha \leq N S_0(\delta) a_2^\alpha \cdot K. \tag{7.5}$$

Since K is an arbitrary small number, this means that

$$\alpha > \alpha_{1n}, \quad \text{i.e.,} \quad \alpha_{2n} \geq \alpha_{1n}.$$

3. Similarly, we may start with a cover $\{U_{1j}\}$ of the set A_{1n} and obtain the opposite inequality $\alpha_{1n} \geq \alpha_{2n}$. Thus, $\alpha_{1n} = \alpha_{2n}$. Combining it with (7.2), we obtain the desired result. ∎

8. Dimension for Poincaré Recurrences as an Indicator of Synchronized Regimes

Let us define, first, the dimension and capacities for x- and y-Poincaré recurrences. We restrict ourselves to the case of chaotic behavior, i.e., we use the function e^{-t} in the capacity of $\varphi(t)$ in (5.1), (5.2), and consider the spectrum of dimension for the x- and y-Poincaré recurrences. For that, as in Section 5, we consider the sums

$$M_x(\alpha, \varepsilon, e^{-t}, q) = \inf_{G_1} \sum_i e^{-q\tau_x(U_{1i})} (\text{diam } U_{1i})^{\alpha_x} \tag{8.1}$$

and

$$M_y(\alpha, \varepsilon, e^{-t}, q) = \inf_{G_2} \sum_i e^{-q\tau_y(U_{2i})} (\text{diam } U_{2i})^{\alpha_y}, \tag{8.2}$$

where in each sum the infimum is taken over all covers G_1 (resp. G_2) of the set A_1 (resp. A_2) by open sets with diameters $\leq \varepsilon$. The critical values $\alpha_x(q)$ in (5.1) and $\alpha_y(q)$ in (5.2) will be the desired spectra of dimensions for Poincaré recurrences. The main result of the present paper is the following statement.

Theorem 8.1. *If a dynamical system $f^t : X \times Y \to X \times Y$ is (m_0/n_0)-topologically synchronized, then*

$$\dim_P(A_2 \backslash B_2) = \frac{m_0}{n_0} \dim_P(A_1 \backslash B_1), \tag{8.3}$$

where $\dim_P$ *is the dimension for Poincaré recurrences.*

Proof. 1. We shall follow the proof of Theorem 7.1; in particular, we assume that A_{in} are the same sets as the ones over there. Given $\alpha > \alpha_y(q, A_{2n})$ and $K > 0$, consider a finite cover $\{\tilde{U}_{2j}\}$ of the set A_{2n} by open sets, diam $\tilde{U}_{2j} \le \varepsilon \le \varepsilon_0$, such that

$$\sum_j \exp\left(-q\tau\left(\tilde{U}_{2j}\right)\right) \cdot \left(\mathrm{diam}\tilde{U}_{2j}\right)^\alpha \le K. \tag{8.4}$$

By the definition of the spectrum for dimensions, such a cover exists. As in the proof of Theorem 7.1, we choose a cover of A_{1n} by sets $\tilde{U}_{1j}(x, y_{os}),\ x \in X_{y_{os}}$. By using the assumptions of (ii), (iv), (v) in Definition 3.1 and the inequality (8.4), we obtain

$$\begin{aligned}
&\sum \exp\left(-q\frac{n_0}{m_0}\tau_x\left(\tilde{U}_{1j}(x, y_{os})\right)\right) \cdot \left(\mathrm{diam}\ \tilde{U}_{1j}(x, y_{os})\right)^\alpha \\
=&\sum_j \sum_{y_{os}} \sum_{x \in X_{y_{os}}} \exp\left(-q\frac{n_0}{m_0}\tau_x\left(\tilde{U}_{1j}(x, y_{os})\right)\right) \cdot \left(\mathrm{diam}\ \tilde{U}_{1j}(x, y_{os})\right)^\alpha \\
\le& N \cdot S_0(\delta) \cdot a_2^\alpha e^{\bar{\beta} \cdot \frac{n_0}{m_0} q} \textstyle\sum_j \exp\left(-q\tau_y\left(\tilde{U}_{2j}\right)\right) \cdot \left(\mathrm{diam}\ \tilde{U}_{2j}\right)^\alpha \\
\le& N \cdot S_0(\delta) \cdot a_2^\alpha \cdot e^{\frac{n_0}{m_0}\bar{\beta} q} \cdot K.
\end{aligned} \tag{8.5}$$

Since K is an arbitrary small number, the inequality (8.5) means that

$$\alpha_x\left(\frac{n_0}{m_0}q, A_{1n}\right) \le \alpha_y(q, A_{2n}), \quad q > 0. \tag{8.6}$$

2. Starting with a cover $\{U_{1j}\}$ of the set A_{1n} for which

$$\sum_j \exp\left(-q\tau_x\left(U_{1j}\right)\right)\left(\mathrm{diam}\ U_{1j}\right)^\alpha \le K,$$

and repeating the proof above, we obtain that

$$\alpha_y\left(\frac{m_0}{n_0}q, A_{2n}\right) \le \alpha_x(q, A_{1n}), \quad q > 0. \tag{8.7}$$

Now, assume that $q_{on}^{(x)} =: \dim_P(A_{1n})$ is the dimension for Poincaré recurrences of the set A_{1n}, $\alpha_x\left(q_{on}^{(x)}\right) = 0$, and $q_{on}^{(y)} =: \dim_P(A_{2n})$ is the dimension for Poincaré recurrences of the set A_{2n}, i.e., $\alpha_y\left(q_{on}^{(y)}\right) = 0$. Since $\alpha_x(q)$ and $\alpha_y(q)$ are monotone functions, (8.6) implies that $\alpha_x\left((n_0/m_0)\, q_{on}^{(y)}\right) \le 0$, i.e.,

$$q_{on}^{(x)} \ge \frac{n_0}{m_0} q_{on}^{(y)}. \tag{8.8}$$

Similarly, (8.7) implies that $\alpha_y\left((m_0/n_0)\, q_{on}^{(x)}\right) \le 0$, i.e.,

$$q_{on}^{(y)} \ge \frac{m_0}{n_0} q_{on}^{(x)}. \tag{8.9}$$

It follows that $q_{on}^{(y)} = (m_0/n_0)\, q_{on}^{(x)}$, i.e.,

$$\dim_P(A_{2n}) = \frac{m_0}{n_0} \dim_P(A_{1n}). \tag{8.10}$$

3. It follows from [13, Theorem 1.1] and (8.10) that

$$\begin{aligned} \dim_P(A_2\backslash B_2) &= \sup_n \dim_P(A_{2n}) = \frac{m_0}{n_0} \sup_n \dim_P(A_{1n}) \\ &= \frac{m_0}{n_0} \dim_P(A_1\backslash B_1). \end{aligned} \tag{8.11}$$

■

Remark 8.1. We believe that (under some general conditions), $\dim_P(A_2) = (m_0/n_0) \dim_P(A_1)$ as well. Of course, the Poincaré recurrences on the "bad" sets B_1 and B_2 can be different from those on $A_1\backslash B_1$ and $A_2\backslash B_2$. However, since $\overline{\dim}_B B_i < \dim_H(A_i\backslash B_i)$, $i = 1, 2$, by assumption, we believe that a "randomly chosen" point on A_i belongs to $A_i\backslash B_i$. In numerical simulations we may neglect "bad points" and treat the equality (8.3) as an indicator of (m_0/n_0)-synchronization.

9. Concluding Remarks

The asymptotic equality (5.6) shows that we may expect

$$\begin{aligned} &\left\langle \exp\left(-q_0^{(x)}\tau(U_{1j})\right)\right\rangle \sim \varepsilon^{b_1}, \\ &\left\langle \exp\left(-q_0^{(y)}\tau(U_{2j})\right)\right\rangle \sim \varepsilon^{b_2}, \end{aligned} \tag{9.1}$$

where $q_0^{(x)} = \dim_P(A_1)$, $q_0^{(y)} = \dim_B(A_2)$ and $b_i = \dim_P A_i$, $i = 1, 2$. We may expect also that $\dim_B(A_i) = \dim_H(A_i)$. In this case, (9.1) may imply the asymptotic equalities

$$\begin{aligned} \langle \tau(U_{1j}) \rangle &\sim -\frac{b}{q_0^{(x)}} \ln \varepsilon, \\ \langle \tau(U_{2j}) \rangle &\sim -\frac{b}{q_0^{(y)}} \ln \varepsilon, \end{aligned} \tag{9.2}$$

where $b = b_1 = b_2$, and $\{U_{ij}\}$ is a cover of A_i by balls of diameter ε, $i = 1, 2$. The formulas (9.2) can serve as the basic ones for some algorithms for the indication of synchronized regimes. Some preliminary results with W.-W. Lin show that such algorithms work, and we hope to present the results in the nearest furture.

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